2 Fair Distribution

2.1 Four Principles of Distributive Justice

In this chapter and the next, a *benevolent dictator* representing the public authority seeks a reasonable compromise between the conflicting interests of the parties involved in a given problem of distribution. Reasonable means "for a reason," and the axioms are the formal expression of such reasons. The manager of a firm, the parents in a family, and the judge in a litigation are all acting as benevolent dictators.[1]

Recall Aristotle's maxim, sometime called the formal principle of distributive justice: "Equals should be treated equally, and unequals unequally, in proportion to the relevant similarities and differences." The term "proportion" should not be taken too literally here; the interesting point is to draw our attention to the "relevant" similarities and differences.

Four principles guide the definition of "relevance," and are not exclusive of one another. They are *compensation, reward, exogenous rights,* and *fitness*. The canonical story[2] is that of a flute that must be given to one of four children. The first child has much fewer toys than the other three, hence should get the flute by the compensation principle. The second child worked hard at cleaning and fixing it, so he should get it as a reward. The third child's father owns the flute (although the father does not care for it), so he has the right to claim it. The fourth child is a flutist, so the flute must go to him because all enjoy the music (fitness argument).

Compensation, reward, and exogenous rights belong squarely to the principles of fairness. As explained below, fitness is related to fairness as well as to welfare.

Compensation and Ex post Equality

Certain differences in individual characteristics are involuntary, morally unjustified, and affect the distribution of a *higher-order* characteristic that we deem to equalize. This justifies unequal shares of resources in order to compensate for the involuntary difference in the primary characteristic and achieve equality of the higher-order characteristic.

Nutritional *needs* differ for infants, pregnant women, and adult males, and hence call for different shares of food. The ill *needs* medical care to become *as* healthy as a "normal" person. The handicapped *needs* more resources to enjoy certain "primary" goods, such as transportation or access to public facilities. A socioeconomic disadvantage calls for more educational resources to restore equal access to the job market. Economic needs are the central justification of the macroeconomic redistributive policies, taking the form of tax breaks, welfare support, and medical aid programs.

1. The discussion of cost- and surplus-sharing in chapter 5 takes place in the benevolent dictator context.
2. The story goes back to Plato.

Formally, the compensation principle is implemented by defining an index v_i representing the level of the higher-order characteristic enjoyed by agent i, and a function u_i transforming her share of resources y_i into her index $v_i = u_i(y_i)$. For instance, v_i is the level of satisfaction of i's nutritional needs (with $v_i = 1$ and $v_i = 0$ representing full satisfaction and starvation, respectively) and y_i is the amount of food she eats (where for simplicity food is measured along a unidimensional scale, e.g., calories). Thus a pregnant woman i and an elderly male j who eat the same amount of food y are not nourished equally, $u_i(y) < u_j(y)$, or equivalently, it takes more food to bring i at the same level of nourishment than j, $u_i(y_i) = u_j(y_j) \Rightarrow y_i > y_j$. The other examples are similar: if v_i is the level of health, and y_i the amount of medical care devoted to agent i, a healthy person i needs no care at all $u_i(0) = 1$, and the amount of care y_j it takes to restore j's health measures the seriousness of his condition. The definition of the function u_i is "objective," and agent i bears no responsibility in its shape. This feature is essential to the benevolent dictator interpretation.

Equality ex post can be applied to many other indexes than the satisfaction of basic needs. Handicaps in a horse race restore equal chances of winning by an unequal distribution of weight, tax breaks to certain businesses restore their compensation, travel subsidies for conference participants restore equality of the cost of attending, and so on.

Reward

Differences in individual characteristics are morally relevant when they are viewed as voluntary and agents are held responsible for them. They justify unequal treatment.

Past sacrifices justify a larger share of resources today (veterans). Past wrongdoings justify a lesser share: reckless drivers should pay more for insurance, no free healthcare for the substance abuser, no organ transplant for the criminal. Past hardships to my ancestors justify, vicariously, a compensation transfer today: affirmative action.

Merit by extraordinary achievement calls for reward: prizes to a creator, an athlete, a peacemaker, and other outstanding individuals.

A central question of political philosophy is the fair reward of individual productive contributions: the familiar Lockean argument entitles me to the fruit of my own labor, but this hardly leads to a precise division rule except when the production of output from the labor input unambiguously separates the contributions of the various workers. Separating the fruit of my labor from that of your labor is easy only when your labor creates no externality on mine, and vice versa. If we are fishing in the same lake, cutting wood from the same forest, or sharing any other kind of exhaustible resources, this separation is no longer possible hence the fair reward of one's labor is not a straightforward concept. The same difficulty arises when sharing joint costs or the surplus generated by the cooperation of actors with different input contributions: some bring capital, some bring technical skills, and so on. This question is the subject of chapter 5; it is also discussed in this chapter (e.g., example 2.4).

Exogenous Rights

Certain principles guiding the allocation of resources are entirely *exogenous* to the consumption of these resources and to the responsibility of the consumers in their production. Such is the right of private property in the flute story: the point is that ownership is independent from the consumption of the flute (and the related questions who needs it?, who deserves it?, who will make the best use of it?).

A paramount instance of exogenous rights is the fairness principle of equality in the allocation of certain basic rights such as political rights, the freedom of speech and of religion, or access to education. My right to vote and to be eligible for office equals yours, despite the fact that I don't care to vote or to run for office, or that I will use my vote irrationally or wastefully, such as by voting according to the phases of the moon. My right to education is not related to my IQ, nor to the admirable deeds I perform when I am not in school.

Equal exogenous rights correspond to *equality ex ante,* in the sense that we have an equal claim to the resources (be they the ability to vote and the weight of one's vote, the duty to be drafted, the right to police protection, the access to a public beach, etc.) regardless of the way they affect our welfare and that of others. This stands in sharp contrast with equality ex post suggested by the compensation principle.

Examples of unequal exogenous rights are numerous and important as well. Beside private ownership (see above), there is also the difference in status brought about by social standing or by seniority. When the beneficiaries of the distribution are institutions or represent groups of agents, the inequality in their exogenous rights is commonplace: shareholders in a publicly traded firm, or political parties with different size of representation in the parliament, should have unequal shares of decision power; creditors in the American bankruptcy law are prioritized, with the federal government coming first followed by the trustees, and the shareholders come last; and so on.

Fitness

Resources must go to whomever makes the best use of them, flutes to the best flutist, the child to his true mother (Solomon), the book in Japanese to whomever can read Japanese, the cake to the glutton, and so on.

Thus *fitness* justifies unequal allocation of the resources independently of needs, merit, or rights. Formally, fitness can be expressed in two conceptually different ways, *sum-fitness* and *efficiency-fitness*.

The concept of sum-fitness relies on the notion of utility, namely the measurement of the higher-order characteristic that is relevant to the particular distributive justice problem at hand. Going back to the discussion of the compensation principle above, the central object is the function transforming resources into utility. If we distribute medical care among a group of patients, the index v_i represents the health level of patient i and $v_i = u_i(y_i)$ is the function telling us what health level is achieved by what level of care. If we distribute food, v_i is i's level of nourishment (satisfaction of nutritional needs) and y_i her share of food. If we divide a cake, v_i is the degree of "pleasure" accruing to i when he eats the share y_i.

Sum-fitness allocates resources so as to maximize total utility of the concerned agents. Sum-fitness is a fairness principle, however unsettling and radical its recommendations may be at times. The critical comparison of sum-fitness—maximizing the sum $\sum_i u_i$ of individual utilities—and of compensation—equalizing u_i across all agents—accounts for a familiar trade-off of distributive justice.

Consider the flute example. The only use of the flute is to play music and music can be heard by everyone. Say that the utility of child i has two components: the objective quality a_i of the music being played, and the pleasure b he derives from playing the instrument (the same for every child). With n being the number of children, total utility when the flute is given to child i is $n \cdot a_i + b$, where a_i measures how well i plays the flute. Here sum-fitness unambiguously recommends giving the flute to the most talented flutist. The compensation principle, on the other hand, would sometime time-share, allowing the children to take turn playing the flute.[3]

In section 2.5 we compare sum-fitness and compensation in a simple model of fair division: often their recommendations differ sharply, yet at a deeper level of analysis the two principles can be viewed as two faces of the same coin.

The more general concept of efficiency-fitness (or simply efficiency, or Pareto optimality) is the central normative requirement of collective rationality; section 1.3. Efficiency-fitness is developed in chapter 3, and it plays a leading role in the subsequent chapters.

Efficiency-fitness typically imposes much looser constraints than sum-fitness on the allocation of resources. For instance, in the simple models of this chapter,[4] efficiency-fitness is automatically satisfied. In the general welfarist approach of chapter 3, efficiency-fitness is compatible with sum-fitness—also called classical utilitarianism—with compensation—in the form of the egalitarian collective utility function—and with many other compromises between these two extremes.

3. If each child plays $1/n$th of the time, each will enjoy the utility $b/n + Ea$, where Ea is the average skill of the children. For some values of b and a_i, $b/n + Ea > \max_i a_i$ so that time-sharing improves upon the utility of all agents but one (the best flutist) and yields a more egalitarian distribution of net utilities.

4. Here the resources are "one-dimensional," meaning that, a given amount of a single divisible commodity is distributed.

We now discuss some examples, contrasting the four principles of distributive justice.

Example 2.1 Lifeboat Consider the allocation of a single indivisible "good": each agent can either have it or not. The benevolent dictator must choose, under some constraints, who will get it and who will not. The paradigmatic example is access to the lifeboat when the ship is sinking: the lifeboat is too small to accommodate everyone. Other dramatic examples include medical triage—who will receive medical attention, in a war or a natural disaster—the allocation of organs for transplant, and immigration policies.

We start by the genuine lifeboat story, where seats in the boat must be rationed. The simplest version of exogenous rights is strict equality: we draw lots to pick who should be sacrificed. Alternatively, exogenous rights amount to an exogenous priority ranking: keep the good citizen (respected scientist, politician, or whatever, provided that his skills are not useful in these circumstances) and throw out the bad one (criminal). Compensation suggests letting the "strong" men take their chance by swimming, whereas the "weak" women and children stay on the boat, thus equalizing ex post chances of survival. The reward viewpoint would dispose of the one who causes the ship to sink. Finally fitness commands to keep on board the crew (for their navigation skills) or the women and children (for the sake of future humankind: the child has more potential for welfare than the old; the women can bear children).

Another example is food rationing in a besieged town. Compensation says to give more food to the sick and the children; fitness favors those who fight in defense of the town, whereas reward favors those who risked their lives to get the supplies; finally exogenous rights enforces either strict equality of rations or make the size of one's ration depend on social status.

In medical triage, compensation gives priority to the most severely wounded, reward gives priority to the bravest soldiers, exogenous rights enforces strict equality or priority according to rank in the hierarchy, and fitness maximizes the expected number or recoveries (where recovery refers to the ability to fight) implying that one badly wounded soldier who needs intensive care is sacrificed in favor of several soldiers to whom recovery can be guaranteed with few medical resources.[5] An alternative interpretation is priority according to rank (a general is more important to victory than a private). A variant is medical triage where, after an earthquake, fitness gives priority to doctors and engineers who produce the most social value under the circumstances; reward is not relevant, unless we want to punish looters.

In the allocation of organs for transplant, compensation gives priority to whose who can survive the shortest time or whose life is most difficult without a new organ; reward gives

5. Later in example 2.7 we have a model of medical triage where the decision is not simply "in" or "out" but the quantity of medical resources allocated to each wounded soldier.

priority according to seniority on the waiting list (first come, first served); exogenous rights enforces strict equality of chances (lottery) or priority according to social status, or wealth (if the donation of the wealthy patient does not increase the availability or organs); fitness maximizes medical fitness, namely chances of success of transplant.

In the next three examples, the decision does not bring life or death. But the problem is formally equivalent in the sense that we must decide who is "in" and who is "out."

