Exercise 10.3 *A taxpayer has income y that should be reported in full to the tax authority. There is a flat (proportional) tax rate γ on income. The reporting technology means that that taxpayer must report income in full or zero income. The tax authority can choose whether or not to audit the taxpayer. Each audit costs an amount φ and if the audit uncovers under-reporting then the taxpayer is required to pay the full amount of tax owed plus a fine F.*

1. *Set the problem out as a game in strategic form where each agent (taxpayer, tax-authority) has two pure strategies.*
2. *Explain why there is no simultaneous-move equilibrium in pure strategies.*
3. *Find the mixed-strategy equilibrium. How will the equilibrium respond to changes in the parameters γ, φ and F?*

Outline Answer

1. See Table 10.3.

			Tax-Authority s_1^b *Audit*	s_2^b *Not audit*
Taxpayer	s_1^a	*conceal*	$[1-\gamma]\,y-F,\ \gamma y+F-\varphi$	$y,\ 0$
	s_2^a	*report*	$[1-\gamma]\ y,\ \gamma y-\varphi$	$[1-\gamma]\,y,\ \gamma y$

Table 10.3: The tax audit game

2. Consider the best responses:
   - Tax-Authority's best response to "conceal" is "audit"
   - Taxpayer's best response to "audit" is "report"
   - Tax-Authority's best response to "report" is "not audit"
   - Taxpayer's best response to "not audit" is "conceal"

3. Suppose the taxpayer conceals with probability π^a and the tax authority audits with probability π^b.

   (a) Expected payoff to the taxpayer is

$$\begin{aligned} \upsilon^a &= \pi^a\left[\pi^b\left[[1-\gamma]\,y-F\right]+\left[1-\pi^b\right]y\right] \\ &\quad +[1-\pi^a]\left[\pi^b\,[1-\gamma]\,y+\left[1-\pi^b\right][1-\gamma]\,y\right], \end{aligned}$$

   which, on simplifying, gives

$$\upsilon^a = [1-\gamma]\,y+\pi^a\left[1-\pi^b\right]\gamma y-\pi^a\pi^b F.$$

   So we have

$$\frac{d\upsilon^a}{d\pi^a} = \left[1-\pi^b\right]\gamma y-\pi^b F$$

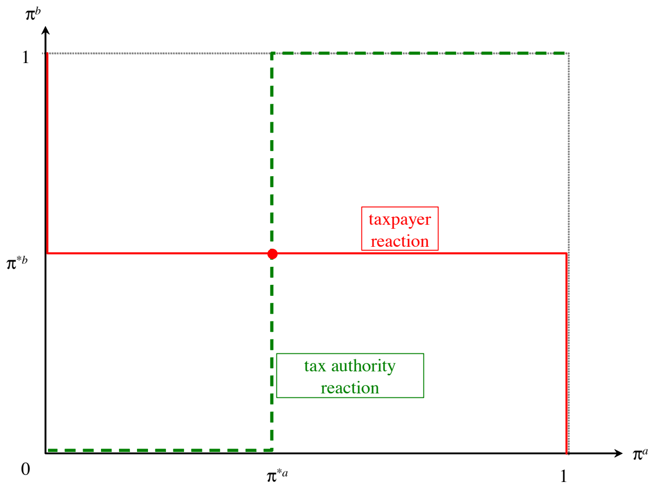

Figure 10.1: The tax audit game

It is clear that $\frac{d\upsilon^a}{d\pi^a} \gtreqless 0$ as $\pi^b \lesseqgtr \pi^{*b}$ where

$$\pi^{*b} := \frac{\gamma y}{\gamma y + F}. \tag{10.1}$$

So the taxpayer's optimal strategy is to conceal with probability 1 if the probability of audit is too low ($\pi^b < \pi^{*b}$) and to conceal with probability zero if the probability of audit is high.

(b) Expected payoff to the tax-authority is

$$\begin{aligned} \upsilon^b &= \pi^b \left[\pi^a \left[\gamma y + F - \varphi\right] + \left[1 - \pi^a\right]\left[\gamma y - \varphi\right]\right] \\ &\quad + \left[1 - \pi^b\right]\left[\pi^a \left[0\right] + \left[1 - \pi^a\right]\left[\gamma y\right]\right] \end{aligned}$$

which, on simplifying, gives

$$\upsilon^b = \left[1 - \pi^a\right]\gamma y + \pi^a \pi^b \left[\gamma y + F\right] - \pi^b \varphi$$

So we have

$$\frac{d\upsilon^b}{d\pi^b} = \pi^a \left[\gamma y + F\right] - \varphi$$

It is clear that $\frac{d\upsilon^b}{d\pi^b} \gtreqless 0$ as $\pi^a \gtreqless \pi^{*a}$ where

$$\pi^{*a} := \frac{\varphi}{\gamma y + F}. \tag{10.2}$$

So the tax authority's optimal strategy is to audit with probability 0 if the probability of the taxpayer concealing is low ($\pi^a < \pi^{*a}$) and to audit with probability 1 if the probability of concealment is high.