In immigration policy, compensation admits political or economic refugees; exogenous rights is blind equality (lottery) or priority based on an ethnic, religious, or racial characteristic (e.g., Germany, Israel); reward gives priority to those with a *record* of "good deeds" for the country in question (even if they will retire after immigration, and be a net burden to their host) such as investments and political support; fitness gives priority to those with an *expectation* of good deeds, with useful skills, with a commitment to invest, and the like, or priority to those whose relatives have already immigrated.

In admission to colleges, compensation gives priority to applicants with disadvantageous socioeconomic background; exogenous rights give equal right to admission (often the rule in European universities—France, Netherlands—where rationing can take the form of a lottery); unequal exogenous rights include quotas favoring minority students (regardless of their own circumstances), children of alumni, or citizens of foreign countries; reward gives priority to the student's academic record as it reflects *past* efforts and achievements; fitness also uses the academic record but as a signal correlated to *future* success in the college itself (note the analogy with the previous example).

In tickets for an overdemanded musical performance or sporting event, compensation gives priority to out-of-town residents or to applicants who have not been attending any of the previous events; for exogenous rights a lottery is faultlessly egalitarian, or alternatively, priority to politicians, honor students in the local high school, or to any group whose distinctive characteristic bears no relation to the event; reward gives priority to sponsors of the orchestra or team; fitness favors musicians or music teachers, or athletes.

Example 2.2 Queuing and Auction Two common methods used to ration seats in concert halls but also in planes and in private clubs are queuing and auction. Queuing (with a real waiting cost) rewards effort and effort is correlated to benefit from the good in question; therefore it meets the sum-fitness criterion better than a lottery because seats will go to their most eager consumers. On the other hand, queuing is an inefficient use of time. Auctioning the goods is in one sense the best system from the fitness angle because the goods go to those who value them most without any waste of resources; hence total utility is maximized. This argument is rigorous only if all agents are of comparable wealth: the most eager opera fan will be denied access if she is short of cash. Thus auctioning is unpalatable because it favors the rich (think of the right to buy off conscription), a criterion orthogonal to merit (reward) and to fitness. Only if agents have comparable wealth, will auctioning maximize sum-fitness *and* efficiency-fitness. Yet the common practice of bumping passengers off a plane by auction shows that wealth differences are not always viewed as an ethical obstacle to the fairness of the auctioning method. The choice to be bumped is voluntary, which makes the method more acceptable than if the airline was auctioning the right to *stay* on the plane (with the proceeds being redistributed to passengers who are bumped).

Example 2.3 Political Rights Plato (in the *Republic*) invoked the fitness argument when proposing to place philosophers at the reins of government. As recently as one hundred years ago, inequalities in voting rights (or eligibility to a political office) based on wealth, land ownership, or literacy were more common than the universal suffrage (and eligibility to office) that has become the modern norm. The moral basis of these unequal voting rights was a combination of fitness (uneducated and/or poor citizens cannot form a reasonable opinion) and reward (the wealthier I am, the higher my contribution to the commonwealth, hence the higher my stake in the decisions being taken). Both arguments have been swept aside to leave room for the strict exogenous equality of individual political rights, an inalienable component of membership in the political community. The age limit and the denial of voting rights to the insane are two fitness arguments still in place. Denial of rights to criminals is a reward argument.

In many voting bodies, equality of voting rights is not warranted: members of the European union, and shareholders in a board meeting, are given unequal voting weights because they represent unequal population sizes or capital investment. An interesting and important question is the just distribution of weights. Simple proportionality does not work because a small agent may end up with no influence whatsoever on the decision process, and we must rely on other normative principles.[6]

2.2 A Simple Model of Fair Distribution

The model discussed in this and the next two sections is the simplest formal model of distributive justice. There is a given amount t of a commodity to be divided among a given set of agents, and each agent i is endowed with a claim x_i. The commodity can be a "good" (valuable resource) or a "bad" (a cost to be shared, e.g., a tax burden): if the former, we call x_i the demand of agent i; if the latter, we speak of his liability.

The problem is that t, the available resource, differs from the total sum x of claims: $t \neq x_N = \sum_i x_i$. If there is equality, we simply meet each agent's demand, or assign his liability to each agent.

6. The most popular method is an application of the Shapley value.

We distinguish the cases where t is smaller or larger than x_N: we speak of a *deficit,* of a rationing situation, in the former case, of *excess* in the latter case. The most frequent case is when the commodity is a good and the resource t falls short of x_N. One example is rationing an overdemanded good, as in rationing prescription drugs: x_i is the quantity prescribed to agent i and t is the pharmacist's supply. Two other examples are bankruptcy (x_i is the bankrupt firm's debt to creditor i, t is its liquidation value; see below) and inheritance (x_i is agent i's deed and t the value of the estate).

The case with a "good" commodity and resources in excess of the claim is illustrated in example 2.4 about a joint venture: x_i is agent i's opportunity cost of joining the venture (in our story, we speak of stand-alone salaries) and t its total revenue. Of course, both cases $t > x_N$ (excess) and $t < x_N$ (deficit) are possible here.

The case where the commodity is a bad is no less interesting. The design of a taxation schedule (section 2.4) is a key example: x_i is agent i's taxable income and t total tax to be levied.[7] Other examples discussed below include a fund-raising story and the distribution of chores. In the fund-raising story, x_i is agent i's pledge and t is the amount that must be raised; thus both cases of an excess $t > x_N$ or a deficit $t < x_N$ are plausible. In the distribution of chores, x_i is the amount for which i is responsible and t is the actual workload.

Throughout the rest of chapter 2 we assume equal exogenous rights, namely the differences in their claims is the only reason to give different shares to the agents. In particular, two agents with identical claims must receive the same share. Also fitness plays no role, with the exception of the model in section 2.5.[8] As either every agent wants more of the good or every agent wants less of the bad, efficiency-fitness is automatically satisfied. Moreover we identify an agent's share with her welfare; therefore sum-fitness has no bite either. Thus our discussion bears on the principles of compensation and reward.

Example 2.4a Joint Venture: Excess Teresa is a pianist and David is a violinist. They work as a full-time duo. Before the duo was formed, Teresa was earning \$50K a year as a teacher and solo artist, and David \$100K as the first violinist of a symphony orchestra. After one year of performing together, the net revenue of their duo is \$210,000. What is a fair split of this revenue?

The key to the example is the interpretation of the cooperation technology.

One viewpoint is that the stand-alone salaries are relevant to the cooperative process, and agents are held responsible for them. The presumption is that the input of each instrument is to some extent separable in the final product; it makes sense to take stand-alone salaries as a

7. In section 2.4 we consider a dual interpretation where $x_N - t$ is the tax and t is the net salary mass; the taxation problem becomes an instance of dividing an overdemanded good.

8. Here we model explicitely a utility function transforming a share of resources into welfare. This gives the sum-fitness property some bite.

proxy for the value of their respective contributions, and to divide profit in the corresponding proportions. To make this interpretation more plausible, consider the case of a famous singer and her unknown accompanying pianist, so their ex ante earnings are very different.

This first solution is called the *proportional* solution, and its mathematical formulation is transparent. Teresa and David in our example receive 70K and 140K respectively. More generally, if x_i is agent's stand-alone salary and t total revenue of the joint venture, agent i's share is

$$y_i = \frac{x_i}{\sum_N x_j} t \tag{1}$$

Here is another plausible solution: taking their stand-alone salaries as the "status quo ante" outcome, the agents divide equally the surplus (in excess of the status quo) generated by their cooperative venture: in this view the difference in the voluntary characteristics (stand-alone salaries) is preserved (at 50K). Teresa and David get 80K and 130K respectively, and more generally,

$$y_i = x_i + \frac{1}{n}\left(t - \sum_N x_j\right) \tag{2}$$

This is the *equal surplus* solution that is always more (resp. less) advantageous to the agent with the smallest (resp. largest) value x_i than the proportional solution above.[9]

The third solution of interest pushes the egalitarian criterion one step further. The stand-alone salary sets a floor on an agent's share because no one should be penalized for joining the cooperative venture. Except for this constraint, the revenue is shared equally among all agents. This solution regards the individual contributions as no more separable than that of the left and right hands clapping, hence stand-alone salaries as irrelevant to the production process—if not to the division of the proceeds. In our example Teresa and David get 105K each. With a total revenue of 190K, they would get 90K and 100K respectively: as long as total revenue is below 200K, David's share stays put at 100K and Teresa gets all the surplus. When revenue is above 200K, it is split equally.

The mathematical expression of this third solution is slightly more involved. Agent i receives a common share λ or his stand-alone salary, whichever is largest: $y_i = \max\{\lambda, x_i\}$. The common share λ is computed by solving the equation

$$\sum_N \max\{\lambda, x_i\} = t \tag{3}$$

9. This mathematical fact is easy to check using the two formulas above; see exercise 2.5.

This solution is called the *uniform gains* solution. It is more (resp. less) advantageous to the agent with the smallest (resp. largest) value than the equal surplus solution (exercise 2.5).

Example 2.4b Joint Venture: Deficit Now suppose that total revenue falls short of 150K, the sum of the stand-alone salaries. We must divide a *deficit* instead of a surplus. Her stand-alone salary is an upperbound on an agent's share because everyone must bear a share of the deficit.

The three solutions are easily adapted to the deficit case. The proportional solution is given by the same formula (1).

The uniform gain solution pursues the same egalitarian goal, but this time the share y_i must not exceed x_i. Each agent receives a common share λ or x_i, whichever is less: $y_i = \min\{\lambda, x_i\}$. The common share λ is the solution of the equation:

$$\sum_N \min\{\lambda, x_i\} = t \tag{4}$$

Finally the equal surplus solution becomes the *uniform losses* solution, which aims at subtracting the same amount from every stand-alone salary. Say that total revenue is 90K: the deficit 60K is shared equally between Teresa and David who end up with 20K and 70K. But if total revenue is very low, say 40K, the deficit 110K cannot be split equally lest Teresa ends up paying David from her own pocket! In other words, equalization of the losses must be adjusted to take into account the constraint $y_i \geq 0$. In the example Teresa ends up with nothing at all, and David keeps the 40K. This contradicts even the mildest version of the reward principle, as Teresa gets nothing for her work!

If the uniform losses solution is implausible in the joint venture problem, for the reasons given above, it is very convincing in other contexts, examples of which are provided below. Its mathematical expression is as follows: if the common loss is μ, agent's share y_i is $y_i = \max\{x_i - \mu, 0\}$; in other words, i's loss is the smallest of the two numbers μ and x_i. The common loss μ is the solution of the equation

$$\sum_N \max\{x_i - \mu, 0\} = t \tag{5}$$

Figure 2.1 illustrates our three solutions, proportional, uniform gains and equal surplus/ uniform losses in the case of two agents with claims x_1, x_2. The vector $x = (x_1, x_2)$ is fixed and the total t to be divided varies from 0 to infinity, generating a path for the vector of shares $y = (y_1, y_2)$.

The figure illustrates the three solutions in a deficit case (with total resources t, solutions a, b, and c) and in an excess case (t', a', b', c'). In the deficit case, agent 2 with the smallest claim of the two prefers his share under the uniform gains solution (point c) to his

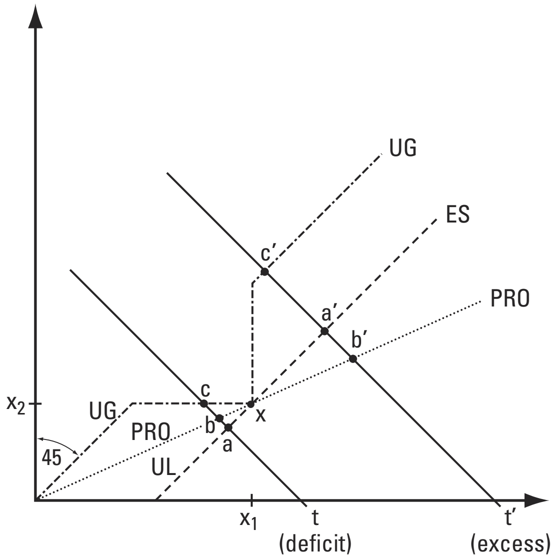

Figure 2.1
Three basic rationing/surplus-sharing methods

proportional share (point b), and the latter to his uniform losses share (point a). In the excess case, his first choice is uniform gains (c'), and his last is proportional (b'). Exercise 2.5 generalizes to an arbitrary number of agents these preferences of the agent with the largest or smallest claim.

We now discuss three examples where the three solutions appeal differently to our intuition.

First, consider *bankruptcy:* creditor i has a claim $\$x_i$ and the total liquidation value t of the firm is smaller than the sum of the debts. If our creditor have equal exogenous rights, the proportional solution is compelling. It is the legal solution as well.

An important—indeed a characteristic—feature of this solution is its robustness to the transfer of claims. If creditors i, j with claims x_i, x_j reallocate the entire claim to agent i—effectively merging the two claims in one—their total share $y_i + y_j$ is unaffected. The same is true if a creditor i splits into two creditors $i_1 + i_2$, and divides his claim x_i as $x_i = x_{i_1} + x_{i_2}$ in arbitrary shares for i_1 and i_2: the splitting operation leaves the total share of $\{i_1, i_2\}$ identical to that of i. Contrast this with the uniform gains solution, where the merging of two claims x_i, x_j in one of size $x_i + x_j$ results in either a smaller or the same share for the "merged" agent; symmetrically the splitting of one claim x_i into two subclaims x_{i1}, x_{i2}

can only increase the total share ($y_{i_1} + y_{i_2} \geq y_i$). The reverse statements (i.e., merging is advantageous and splitting is disadvantageous) hold for the uniform losses solution in the case of a deficit. See exercise 2.2.

It is not too difficult to show that the proportional solution is the only solution in the deficit problem robust with respect to the transfer of claims (exercise 2.2). Hence it is compelling in any situation where claims are akin to anonymous bonds, as is nearly the case in a bankruptcy situation (e.g., stocks are transferable), except for some creditors with special status such as the federal government.

Next, consider rationing *medical supplies*. The pharmacist has t units of a certain drug—say insulin—in stock and patient i shows a prescription for x_i units; t falls short of x_N. The uniform losses solution is appealing here: x_i represents an objective "need," and it seems fair to equalize net losses, assuming that a loss of k units of insulin below the optimal level is equally harmful to every patient. On the other hand, if the demand for the drug is not based on an objective need, say sleeping pills or diet pills, uniform gains seems most fair: it corresponds to the familiar rationing by individual coupons—the same number of coupons per person. In both cases—insulin and diet pills—the proportional solution is not especially appealing.

Third, consider a *fund-raising* situation. Donor i offers to contribute $\$x_i$ to the project in need of funding. If the actual cost t exceeds the sum of x_i, and it must be raised among these same donors, the proportional solution will unfairly penalize the most generous donor, unless we can view x_i as a proxy of his ability to pay. Equal surpluses is not appealing even if all donors are viewed as equally able—if unequally willing—to pay for the project. In this case the uniform gains solution is the most plausible of the three, charging the extra cost to the most timid donors first.

Symmetrically, suppose that the actual cost t falls short of the sum of the pledges x_i: How should we allocate rebates? Here the uniform gains is quite appealing. If the pledges are ranked as $x_1 \geq x_2 \geq x_3 \geq \ldots$, this solution gives the first $(x_1 - x_2)$ dollars of rebate to donor 1, splits equally the next $2(x_2 - x_3)$ dollars of rebate between donors 1 and 2, and so on.[10] If the donors have similar ability to pay, uniform gains rewards first the most generous agents, meaning the highest contributors. On the other hand, uniform losses is not palatable for symmetric reasons as it gives an equal rebate irrespective of the size of x_i. If all agents have the same ability to pay, this rewards bad behavior; if, on the contrary, x_i is a proxy of their ability to pay, a proportional rebate is the most natural compromise.

Example 2.5 We illustrate the three solutions by a numerical example with five agents with respective claims and 20, 16, 10, 8, and 6, so the total claim is 60. The table gives the shares for five values of total resource t: $t = 20, 40, 50$ yield a deficit and $t = 80, 120$ an excess.

10. This algorithm is described in exercise 2.6.

Claims		20	16	10	8	6
$t = 20$	PRO shares	6.7	5.3	3.3	2.7	2
	UG shares	4	4	4	4	4
	UL shares	11.3	7.3	1.3	0	0
$t = 40$	PRO shares	13.3	10.7	6.7	5.3	4
	UG shares	8.7	8.7	8.7	8	6
	UL shares	16	12	6	4	2
$t = 50$	PRO shares	16.7	13.3	8.3	6.7	5
	UG shares	13	13	10	8	6
	UL shares	18	14	8	6	4
$t = 80$	PRO shares	26.7	21.3	13.3	10.7	8
	UG shares	20	16	14.7	14.7	14.7
	ES shares	24	20	14	12	10
$t = 120$	PRO shares	40	32	20	16	12
	UG shares	24	24	24	24	24
	ES shares	32	28	22	20	18

We turn to the issue of computing two of our three solutions, the uniform gains and uniform losses solution, of which the mathematical definitions (3), (4), and (5) are not entirely transparent. A simple algorithm to compute the uniform gains solution works as follows. Divide t in equal shares and identify agents whose claims are on the "wrong" side of t/n. If we have a deficit, this means those agents with $x_i \leq t/n$; if we have an excess, it means those with $x_i \geq t/n$. Give their claim x_i to those agents, decrease the resources accordingly, and repeat the same computation among the remaining agents, with the remaining resources.

A similar algorithm delivers the uniform losses solution in the deficit case: apply formula (2) and identify all agents who receive $y_i \leq 0$, give zero to these agents. Repeat the algorithm among the remaining agents.

In the deficit case the algorithm computing the uniform gains solution reveals that agent i's share must be at least t/n or x_i, whichever is smaller: $y_i \geq \min\{x_i, t/n\}$. Indeed, an agent who is on the wrong side of t'/n', at any stage of the algorithm where t' units remain to be shared among n' agents, receives $y_i = x_i$. Moreover the sequence of per capita shares $t/n, t'/n', t''/n''$, is nondecreasing because at each step the claims of the agents who are dropped are below the per capita share. Therefore an agent who is always on the right side of t'/n' receives no less than t/n. Exercise 2.8 elaborates on such lower bound for the uniform gains shares, and stresses that neither uniform losses nor proportional meets any nontrivial lower bound. Exercise 2.6 offers two more algorithms for computing the uniform gains and uniform losses solutions.

Our final example shows the versatility of the three basic solutions, which can be adapted to a distribution problem with indivisible units and lotteries.

***Example 2.6 Scheduling** A server processes one job per unit of time. User i demands x_i jobs. For every user, the earlier a job is done, the better. The server must schedule total demand $x_N = \sum_i x_i$, namely decide in what order the x_N jobs will be processed. A scheduling sequence is a list $\{i_1, i_2, \ldots, i_t, \ldots, i_{x_N}\}$, where for all t, i_t is one of the users and where user i appears exactly x_i times in the sequence. For the sake of fairness, the server randomizes the choice of the sequence $\{i_1, \ldots, i_{x_N}\}$.

The link between this scheduling problem and the rationing of an overdemanded commodity is apparent if we fix a date t and consider the number y_i of user i's jobs processed up to date t. The vector (y_i) is a division of t units among users with demand profile (x_i); it is a solution to a rationing problem where the resources come in indivisible units and their allocation is random.

The proportional solution works by filling an urn with x_N balls, where x_i balls are labeled i, then drawing balls from the urn successively and without replacement: in other words, all sequences $\{i_1, \ldots, i_{x_N}\}$, where each i appears x_i times are equiprobable. An alternative definition of proportional scheduling goes as follows: if at time t user i has z_i jobs still unserved, the $(t+1)$th job will be given to user i with probability z_i/z_N. To see why this method corresponds to the proportional solution of the rationing problem, observe that the *expected* number of i's jobs served in the first t periods is $(x_i/x_N) \cdot t$.

Proportional scheduling has been deemed an unfair solution because a small demand x_i is swamped by a much larger demand x_j. If x_j becomes arbitrarily large, the expected share of agent i up to a fixed date t dwindles to zero (his expected waiting time until completion of all his jobs becomes arbitrarily large). The simple method known as "fair queuing" avoids this problem by giving an equal chance of receiving the first (most preferred) unit of service, irrespective of the sizes of their demands.

Specifically, fair queuing serves first one job of each user i such that $x_i \geq 1$, using an ordering of these agents selected at random, with uniform probability on all orderings; next all users i such that $x_i \geq 2$ are served a second unit in random order, and so on. In other words, the method empties a series of urns where each agent is allowed to throw at most one ball in each urn.

Figure 2.2 illustrates the method in the two agents' case, and suggests its relation to the uniform gains solution. In the first t periods, the expected number of i's jobs served is precisely given by the uniform gains shares.

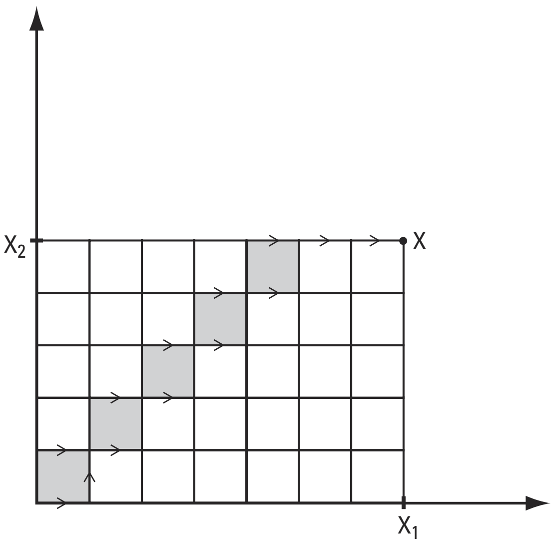

Figure 2.2
Fair queuing in example 2.6

Under fair queuing a small demand is not swamped by a large demand, as with proportional scheduling. In fact an agent's delay is unaffected when another agent with a larger demand raises his demand even further.

A related observation is the following incentive property of fair queuing. Suppose that agent i needs x_i jobs. By inflating his demand to x'_i jobs, $x'_i > x_i$, she will not affect in any way the (random) scheduling of her x_i first jobs: hence this move is not profitable (and neither is a symmetric reduction of her demand, obviously). By contrast, artificially inflating one's demand is always profitable if the proportional scheduling method is used, in the sense that the first x_i jobs (the true demand) will be served earlier.

Our third scheduling method works by simply reversing the scheduling sequence of fair queuing and for this reason is called its dual method[11] and denoted fair queuing*. This means that fair queuing* selects the sequence $\{i_1, i_2, \ldots, i_{x_N}\}$ with the same probability as fair queuing selects the sequence $\{i_{x_N}, i_{x_N - 1}, \ldots, i_2, i_1\}$; see figure 2.3.