(c) This yields a unique mixed-strategy equilibrium $\left(\pi^{*a}, \pi^{*b}\right)$ as illustrated in Figure 10.1.

(d) The effect of a change in any of the model parameters on the equilibrium can be found by differentiating the expressions (10.1) and (10.2). we have

$$\frac{\partial \pi^{*a}}{\partial \gamma} = -\frac{\varphi y}{[\gamma y + F]^2} > 0; \quad \frac{\partial \pi^{*b}}{\partial \varphi} = \frac{Fy}{[\gamma y + F]^2} > 0.$$

$$\frac{\partial \pi^{*a}}{\partial \varphi} = \frac{1}{\gamma y + F} > 0; \quad \frac{\partial \pi^{*b}}{\partial \varphi} = 0.$$

$$\frac{\partial \pi^{*a}}{\partial F} = -\frac{\varphi}{[\gamma y + F]^2} < 0; \quad \frac{\partial \pi^{*b}}{\partial F} = -\frac{\gamma y}{[\gamma y + F]^2} < 0.$$

Exercise 10.4 *Take the "battle-of-the-sexes" game in Table 10.4*

.

		s_1^b [West]	s_2^b [East]
s_1^a	[West]	2,1	0,0
s_2^a	[East]	0,0	1,2

Table 10.4: "Battle of the sexes" – strategic form

1. *Show that, in addition to the pure strategy Nash equilibria there is also a mixed strategy equilibrium.*
2. *Construct the payoff-possibility frontier . Why is the interpretation of this frontier in the battle-of-the-sexes context rather unusual in comparison with the Cournot-oligopoly case?*
3. *Show that the mixed-strategy equilibrium lies strictly inside the frontier.*
4. *Suppose the two players adopt* ***the same*** *randomisation device, observable by both of them: they know that the specified random variable takes the value 1 with probability π and 2 with probability $1-\pi$; they agree to play $\left[s_1^a, s_1^b\right]$ with probability π and $\left[s_2^a, s_2^b\right]$ with probability $1-\pi$; show that this* ***correlated mixed strategy*** *always produces a payoff on the frontier.*

Outline Answer

1. Suppose a plays [West] with probability π^a and b plays [West] with probability π^b. The expected payoff to a is

$$\begin{aligned} v^a &= \pi^a \left[\pi^b [2] + \left[1-\pi^b\right][0]\right] + [1-\pi^a]\left[\pi^b [0] + \left[1-\pi^b\right][1]\right] \\ &= 2\pi^a\pi^b + [1-\pi^a]\left[1-\pi^b\right] \\ &= 1 - \pi^a - \pi^b + 3\pi^a\pi^b \qquad (10.3) \end{aligned}$$

So we have

$$\frac{dv^a}{d\pi^a} = -1 + 3\pi^b$$

It is clear that $\frac{dv^a}{d\pi^a} \gtreqless 0$ as $\pi^b \gtreqless \frac{1}{3}$. The expected payoff to b is

$$\begin{aligned} v^b &= \pi^b \left[\pi^a [1] + [1-\pi^a][0]\right] + \left[1-\pi^b\right]\left[\pi^a [0] + [1-\pi^a][2]\right] \\ &= \pi^a\pi^b + 2[1-\pi^a]\left[1-\pi^b\right] \\ &= 2 - 2\pi^a - 2\pi^b + 3\pi^a\pi^b. \qquad (10.4) \end{aligned}$$

And so

$$\frac{dv^b}{d\pi^b} = -2 + 3\pi^a$$

It is clear that $\frac{dv^b}{d\pi^b} \gtreqless 0$ as $\pi^a \gtreqless \frac{2}{3}$. So there is a mixed-strategy equilibrium where $\left(\pi^a, \pi^b\right) = \left(\frac{2}{3}, \frac{1}{3}\right)$.