This definition is not very intuitive, but fortunately a more direct one is available: fair queuing* gives the first job to one of the agents with largest demand x_i(with equal probability among these agents if they are two or more); the $(t + 1)$th job goes (with equal

11. More on duality appears in the next section.

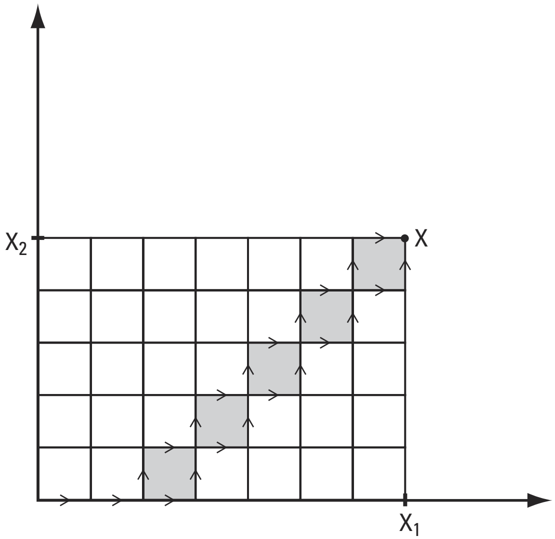

Figure 2.3
Fair queuing* in example 2.6

probability) to one of the agents with the largest remaining demand z_i after the first t periods. This algorithm is entirely similar to the algorithm provided in exercise 2.6 to compute the uniform losses solution. Hence, in particular, the expected distribution of jobs served after t periods is given by the uniform losses formula (5).

The scheduling example uncovers several properties of great relevance for the general deficit-sharing problem when the resources are divisible and shares are deterministic—formulas (1), (4), and (5). First of all it suggests a dynamic interpretation of the three basic methods, proportional, uniform gains and uniform losses in the case $t < x_N$. Think of the resources t as being distributed progressively, at a rate of 1 unit per unit of time. Uniform gains shares the incremental resources dt equally among all agents whose demand x_i is not yet met at date t. Uniform losses shares dt equally among all agents with the largest remaining demand at date t. Proportional simply shares dt in proportion to x_i at all time.

Next we check that the two properties of fair queuing discussed above apply to the uniform gains solution as well. First, consider the effect on agent 1 of raising the claim/demand of another agent, say agent 2, whose demand was larger than agent 1's in the first place. That is, we start from claim $x_1, x_2, x_3, \ldots$ with $x_1 \leq x_2$, and raise x_2 to $x'_2, x'_2 > x_2$, leaving all other claims and the resources t unchanged. Under the uniform gains solution, agent

$1'$s share does not change either. To see this, consider equation (4), with solution λ before and λ' after the raise of agent 2's claim. If $\lambda \leq x_2$, then $\lambda' = \lambda$ solves (4) in both cases; hence agent 1's share $\min\{\lambda, x_1\}$ does not change. If $\lambda > x_2$, λ' is smaller than λ but cannot fall below $x_2 : \lambda' \geq x_2$ (exercise: Why?). The assumption $x_1 \leq x_2$ now implies that agent 1's share is x_1 before and after the raise. Exercise 2.3 shows that this property (known as *independence of higher claims*) is in fact characteristic of the uniform gains solution.

The second idea emerging from the discussion of example 2.5 is the immunity to a strategic misreport of one's characteristics, in this case the report of an inflated demand: under the proportional method such a report is profitable, under fair queuing/uniform gains it is not. The immunity in question is called *strategy-proofness* in microeconomic jargon; it plays a central role in chapters 4 and 6. It turns out that uniform gains is the only equitable and strategyproof division method. Exercise 2.4 explains this remarkable property.

*2.3 Contested Garment Method

Consider an inheritance or bankruptcy problem where agent i's claim x_i is the debt he holds on the bankrupt firm or a legitimate deed he received from the deceased. The liquidation value of the firm—or the actual value of the estate—is t and $t < x_N$.

Suppose that agent i holds a claim x_i and $x_i \geq t$. This means that this agent claims the entire estate. In this case the uniform gains solution does not pay attention to the *unfeasible* claim $x_i - t$. For instance, two agents i, j such that $x_i \geq t$ and $x_j \geq t$ must receive the *same* share under uniform gains, though x_i and x_j may be very different. Formally, the uniform gains solution is unchanged when we replace claim x_i by $x_i' = \min\{x_i, t\}$. This follows at once from the fact that the solution λ of equation (4) cannot exceed t (exercise: Why?). Therefore $\min\{\lambda, x_i\} = \min\{\lambda, \min\{x_i, t\}\}$, and λ is still the solution of (4) in the problem with claims x_i'.

We call the *truncation property* the fact that we can truncate any claim larger than t at the level t without affecting the distribution. Obviously neither the proportional solution nor the uniform losses have the truncation property, but we have seen that the uniform gains solution does.

A property related to truncation (the link will become clear only after we define the duality operation below) rests on the idea of *concession*. In the problem x, t, the quantity $t - x_{N\backslash i}$ represents what is left of the resources after all agents but i have received their full claim. Naturally this number may be negative or zero, but if it is positive, it is a share of the resources that agent i will necessarily receive.[12] We write $c_i = \max\{t - x_{N\backslash i}, 0\}$ and call this quantity the concession by $N\backslash i$ to agent i.

12. This is because $y_j \leq x_j$ for all j and $t - x_{Ni} < x_i$.

In the example 2.5 nobody gets a concession if $t = 20$ or $t = 40$. However, if $t = 50$, we have $c_1 = 10, c_2 = 6, c_i = 0$, for $i = 3, 4, 5$. We take away these concessions from the initial claims and distribute the $50 - 10 - 6 = 34$ remaining units given the profile of reduced claims (10, 10, 10, 8, 6). Under the uniform losses solution we can simply compute the shares for this reduced problem, namely (8, 8, 8, 6, 4) where everyone loses two units. The solution to the initial problem obtains by adding the profile of concessions, namely (10, 6, 0, 0, 0).

The decomposition above does not work for the uniform gains (or proportional) solution because the uniform gains shares of the reduced problem are (7, 7, 7, 7, 6), and

$$(7, 7, 7, 7, 6) + (10, 6, 0, 0, 0) \neq (13, 13, 10, 8, 6)$$

We call the *concession* property the fact that the distribution of t units can be done in two steps, first giving concession c_i to each agent i, next sharing $t - c_N$ units according to the profile of reduced claims $x_i - c_i$. We note that the uniform losses solution satisfies the concession property,[13] but neither uniform gains nor proportional does.

The *contested garment* solution to the deficit problem t, x_1, x_2 takes its name from the following passage in the Talmud: "Two people cling to a garment; the decision is that one takes as much as his grasp reaches, the other takes as much as his grasp reaches, and the rest is divided equally among them."

We interpret agent i's concession c_i as his "grasp," namely the part of the garment that the other agent is not claiming. In a two-person problem the definition of c_i is

$$c_i = \max\{t - x_j, 0\} = t - \min\{x_j, t\}, \qquad \text{where } \{i, j\} = \{1, 2\}$$

The contested garment (CG) solution gives concession c_i to agent i for $i = 1, 2$, and it divides the remaining resources $t - (c_1 + c_2)$ equally: agent i receives $c_i + (t - c_1 - c_2)/2$. Rearranging this expression with the help of the formula for c_i above, we get the *contested garment shares:*

$$\begin{aligned} y_1 &= \tfrac{1}{2}(t + \min\{x_1, t\} - \min\{x_2, t\}) \\ y_2 &= \tfrac{1}{2}(t - \min\{x_1, t\} + \min\{x_2, t\}) \end{aligned} \tag{6}$$

Recall that in example 2.4b, we had $x_T = 50\text{K}$, $x_D = 100\text{K}$, and $t = 90\text{K}$. Here $c_T = 0$, $c_D = 40$, and (6) gives $y_T = 25$, $y_D = 65$.

Unlike any of our three earlier solutions, the contested garment solution meets *both* the concession property and the truncation property. This is clear for the truncation property,

13. To prove this fact, observe that a solution r meets the concession property if and only if its dual r^* (defined below) meets truncation.

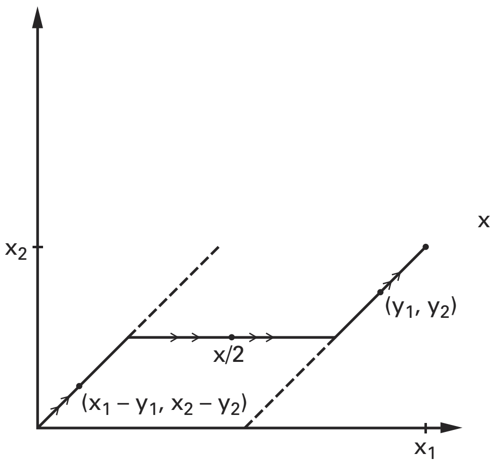

Figure 2.4
Contested garment method

since formula (6) depends only on $\min\{x_i, t\}$, $i = 1, 2$. It takes some work to check the concession property. Moreover the the contested garment solution is the *only* two-person solution for which both concession and truncation hold: exercise 2.9 explains this axiomatization of the contested garment.

Figure 2.4 depicts the path among which the vector of shares (y_1, y_2) given in (6) varies as the resources t vary from zero to $x_1 + x_2$. It reveals that our new solution coincides with uniform gains for small t (i.e., when $t \leq \min\{x_1, x_2\}$) and with uniform losses for large t (when $t \geq \max\{x_1, x_2\}$). These two observations follow respectively from the truncation and concession properties (exercise: Why?).

In figure 2.4 we see that the path between 0 and x is symmetric around the midpoint $x/2$. If (y_1, y_2) is on this path, so is $(x_1 - y_1, x_2 - y_2)$. In words, if the contested garment solution divides t as (y_1, y_2), it divides $x_1 + x_2 - t$ as $(x_1 - y_1, x_2 - y_2)$, meaning it also divides a *deficit* of t as (y_1, y_2)—receiving $x_1 - y_1$ is the same for agent 1 as incurring a deficit y_1.

This property is called *self-duality:* the method divides a deficit exactly as it divides a gain (so a bottle half full is really the same as a bottle half empty). The relevant concept here is the *duality* operation. Given a solution $y = r(t, x)$, where x stands for the vector (x_i) of claims, and y for the vector (y_i) of shares, the dual solution r^* is defined as follows:

$$r_i^*(t, x) = x_i - r_i(x_N - t, x)$$

Thus r^* divides t units of "gain" exactly as r divides t units of deficit. For instance, the proportional solution is self-dual because it divides gains and deficits alike in proportion to claims.

The dual of the uniform gains method is the uniform losses method, and vice versa. Taking example 2.5, let us compare uniform gains and uniform losses at $t = 20$ and $t = 40$, meaning 20 units of gains and 20 units of deficit. Agent 1 with claim 20 has a uniform losses share 11.3 at $t = 20$ and a uniform gains share $8.7 = 20 - 11.3$ at $t = 40$. Similarly he has a uniform gains share 4 at $t = 20$ and uniform losses share $16 = 20 - 4$ at $t = 40$. The same comparison applies to all agents.

The normative appeal of a self-dual method ($r = r^*$) is to eliminate the difference between a gain and a loss with respect to the individual claims. The choice of the reference point (at the full or null satisfaction of one's claim) does not matter.

In particular, if the bottle is exactly half full, $t = x_N/2$, a self-dual method gives half of his claim to every agent, $y_i = x_i/2$. The method requires one to be oblivious to the orientation of the units to be divided as gains or losses.

How can we generalize the two-person contested garment solution to an arbitrary number of agents? There are two natural ways to do so. They both preserve the truncation and concession properties, as well as self-duality. In view of truncation, whenever every agent claims the entire resources ($t \leq x_i$ for all i), the resources t are split equally just like uniform gains does. By concession, whenever the deficit can be covered by any agent ($x_N - t \leq x_i$ for all i), this deficit is split equally, as under uniform losses.[14]

The first idea to generalize contested garment is *random priority*. Taking a two-agent problem, let us suppose that the two agents toss a fair coin to decide whose claim has absolute priority over the other claim. If agent 1 wins, the shares are $y_1 = \min\{x_1, t\}$, $y_2 = t - \min\{x_1, t\}$; if agent 2 wins, the shares are $y'_1 = t - \min\{x_2, t\}$; $y'_2 = \min\{x_2, t\}$. The average of these two vectors is precisely the vector of shares (6). Exercise 2.10 describes the application of the random priority idea with an arbitrary number of agents.

The second generalization of the contested garment solution to any number of agents is a clever hybrid between a uniform gains solution whenever the bottle is more than half empty $-t \leq x_N/2$, and a uniform losses solution when it is more than half full $-t \geq x_N/2$. It is the subject of exercise 2.11.

14. Indeed, $c_i = t - x_{N\setminus i} \geq 0$; hence after distribution of c_N, agent i's remaining claim is $x_i - c_i = x_N - t$. In the reduced problem each agent has the same claim, so each gets $1/n$ of $t - c_N = (n-1)(x_N - t)$. This in turn yields the UL shares.

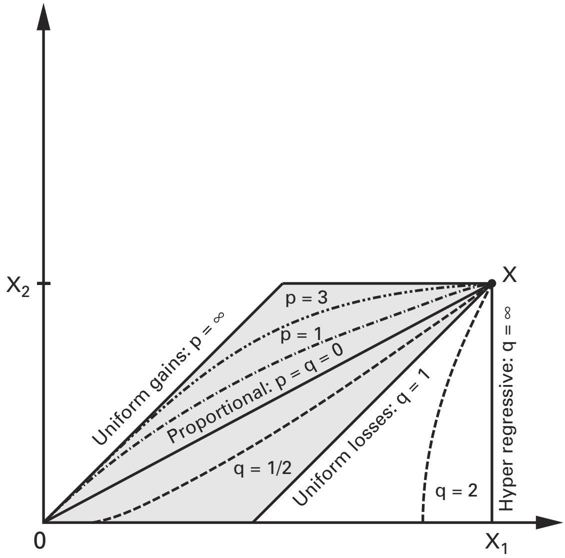

Figure 2.5
Progressive and regressive methods

*2.4 Equal Sacrifice in Taxation

In the taxation problem, x_i represents agent i's taxable income, and a given amount of tax must be divided among the n agents. We choose to write t for the total aftertax income so that $x_N - t$ is the total tax to be levied, and the share y_i is agent i's aftertax income.[15]

The simple property called *fair ranking* places some minimal equity constraints on tax shares:

$$x_i \leq x_j \Longrightarrow y_i \leq y_j \quad \text{and} \quad x_i - y_i \leq x_j - y_j \tag{7}$$

A higher taxable income warrants a higher after-tax income *as well as* a higher tax burden. In particular, equal incomes are equally taxed. In figure 2.5 the shaded area represents, in the case $n = 2$, the vectors (y_1, y_2) circumscribed by inequalities (7). Notice that the path of the uniform gains solution forms the northeastern boundary of this region, while that of the uniform losses solution forms the southwestern boundary.

15. The dual representation where t is total tax and y_i is i's tax share can be used just as easily.

Next consider the familiar ideas of progressivity and regressivity:

$$\begin{aligned} &\text{progressivity:}\quad x_i \le x_j \Rightarrow \frac{x_i - y_i}{x_i} \le \frac{x_j - y_j}{x_j} \\ &\text{regressivity:}\quad x_i \le x_j \Rightarrow \frac{x_i - y_i}{x_i} \ge \frac{x_j - y_j}{x_j} \end{aligned} \tag{8}$$

Under a progressive tax scheme, the higher the (taxable) income, the higher will be the tax rate, namely the fraction taxed away. The opposite statement holds true under a regressive scheme. Observe that the proportional solution (flat tax) is *both* progressive and regressive.

In figure 2.5 progressivity means that the vector (y_1, y_2) must be above the straight line from 0 to (x_1, x_2), and regressivity means that it must be below this line. Thus the uniform gains solution is progressive, whereas uniform losses is a regressive solution.

Uniform gains (resp. uniform losses) is in fact the most progressive (resp. the most regressive) solution among those meeting fair ranking. This is intuitively clear in figure 2.5, and exercise 2.7 gives a precise statement of these facts for an arbitrary number of agents.

The idea of equal sacrifice yields a rich family of taxation schemes that contains our three basic solutions, proportional, uniform gains, uniform losses, and much more. J. S. Mill introduced this idea first in the context of taxation: "Equality of taxation means equality of sacrifice. It means apportioning the contribution of each person towards the expenses of government, so that he shall feel neither more nor less inconvenience from his share of the payment than every other person experiences from his."

We pick an arbitrary increasing (continuous) function $z \to u(z)$ representing the conventional "utility" associated with the income z. Then the u-equal sacrifice method chooses aftertax incomes y_i so as to satisfy

$$u(x_i) - u(y_i) = u(x_j) - u(y_j) \qquad \text{for all } i, j \tag{9}$$

For a given vector of taxable incomes (x_i) and total after-tax income t, we may or may not be able to find a vector (y_i) satisfying the system above and $\sum_i y_i = t$. For instance, with the function $u(z) = \log z$ it reads $x_i/y_i = x_j/y_j$ and yields the proportional solution. On the other hand, if we set $u(z) = z$, the u-equal sacrifice method resembles the uniform losses solution but the system (9) may yield some negative shares y_i. In order to guarantee a solution meeting the constraint $y_i \ge 0$ for all i, we modify the system as follows:

$$\text{for all } i\text{:}\quad y_i > 0 \Rightarrow u(x_i) - u(y_i) = \max_j \{u(x_j) - u(y_j)\} \tag{10}$$

Only those agents who get a positive after-tax income incur the largest sacrifice. The system (10), together with $\sum_i y_i = t$, always has a unique solution (exercise: prove this claim). With $u(z) = z$ this solution is uniform losses.

An equal sacrifice method always meets half of the fair ranking property (7), namely $x_i \leq x_j \Rightarrow y_i \leq y_j$. The other half, $x_i \leq x_j \Rightarrow x_i - y_i \leq x_j - y_j$, is satisfied if and only if u is a *concave* function ($u'(z)$ nonincreasing in z).

The u-equal sacrifice method is progressive if and only if $z \cdot u'(z)$ is nonincreasing in z, namely u is more concave than the log function;[16] it is regressive if and only if $z \cdot u'(z)$ is nondecreasing in z, meaning that u is less concave than the log function.[17]

The simplest family of equal sacrifice methods comes from taking for u a power function. This allows the u-equal sacrifice method to be *scale invariant:* if we multiply all taxable incomes as well as the aftertax total income by a common factor, the corresponding aftertax incomes are also multiplied by the same factor. Thus only the relative incomes x_i/x_j and total tax ratio $(x_N - t)/x_N$ matter. Two subfamilies arise.

First consider the utility function $u(z) = -1/z^p$, where p is a positive parameter. This function is increasing, concave, and more concave than the log function. Therefore it defines a progressive taxation method. Notice that this method never gives $y_i = 0$ whenever $x_i > 0$, and guarantees equal sacrifice for all: $1/y_i^p - 1/x_i^p = 1/y_j^p - 1/x_j^p$ for all i, j; together with $y_N = t$, the system (9) has a unique solution with $y_i > 0$ for all i.

In the case $n = 2$, the path $t \to (y_1, y_2)$ is depicted in figure 2.5 for $p = 1$ and $p = 3$. Two important facts: the p-method approaches the proportional one when p goes to zero; it approaches uniform gains when p goes to infinity. Thus the positive parameter p adjusts the degree of progressivity of our methods, and the u-methods connect smoothly the proportional to the uniform gains method.

Next we consider the utility function $u(z) = z^q$, where q is a positive parameter, $0 \leq q \leq 1$. This function is increasing and concave, and less concave than the log function; therefore it defines a regressive taxation method. Note that the system (10) must be used, because for small values of t some agents end up with $y_i = 0$. The corresponding path $t \to (y_1, y_2)$ is depicted in figure 2.5 for $q = \frac{1}{2}$. When q goes to zero, the u-method approaches the proportional one, and it shows uniform losses when $q = 1$.

When $q > 1$, the function $u(z) = z^q$ is convex instead of concave, so the u-equal sacrifice method violates fair ranking. Indeed, (9) implies that the smaller the income x_i, the higher is the tax $x_i - y_i$. For instance, when q goes to infinity, the u-method approaches the *hyperregressive* method, taxing exclusively the poor: the tax burden $x_N - t$ is allocated first to the agent(s) with the smallest x_i; if $x_i < x_N - t$ (i.e., taxing away all of agent i's income is not enough), the method taxes the next smallest x_j, and so on. See exercises 2.7 and 2.14.

16. This means that we can write u as $u(z) = a(\log z)$ for all z, where a is concave and increasing.

17. In other words, we can write $u(z) = b(\log z)$, where b is convex and increasing.

2.5 Sum-Fitness and Equality

The principles of compensation and of sum-fitness come into play with interesting differences in the simple utilitarian model of resource allocation that is our subject in this section. This model is a prelude to the more general welfarist approach in the next chapter (in particular, section 3.4). It is different from but related to the model of sections 2.2 to 2.4. The benevolent dictator must share t units of resources between n agents, and each agent has his own utility function u_i to "produce" utility from resources: $u_i(y_i)$ is agent i's utility when consuming the share y_i.

The function u_i is a personalized measurement of the benefitness derived by this agent from any possible share of resources. Depending on the context, this measure may be subjective or objective. In one instance, t may be the size of a cake and $u_i(y_i)$ the subjective pleasure derived by child i from a piece of size y_i. In another, t measures some medical resource (e.g., blood or a certain drug) and $u_i(y_i)$ is patient i's objective chance of recovery (measured before treatment) if he receives the quantity y_i.

The two principles of compensation and sum-fitness (section 2.1) correspond to, respectively, the solution that equalizes individual utilities and the solution that maximizes the sum of individual utilities:

$$\begin{aligned} &\text{egalitarian solution:} \quad \text{find } y_i \geq 0 \text{ such that } u_i(y_i) = u_j(y_j) \text{ and } y_N = t \\ &\text{(classical) utilitarian:} \quad \text{find } y_i \geq 0 \text{ maximizing } \sum_i u_i(y_i) \text{ under } y_N = t \end{aligned} \tag{11}$$

If the utilitarian solution is always well-defined mathematically (provided that each function u_i is increasing and continuous), the egalitarian one is not. For instance, the ranges of the functions u_1 and u_2 may not overlap. The proper formulation is that some agents may receive zero, $y_i = 0$, but only if they enjoy the largest utility level:

$$\begin{aligned} &\text{egalitarian* solution:} \quad \text{find } y_i \geq 0 \text{ such that } y_N = t \text{ and for all } i \\ &\qquad\qquad y_i > 0 \Rightarrow u_i(y_i) = \min_j u_j(y_j) \end{aligned} \tag{12}$$

Whenever each function u_i is continuous and nondecreasing, this definition is unambiguous. See exercise 2.15 for the mathematical discussion of this fact.

A crucial factor influencing the comparison of the egalitarian* and (classical) utilitarian solutions is whether or not the marginal utility functions decrease, namely whether or not the functions u_i are concave. They are if consuming one more unit of resources always increases an agent's utility less than did the previous unit.

This fundamental property of utility functions plays a central role in chapters 5 and 6, as in most of economic analysis. It is quite plausible in the cake-tasting example: the first bite is always the most enjoyable! Much less so in the case of medical drugs (one pill of antibiotics won't do any good but 20 may cure you) and other commodities whose consumption must reach a certain threshold in order to have an impact.

We will discuss the utilitarian model of resources allocation first in the case where all utility functions are concave. We emphasize that the classical utilitarian and egalitarian* solutions are different and yet that at a deeper level they are identical. We will show that the three solutions discussed earlier—proportional, uniform gains, uniform losses—are simple special cases of this model.

However, if utility functions are not concave, the egalitarian* and classical utilitarian solutions are irreconcilable, and they lead to radically different conceptions of distributive justice.