2. See Figure 10.2. Note that, unlike oligopoly where the payoff (profit) is transferable, in this interpretation the payoff (utility) is not so the frontier has not been extended beyond the points (2,1) and (1,2). The lightly shaded area depicts all the points in the attainable set of utility can be "thrown away." The heavily shaded area in Figure 10.2 shows the expected-utility outcomes achievable by randomisation. The frontier is given by the broken line joining the points (2,1) and (1,2).

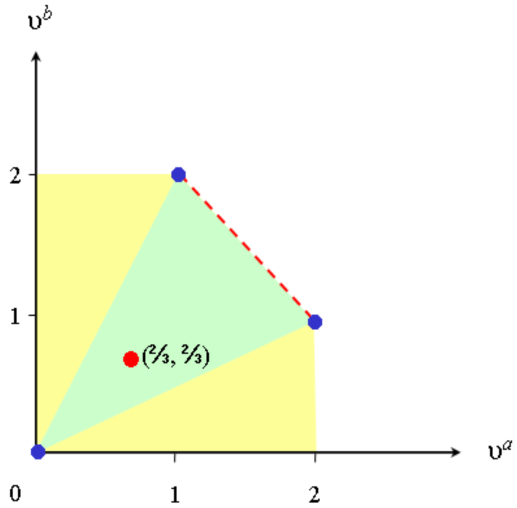

Figure 10.2: Battle-of-sexes: payoffs

3. The utility associated with the mixed-strategy equilibrium is $\left(\frac{2}{3}, \frac{2}{3}\right)$ and clearly lies inside the frontier in Figure 10.2.

4. Given that the probability of playing [West] is π, the expected utility for each player is

$$\upsilon^a = 2\pi + [1 - \pi] = 1 + \pi$$

$$\upsilon^b = \pi + 2[1 - \pi] = 2 - \pi$$

If we allow π to take any value in $[0, 1]$ this picks out the points on the broken line in Figure 10.2.

Exercise 10.5 *Rework Exercise 10.4 for the case of the Chicken game in Table 10.5.*

	s_1^b	s_2^b
s_1^a	$2, 2$	$1, 3$
s_2^a	$3, 1$	$0, 0$

Table 10.5: "Chicken" – strategic form

Outline Answer

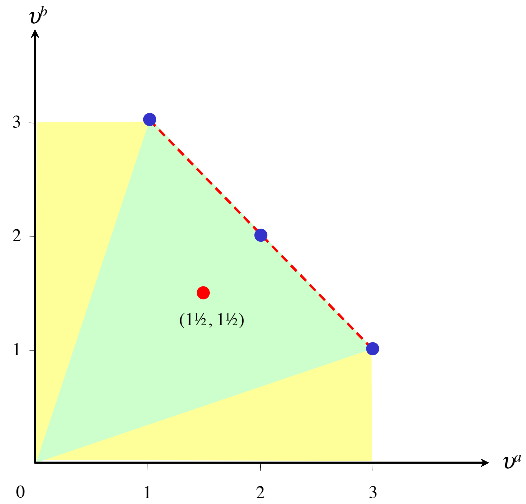

Figure 10.3: Chicken: payoffs

1. Suppose a plays s_1^a with probability π^a and b plays s_1^b with probability π^b. The expected payoff to a is

$$\begin{aligned} \upsilon^a &= \pi^a \left[\pi^b [2] + \left[1 - \pi^b\right] [1]\right] + [1 - \pi^a] \left[\pi^b [3] + \left[1 - \pi^b\right] [0]\right] \\ &= \pi^a + 3\pi^b - 2\pi^a\pi^b \qquad (10.5) \end{aligned}$$

So we have

$$\frac{d\upsilon^a}{d\pi^a} = 1 - 2\pi^b$$

It is clear that $\frac{d\upsilon^a}{d\pi^a} \gtreqless 0$ as $\pi^b \lesseqgtr \frac{1}{2}$. The expected payoff to b is

$$\begin{aligned} \upsilon^b &= \pi^b \left[\pi^a [2] + [1 - \pi^a] [1]\right] + \left[1 - \pi^b\right] \left[\pi^a [3] + [1 - \pi^a] [0]\right] \\ &= \pi^b + 3\pi^a - 2\pi^a\pi^b \qquad (10.6) \end{aligned}$$

And so

$$\frac{d\upsilon^b}{d\pi^b} = 1 - 2\pi^a$$

It is clear that $\frac{dv^b}{d\pi^b} \gtreqless 0$ as $\pi^a \lesseqgtr \frac{1}{2}$. So there is a mixed-strategy equilibrium where $\left(\pi^a, \pi^b\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

2. See Figure 10.3. The lightly shaded area depicts all the points in the attainable set of utility can be "thrown away." The heavily shaded area shows the expected-utility outcomes achievable by randomisation

3. The utility associated with the mixed-strategy equilibrium is $\left(1\frac{1}{2}, 1\frac{1}{2}\right)$ and clearly lies inside the frontier.