Example 2.7 Common Utility and Unequal Endowments The base utility function is u and agent i is initially endowed with x_i units of the resources. Upon receiving the share y_i of the resources, her final utility is $u_i(y_i) = u(x_i + y_i)$. One interpretation is redistribution of income: x_i is the income before the division of the subsidy t.

Assume that u is increasing and concave. A simple observation, due to Mill, is that the egalitarian* and classical utilitarian programs coincide in this case. Their recommendation is to equalize the net income $x_i + y_i$, taking into account the nonnegativity constraint on y_i. We compute the egalitarian* solution first. The system (12) gives

$$\text{for all } i: \quad y_i > 0 \Rightarrow x_i + y_i = \min_j \{x_j + y_j\} \tag{13}$$

Upon writing $z_i = x_i + y_i$ for the net income, we recognize here the uniform gains solution which allocates the resources $s = x_N + t$ (the surplus t) given the claims x_i.

The classical utilitarian solution maximizes $\sum_i u(x_i + y_i)$ under the constraints $y_i \geq 0$, $y_N = t$. Because u is a concave function, the first-order optimality conditions capture the optimal solution:

$$y_i > 0 \Rightarrow u'(x_i + y_i) = \max_j u'(x_j + y_j)$$

which is the same system as (13) because u' is decreasing.

Next we consider the case of an increasing and strictly *convex* utility function u (strictly increasing marginal utility). The egalitarian* solution is still computed as the uniform gains solution of the problem with claims x_i and resources $x_N + t$. It is entirely independent of the choice of the increasing utility function u. The classical utilitarian solution, on the other hand, allocates the entire subsidy to one agent with the largest initial endowment. In other words, the richest agent takes all!

To check this claim, consider two agents i, j such that $x_i \geq x_j$, and assume that they receive positive shares y_i, y_j. Convexity of u implies that

$$u(x_i + y_i) + u(x_j + y_j) < u(x_i + y_i + y_j) + u(x_j)$$

Hence transferring y_j to the "richer" agent i increases the sum of individual utilities, as required by classical utilitarianism. It is now a simple matter to deduce that an allocation is optimal for the classical utilitarian criterion if and only if it gives all the resources to an agent i with the largest initial endowment x_i (if there is exactly one such agent, the optimal allocation is unique).

In the context of redistribution of income, a convex utility function makes little sense. It does in the medical triage problem: if we have barely enough medicine to save two patients, it is ethically sensible to concentrate on the two most promising patients and ignore the others altogether. Another example is the distribution of subsidies among ailing firms in a regulated economy.

A difficulty of the "richest takes all" solution is its discontinuity with respect to individual characteristics. A small increase in the initial endowment x_i may result in a dramatic shift of the share y_i: it may give agent i the largest initial endowment, thus shifting all the resources onto his plate. This unpalatable feature never occurs with the classical utilitarian solution if utilities are concave, or with the egalitarian* solution for any utility functions; see exercises 2.15 and 2.16.

Example 2.8 Constant Utility Ratios The base utility function u is strictly concave, and agent i's utility from taking a bite of cake piece y_i is $u_i(y_i) = a_i u(y_i)$. The constant factor a_i measures agent i's productivity in generating utility. Here the compensation and sum-fitness principles make two opposite recommendations.

Assume for simplicity that $u(0) = 0$. The egalitarian* solution simply equalizes net utilities

$$a_i u(y_i) = a_j u(y_j) \qquad \text{for all } i, j$$

Therefore $a_i > a_j \Rightarrow y_j > y_i$: a larger share compensates the agents with low productivity. By contrast, the classical utilitarian solution rewards productivity and gives a larger share to the agents with a larger coefficient a_i. To see this, we write the first-order optimality condition of the maximization problem:[18]

$$a_i u'(y_i) = a_j u'(y_j)$$

and the conclusion $a_i > a_j \Rightarrow y_i > y_j$ follows because u' is decreasing.

The link between the classical utilitarian and egalitarian solutions when individual utility functions are concave is apparent when we write the first-order optimality conditions of the

18. In order to avoid boundary solutions, let us assume, for instance, that $u'(0) = +\infty$. Then the first unit of the good is infinitely more valuable than the next one.

classical utilitarian program (11). Because each u_i is concave, these conditions completely characterize the optimal solution. They are written as follows:

$$\text{for all } i: \quad y_i > 0 \Rightarrow u_i'(y_i) = \max_j u_j'(y_j)$$

Hence the utilitarian solution with utilities u_i equals the egalitarian solution with utilities $-u_i'$. Symmetrically the egalitarian solution with utilities u_i equals the classical utilitarian one with utilities $U_i = \int (A - u_i)$, where the constant A is large enough to ensure $u_i(y_i) \leq A$ for all y_i.

An important property shared by the classical utilitarian solution when utilities are concave and the egalitarian* solution for any individual utilities is *resource monotonicity*: when t increases, every individual share y_i increases. The proof is the subject of exercises 2.15 and 2.16.

Given resource monotonicity, we can think of the allocation process with given utility functions u_i and varying t, as one of pouring water into individual vessels of arbitrary shapes. In figure 2.6 are depicted three such vessels connected to a common reservoir. If the height reached by the quantity y_i of water in vessel i equals $u_i(y_i)$—the width at this level being $1/u_i'(y_i)$—the law of gravity delivers the egalitarian* solution for these utility functions.

A property related to resource monotonicity is *population monotonicity*: when an agent absconds and the resources to be divided remain the same, this is good news for all remaining

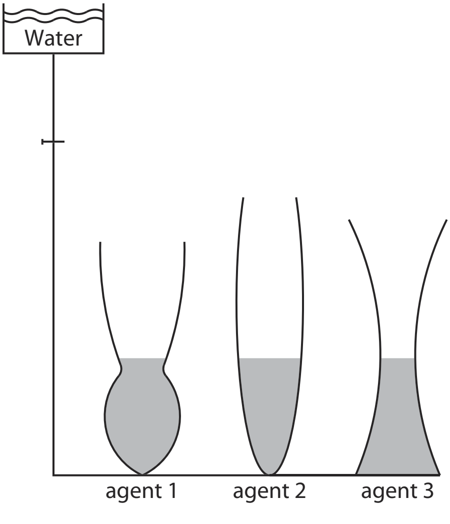

Figure 2.6
Hydraulic method

agents. This property is clear in the hydraulic representation: when agent i's vessel is shut down or destroyed, "his" water share is redistributed to all other vessels, so the level of water does not fall.

We check now that our three basic solutions, proportional, uniform gains, and uniform losses, admit a hydraulic representation. The corresponding vessels are depicted in figures 2.7 to 2.9.

In figure 2.7 the width of agent i's vessel is proportional to her claim, resulting in the proportional solution. In figures 2.8 and 2.9 the vessels are of equal width, or reduce to a tube of insignificant width. In figure 2.8 agent 4 with the largest claim x_4 receives the first $(x_4 - x_3)$ units of water; the next $2(x_3 - x_2)$ units are split equally between agents 3 and 4; the next $3(x_2 - x_1)$ units are split equally among 2, 3, and 4; all additional units are split equally among all four agents. This algorithm delivers precisely the uniform losses solution in the deficit case (see exercise 2.6), and the equal surplus solution is the excess case. We see similarly (again with the help of exercise 2.6) that the hydraulic method in figure 2.9 illustrates the uniform gains method in the deficit as well as excess cases.

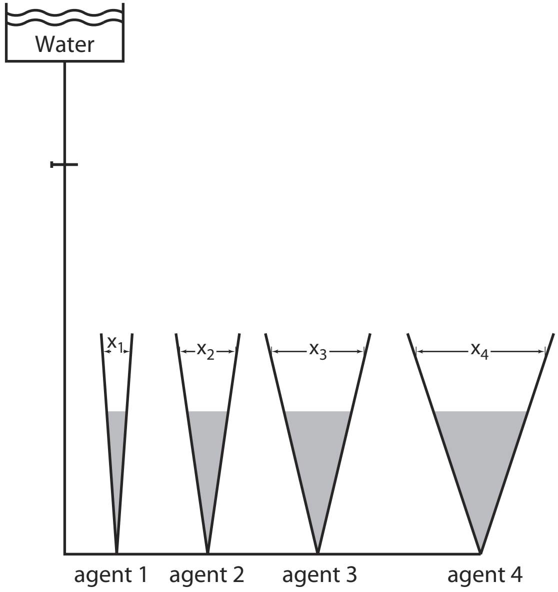

Figure 2.7
Proportional method

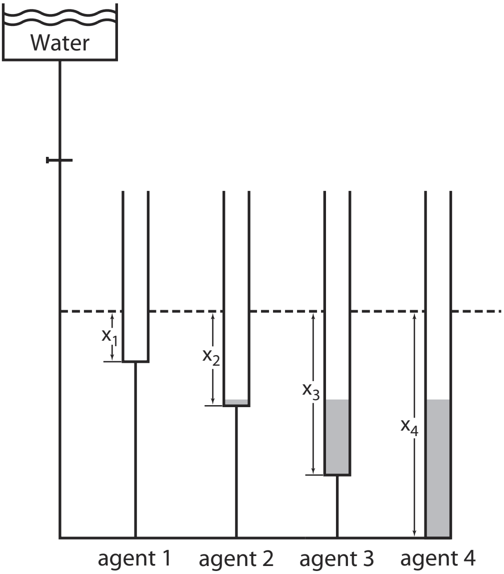

Figure 2.8
Uniform losses/equal surplus

In the hydraulic representation of the proportional solution, the height of agent i's vessel when it contains y_i units of water is $u_i(y_i) = y_i/x_i$, up to the normalization at one when the vessel contains exactly x_i units. For this choice of utilities, this solution is egalitarian* as in system (12). If we choose instead the utility functions $u_i(y_i) = Ay_i - (y_i^2/2x_i)$, where A is larger than the ratio t/x_N, the proportional solution becomes classical utilitarianism as in (11).

Computing similarly the "volume to height" function in figures 2.8 and 2.9 yields a representation of uniform losses/equal surplus and of uniform gains as egalitarian* methods in the sense of (12),[19] or classical utilitarian in the sense of (11). For instance, the uniform losses solution is classical utilitarian for $u_i(y_i) = x_i y_i - (y_i^2/2)$.

We conclude this chapter with an example where individual utilities are not concave, and the classical utilitarian solution is neither resource nor population monotonic.

19. The utility functions are, however, discontinuous, and this creates a minor technical difficulty.

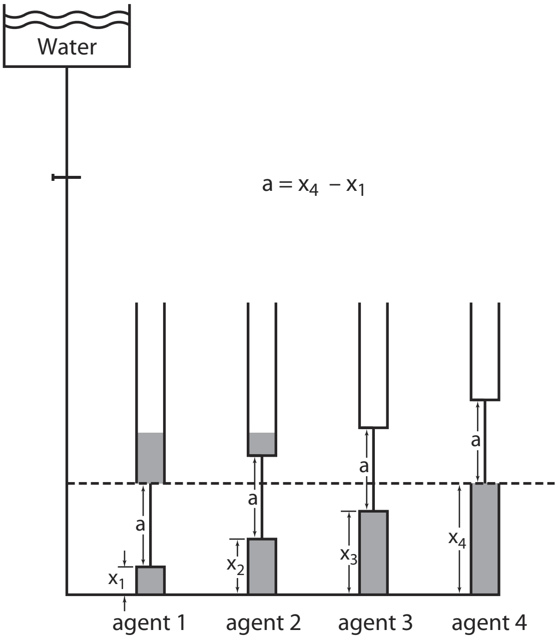

Figure 2.9
Uniform gains

Example 2.9 Failure of Resource and Population Monotonicity Two agents have the following utility functions:

$$u_1(y_1) = 2y_1, \qquad \text{all } y_1 \geq 0$$

$$u_2(y_2) = y_2 \qquad \text{for } 0 \leq y_2 \leq 10$$

$$= 4y_2 - 30 \qquad \text{for } y_2 \geq 10$$

Agent 1's marginal utility is constant and equal to 2; agent 2's is 1 up to ten units and rises to 4 afterward.

The classical utilitarian solution gives all the resources t to agent 1 or all to agent 2:

$$t < 15 \Rightarrow u_1(t) > u_2(t) \Rightarrow y_1 = t,\ y_2 = 0$$

$$t > 15 \Rightarrow u_2(t) > u_1(t) \Rightarrow y_1 = 0,\ y_2 = t$$

Notice that for $t = 15$, the solution can give t to either agent, with no possibility of compromise.

Thus an increase of the resources from $t = 10$ to $t = 20$ wipes out agent 1's share, and resource monotonicity is violated. Compare this with the egalitarian solution:

$t \leq 15 \Rightarrow y_1 = \frac{1}{3}t,\ y_2 = \frac{2}{3}t$

$t \geq 15 \Rightarrow y_1 = \frac{2}{3}t - 5,\ y_2 = \frac{1}{3}t + 5$

To check that the classical utilitarian solution is not population monotonic, fix $t = 18$ and consider a third agent with a marginal utility of 4 up to five units and zero afterward:

$u_3(y_3) = 4y_3 \qquad \text{for } 0 \leq y_3 \leq 5$

$u_3(y_3) = 20 \qquad \text{for } 5 \leq y_3$

The classical utilitarian distribution of 18 units is $y_1 = 13,\ y_2 = 0,\ y_3 = 5$. Upon dropping agent 3, the distribution becomes $y_1 = 0,\ y_2 = 18$. So that agent 1's share vanishes, in violation of population monotonicity.

2.6 Introduction to the Literature

The four principles of section 2.1 are inspired by similar taxonomies in the social psychology literature: see Deutsch (1975), Rescher (1966), and Cook and Hegtvedt (1983).

The lifeboat stories in example 2.1 are discussed by the literature on medical triage, in particular, Winslow (1982). Elster (1992) provides many examples of rationing problems, inspiring some of your examples in section 2.2 as well as in exercise 2.1. The indexes of voting power alluded to in example 2.3 are discussed extensively in two introductory books, Straffin (1980) and Felsenthal and Machover (1998).

The model of fair division developed in sections 2.2 to 2.4 appeared first in the papers by Banker (1981) and O'Neill (1982), and inspired a sizable body of axiomatic research. Recent surveys include Moulin (1988, ch. 6), Herrero and Villar (2001), and Moulin (2001a).

Aumann and Maschler (1985) focused on the contested garment method (section 2.3) and its generalization to an arbitrary number of agents (exercise 2.11). This article, together with O'Neill (1982), stresses the origin of the problem in the Talmudic literature; Rabinovitch (1973) is the historical source from which the contested garment quote is borrowed. See also exercises 2.10.

The entire section 2.4 is inspired by Young's (1988, 1990) work on equal sacrifice methods. He provides axiomatic characterizations of these methods based on the separability property known as consistency, already used by Aumann and Maschler (1985). The general characterization of consistent methods in Young (1987) is related to the hydraulic representation of deficit and surplus-sharing methods in section 2.5. Kaminski (2000) introduces this intuitive representation and explains its link with consistency. Finally the scheduling story in example 2.6 is inspired by the work on fair queuing due to Shenker (1995) and Demers et al. (1990).

Exercise 2.2 borrows ideas from Moulin (1987), and exercises 2.3 and 2.9 from Herrero and Villar (2001). Exercise 2.4 is generalized by Sprumont (1991) into a characterization of the uniform gains solution.

Exercises to Chapter 2

Exercise 2.1

In the following examples, identify the principle or principles in section 2.1 of which a given policy is an example. Find more policies and connect them similarly to the four principles.

a. For the education of which child should we spend more resources?

- The hardworking but not very gifted
- The good tempered and parent loving
- The least academically gifted
- The most academically gifted
- Equally, irrespective of gift or work

b. How to allocate scarce legal resources, namely public defenders?

- Favor defendants with the cleanest criminal record
- Favor defendants accused of the lesser crimes
- Favor defendants accused of the worst crimes
- Equalize the lawyer x hours expense on all defendants
- Minimize the total number of jail years awarded to the group of defendants
- Minimize the maximal number of jail years awarded to any one defendant

c. How to prioritize the restoration of electric power after a storm?

- Easy customers first (near to source)
- Hospitals, fire station first
- Elderly residential customers first
- Big industrial users first

- Small residential users first
- Important citizens first

Exercise 2.2 Merging and Splitting

We start with the resources t and a vector of claims (x_i), $i = 1, \ldots, n$. Both cases $t \leq x_N$ and $t \geq x_N$ are possible. We say that agents i and j merge their claims if j transfers his claim x_j to i so that the new problem has $(n-1)$ agents and agent i's claim is $x_i + x_j$. Symmetrically we say that agent i splits his claim x_i if i is replaced by two agents i_1, i_2 with claims x_{i_1}, x_{i_2} such that $x_{i_1} + x_{i_2} = x_i$, so the new problem has $(n+1)$ agents.

a. Under the proportional solution in both cases, deficit or excess, show that merging or splitting is a matter of indifference.

b. Show that under the uniform gains solution in both cases, merging is bad and splitting is good:

- After-merging share $y'_i \leq$ before-merging shares $y_i + y_j$
- After-splitting shares $y'_{i_1} + y'_{i_2} \geq$ before-splitting share y_i

c. Show that under the equal surplus solution, merging is bad and splitting is good. Show that under the uniform losses solution, merging is good and splitting is bad.

***d.** For the deficit case, $t \leq x_N$, the proportional solution is characterized by the indifference to merging/splitting property.

***Exercise 2.3 Independence of Higher Claims**

For a distribution problem with deficit, $t, x_i, t \leq x_N$, the property independence of higher claims (IHC) is discussed at the end of section 2.2:

for all i, j: $\quad x_i \leq x_j \leq x'_j \Rightarrow y_i = y'_i$

where y_i and y'_i are respectively agent i's share for the initial profile of claims x_i and for the profile where x'_j replaces x_j, everything else equal.

The goal of the exercise is to show that there is only one solution satisfying equal treatment of equals and independence of higher claims, and it is uniform gains. Fix a profile of claims x_i, and label the agents in such a way that $x_1 \leq x_2 \leq \cdots \leq x_n$. Define for $k = 1, 2, \ldots, n$, $z_k = \sum_{j=1}^{k-1} x_j + (n-k+1)x_k$ so that $z_1 = nx_1 \leq z_2 \leq z_3 \leq \cdots \leq z_n = x_N$.

a. Choose t such that $0 \leq t \leq z_1$. By ETE, at the profile x'_i, $x'_i = x_1$ for all i, each agent gets t/n. Show that by ETE and IHC, the same is true at the profile x''_i where $x''_1 = x_1$,

$x_i'' = x_2$ for $i = 2, \ldots, n$. Repeat the argument to show that equal shares still prevail at profile x_i''' where $x_1''' = x_1$, $x_2''' = x_2$, $x_i''' = x_3$, for $i = 3, \ldots, n$. Conclude that a method meeting ETE and IHC coincides with uniform gains whenever $t \le z_1$.

b. Choose t such that $z_1 \le t \le z_2$ and consider first the profile of claims x_i^*, $x_1^* = x_1$, $x_i^* = \frac{t-x_1}{n-1}$ for $i = 2, \ldots, n$. Show by a similar argument that ETE and IHC force the uniform gains solution for the interval $[z_1, z_2]$. Generalize to any t, $0 \le t \le z_n$.

Exercise 2.4 Strategy-Proofness of the Uniform Gains Solution

a. Consider a problem with deficit, $t, x_i, t \le x_N$, and denote the uniform gains shares by (y_i). Show that an agent's share y_i is nondecreasing in his claim: if x_i increases to x_i', everything else (t and x_j, for all $j \ne i$) equal, agent i's new share y_i' is not smaller than y_i.

Show that if y_i is strictly smaller than x_i, and x_i increases to x_i', everything else equal, agent i's share does not change: $y_i' = y_i$.

Deduce that if agent i prefers a larger share to a smaller one—provided they are both below x_i—he cannot benefit from altering (increasing or decreasing) his claim x_i.

b. Consider now a problem with excess, $t \ge x_N$. Show similarly that y_i is nondecreasing in x_i and show the following:

$$\{y_i > x_i \text{ and } x_i' < x_i\} \Rightarrow y_i = y_i'$$

Deduce that if among two shares not smaller than x_i, agent i prefers the smaller one, he cannot benefit from altering his claim x_i.

c. Under the proportional method, check that increasing one's claim is profitable in the deficit case, and that decreasing it is profitable in the excess case (preferences over shares are as in questions *a* and *b* respectively).

d. Under the uniform losses/equal surplus solution, which distortion of one's claim is profitable?

*Exercise 2.5

a. We fix a profile of claims x_i, ranking increasingly as $x_1 \le x_2 \le \cdots \le x_n$. Prove that agent 1 and agent n have unambiguous preferences over the three basic methods, in the sense that the three corresponding shares are always ranked in the same way.

Denoting by $y_i(X)$ agent i's share under the method X, prove the following inequalities:

Deficit

$$y_1(UL) \le y_1(PRO) \le y_1(UG)$$

$$y_n(UG) \le y_n(PRO) \le y_n(UL)$$

Excess

$y_1(PRO) \leq y_1(ES) \leq y_1(UG)$

$y_n(UG) \leq y_n(ES) \leq y_n(PRO)$

b. Find an example of a three person deficit problem, $x_1 < x_2 < x_3$, $t < x_1 + x_2 + x_3$, such that the proportional method is the worst for agent 2:

$y_2(PRO) < y_2(UG)$ and $y_2(PRO) < y_2(UL)$

c. For an arbitrary excess problem with an arbitrary number of agents, show that the equal surplus method cannot be the worst of the three for anyone, namely that the two inequalities

$y_i(ES) < y_i(PRO)$ and $y_i(ES) < y_i(UL)$

are incompatible for i.

Exercise 2.6 Other Algorithms to Compute the Uniform Gains and Uniform Losses Solutions

a. Given is a problem with excess, $t, x_i, x_N \leq t$, where the claims are ordered increasingly, $x_1 \leq x_2 \leq \cdots \leq x_n$. Consider the following algorithm:

Step 1. Increase agent 1's claims by up to $(x_2 - x_1)$ units

Step 2. Increase agent 1, 2's claims by up to $(x_3 - x_2)$ units each

Step 3. Increase agent 1, 2, 3's claims by up to $(x_4 - x_3)$ units each

$\vdots$

The algorithm stops when $(t - x_N)$ units have been distributed.

Show that the outcome is the uniform gains solution.

b. Given is a problem with deficit $t \leq x_N$, and with the claims increasingly ordered $x_1 \leq x_2 \leq \cdots \leq x_n$. The following algorithm reduces the individual claims, starting from the highest claims:

Step 1. Decrease n's claim by up to $(x_n - x_{n-1})$ units

Step 2. Decrease n and $(n-1)$'s claim by up to $(x_{n-1} - x_{n-2})$ units each

Step 3. Decrease n, $(n-1)$ and $(n-2)$'s claim by up to $(x_{n-2} - x_{n-3})$ units each

$\vdots$

We stop whenever total reduction in claims reaches $(x_N - t)$ units. At this point, each agent receives his reduced claim

Show that this algorithm delivers the uniform gains solution.

c. Given a deficit problem as in question b, consider the following algorithm:

Step 1. Give the first $(x_n - x_{n-1})$ units to agent n

Step 2. Split up to $2(x_{n-1} - x_{n-2})$ units equally between agents $n, n-1$

Step 3. Split up to $3(x_{n-2} - x_{n-3})$ units equally between agents $n, n-1, n-2$

$\vdots$

The algorithm stops when t units have been distributed.

Show that the outcome is the uniform losses solution.

***Exercise 2.7**

We are in the deficit case, $t \leq x_N$.

a. The goal is to show formally that uniform gains is the most progressive among all methods meeting fair ranking, which is property (7) in section 2.4. We fix a list of increasing claims $x_1 \leq x_2 \leq \cdots \leq x_n$. Let y_i^* be agent i's share under uniform gains and y_i be his share under an arbitrary method satisfying fair ranking. Prove that

$$y_1 \leq y_1^* \Leftrightarrow \frac{x_1 - y_1^*}{x_1} \leq \frac{x_1 - y_1}{x_1}$$

If $y_1 = y_1^*$, prove similarly that $y_2 \leq y_2^*$; if $y_1 = y_1^*$ and $y_2 \leq y_2^*$, prove that $y_3 \leq y_3^*$; and so on. State and prove an analogue sequence of properties establishing that the uniform losses method is the most regressive among those meeting fair ranking.

b. Consider the hyperregressive solution r alluded to at the end of section 2.4; the profile of shares $y = r(t, x)$ is defined by the following property:

$$\{x_i < x_j \text{ and } y_j < x_j\} \Rightarrow y_i = 0 \qquad \text{for all } i, j$$

and by equal treatment of equals. Show that this corresponds to the definition given at the end of section 2.4. Show that this solution violates fair ranking, property (7). Show that among all solutions r of the deficit problem where $0 \leq y_i \leq x_i$ for all i, the solution above is the most regressive one.

c. Define similarly the hyperprogressive method, the dual of the hyperregressive one, and show that it is the most progressive of all solutions r of the deficit problem.

*Exercise 2.8 Lower Bounds, Upper Bounds

We are in the deficit case, $t < x_N$.

a. We fix t the resources, n the number of agents and x_i the claim of a certain agent i. Show that under uniform gains, agent i's share is bounded below as follows:

$$y_i \geq \min\left\{x_i, \frac{t}{n}\right\}$$

The inequality above holds true for any choice of the variables x_j, $j \neq i$, provided $t \leq x_N$.

b. Show that under uniform losses or proportional, agent i's share can only be bounded below by zero if we do not know the variables x_j, $j \neq i$ (we only know that they satisfy $x_i + \sum_j x_j \geq t$).

c. Now we fix n, x_i and the *deficit* t^* (i.e., $t^* = x_N - t$); we do not know the variables x_j, $j \neq i$ (except that we must have $t^* \leq x_N$). Show that under uniform losses agent i's share is bounded above as follows:

$$y_i \leq \max\left\{x_i - \frac{1}{n}t^*, 0\right\}$$

What is the corresponding upper bound under proportional or uniform gains?

Exercise 2.9 Truncation and Concession

a. Fix a two-person solution for the deficit problems of section 2.3, satisfying equal treatment of equals, truncation, and concession. Fix a profile of claims x_1, x_2 with $x_1 \leq x_2$. For t such that $0 \leq t \leq x_1$, use T and ETE to show that t is split equally. For $x_2 \leq t \leq x_1 + x_2$ use similarly C and ETE to compute the shares. Finally compute the shares for $x_1 \leq t \leq x_2$ and conclude that our method is the contested garment solution.

b. Show that a solution satisfies truncation if and only if its dual (section 2.3) satisfies concession.

Exercise 2.10 Run to the Bank

We are in the deficit case, $t \leq x_N$. Given a rationing problem, we let the agents run to the bank, and we suppose that the ordering of their arrival is random and without bias: each ordering is equally plausible. The bank then serves the agents in the order of their arrival; the first agent receives his full claim or the entire resources, whichever is less; if there is something left after the first agent, the second one gets his full claim or all the remaining resources, whichever is less; and so on.

a. Show that for a two-person problem, "run to the bank" coincides with the contested garment method.

b. Consider the following inheritance problem, due to the Talmudic scholar Ibn Ezra:

> Jacob died and his son Reuben produced a deed duly witnessed that Jacob willed to him his entire estate on his death, his son Simeon also produced a deed that his father willed to him half of the estate, Levi produced a deed giving him one-third and Judah brought forth a deed giving him one-quarter. All of them bear the same date.

Compute the division of the estate under "run to the bank."
Compare it to the divisions under proportional, uniform gains, and uniform losses.

c. In example 2.5 compute the "run to the bank" solution for the values, $t = 20, 40$, and 50.

***d.** Show that run to the bank is self-dual. Show it satisfies truncation and concession (recall from exercise 2.9 that T and C are dual properties).

***e.** To a problem with deficit t, x_i, we associate the following cooperative game (see chapter 5):

$$v(S) = \min\left\{t, \sum_{i\in S} x_i\right\}$$

Show that the Shapley value of this cooperative game is precisely the "run to the bank" solution.

*Exercise 2.11 The Talmudic Solution

We are in the deficit case $t \leq x_N$. The Talmudic solution is a hybrid of the uniform gains and uniform losses solutions. The method divides $t^* = x_N/2$ in proportions to the claims x_i:

$$\text{at} \quad t^* = \frac{x_N}{2} \quad \text{we have} \quad y_i^* = \frac{x_i}{2} \qquad \text{for all } i$$

Then the method follows uniform gains with *respect to the halved claims* for t, between 0 and t^*. It follows uniform losses with respect to the halved claims for t, between t^* and x_N:

$$y_i = UG\left(t; \frac{x_i}{2}, i \in N\right) \qquad \text{if } 0 \leq t \leq t^*$$

$$y_i = UL\left(t - t^*; \frac{x_i}{2}, i \in N\right) + \frac{x_i}{2} \quad \text{if } t^* \leq t \leq x_N$$

a. Check that this method is the contested garment solution when $n = 2$.

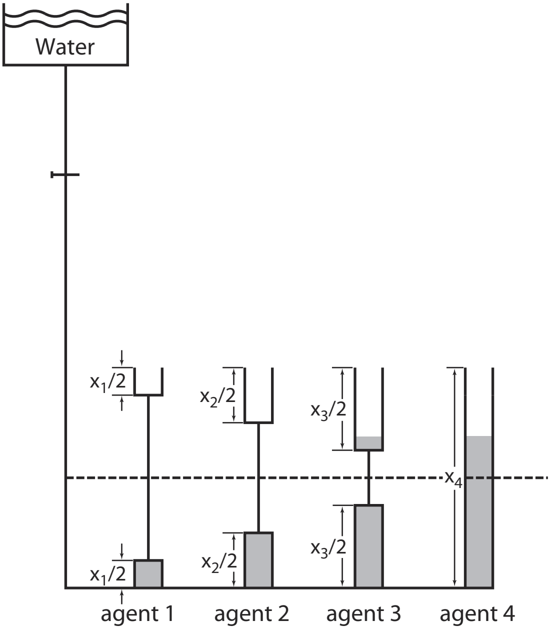

Figure 2.10
Talmudic method

b. Check its hydraulic representation on figure 2.10.

c. Compute the Talmudic solution in the numerical example of question *b* of the previous exercise.

d. Compute the Talmudic solution in example 2.5 for the three values $t = 20, 40$, and 50.

***e.** Show that the Talmudic solution is self-dual. Show it satisfies truncation and concession.

Exercise 2.12

Consider a society $N = \{1, 2, 3, 4, 5, 6, 7\}$ and the method that

- Gives absolute priority to any agent in $\{1, 2\}$ over any agent in $\{3, 4, 5\}$ and absolute priority to any agent in $\{3, 4, 5\}$ over any agent in $\{6, 7\}$
- Between agents 1 and 2 is the proportional method
- Between agents 3, 4, and 5, is the uniform gains method
- Between agents 6 and 7 is the uniform losses method

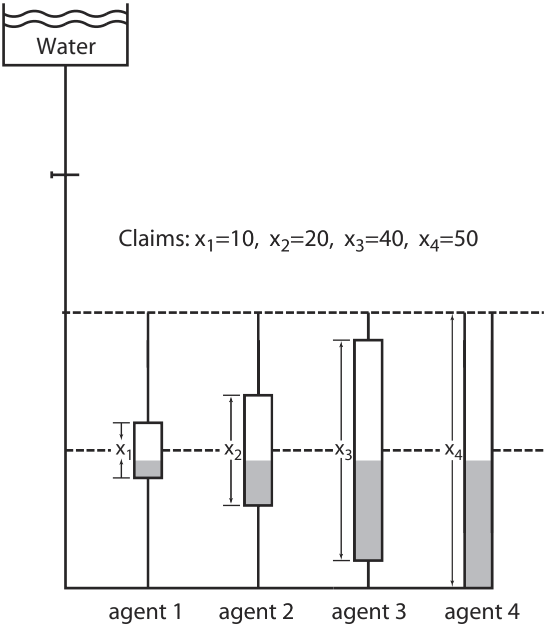

Figure 2.11
Method of exercise 2.13

a. Compute the solution it recommends in the following examples:

Agent	1	2	3	4	5	6	7	Resources
Claim	10	0	0	10	15	0	15	12 or 40
Claim	5	10	80	70	10	5	0	50 or 120
Claim	100	50	10	10	20	15	25	130 or 180 or 200

b. Give, as in section 2.5, a “hydraulic” representation of this method.

Exercise 2.13

Consider the rationing method of which the hydraulic representation is given on figure 2.11.

a. Compute the allocation it recommends for $t = 20$, $t = 50$, $t = 80$, and $t = 90$.

b. Is this method progressive? Regressive?

***c.** Generalize the method to an arbitrary number of agents and give a formula to compute the shares it recommends. *Hint:* Compare it to the Talmudic method in exercise 2.11.

Exercise 2.14

We consider the equal sacrifice methods of section 2.4.

a. Show that the method where $u(z) = z^q$ for all z, converges to the proportional solution when q is positive and approaches zero. To this end, fix a problem t, x_i, with $t < x_N$ and write $y(q)$ its solution under this method. Show that the limit y of $y(q)$ as q approaches zero is the proportional solution. Show that as q goes to infinity, the limit of $y(q)$ is the hyperregressive solution defined in exercise 2.7.

b. Consider the method where $u(z) = -1/z^p$ for all z. Show that it converges to the proportional solution when p is positive and approaches zero, and to uniform gains when p becomes arbitrarily large.

c. Show that the u-method meets fair ranking (7) if and only if u is concave.

d. Show that the u-method is progressive (resp. regressive)—see (8)—if and only if $z.u'(z)$ is nonincreasing (resp. nondecreasing) in z.

***Exercise 2.15**

We consider the egalitarian* solution to the problem t, u_i, defined by the system (12) in section 2.5.

a. Show that if u_i is continuous and strictly increasing, system (12) always has a unique solution. Show that if u_i is continuous and nondecreasing, the system (12) may have several solutions, but they all yield the same utilities $u_i(y_i)$.

b. Assume that u_i is strictly increasing and continuous. Show that the egalitarian* solution is *strictly* resource monotonic: $t < t' \Rightarrow y_i < y'_i$ for all i.

c. Show that the egalitarian* solution is robust to a small change in the utility functions. To this end, assume that u_i takes the form $u_i(a_i, x_i)$ where a_i is a real parameter and that u_i is continuous in the pair (a_i, x_i), as well as strictly increasing in x_i. Show that the egalitarian* solution depends continuously on a_i.

Exercise 2.16

Consider the classical utilitarian solution (11) to the allocation problem t, u_i of section 2.5.

a. Assume that each function u_i is strictly increasing and strictly convex (marginal utility u_i' is strictly increasing). Show that the utilitarian solution gives all the resources to a single agent (this generalizes the argument given in example 2.7).

b. Assume that each function u_i is strictly increasing and strictly concave. Recall from section 2.5 that the utilitarian solution coincides with the egalitarian* solution for the utilities $-u_i'$. Deduce that it is strictly resource monotonic. Show that it is robust to a small change in the utility function u_i, as in question c of exercise 2.15.