4. Once again a correlated strategy would produce an outcome on the broken line.

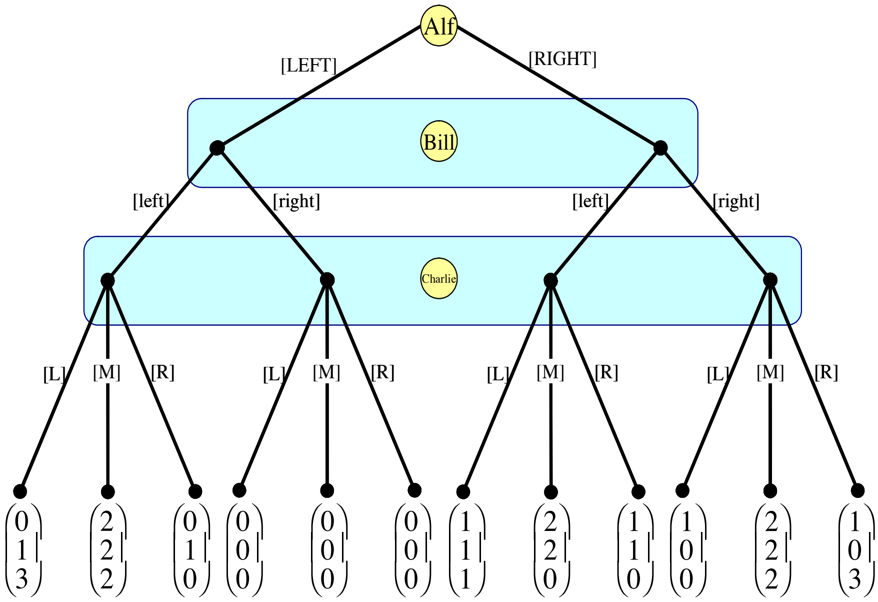

Figure 10.4: Benefits of restricting information

Exercise 10.6 *Consider the three-person game depicted in Figure 10.4 where strategies are actions. For each strategy combination, the column of figures in parentheses denotes the payoffs to Alf, Bill and Charlie, respectively.*

1. *For the simultaneous-move game shown in Figure 10.4 show that there is a unique pure-strategy Nash equilibrium.*

2. *Suppose the game is changed. Alf and Bill agree to coordinate their actions by tossing a coin and playing [LEFT],[left] if heads comes up and [RIGHT],[right] if tails comes up. Charlie is not told the outcome of the spin of the coin before making his move. What is Charlie's best response? Compare your answer to part 1.*

3. *Now take the version of part 2 but suppose that Charlie knows the outcome of the coin toss before making his choice. What is his best response? Compare your answer to parts 1 and 2. Does this mean that restricting information can be socially beneficial?*

Outline Answer

1. The strategic form of the game can be represented as in Table 10.6 from

		s_1^a :[LEFT]			s_2^a :[RIGHT]		
		s_1^c L	s_2^c M	s_3^c R	s_1^c L	s_2^c M	s_3^c R
s_1^b	[left]	$0,1,3$	$2,2,2$	$0,1.0$	$1,1,1$	$2,2,0$	$1,1,0$
s_2^b	[right]	$0,0,0$	$0,0,0$	$0,0,0$	$1,0,0$	$2,2,2$	$1,0,3$

Table 10.6: Alf, Bill, Charlie – Simultaneous move

		s_1^c L	s_2^c M	s_3^c R
Heads	[left,LEFT]	$0,1,3$	$2,2,2$	$0,1.0$
Tails	[right,RIGHT]	$1,0,0$	$2,2,2$	$1,0,3$

Table 10.7: Alf, Bill correlate their play

which it is clear that the best responses for the three players are as follows:

$$\begin{aligned} BR^a(\text{left}, \text{L}) &= \text{RIGHT}* \\ BR^a(\text{left}, \text{M}) &= \{\text{LEFT}, \text{RIGHT}\} \\ BR^a(\text{left}, \text{R}) &= \{\text{LEFT}, \text{RIGHT}\} \\ BR^a(\text{right}, \text{L}) &= \text{RIGHT} \\ BR^a(\text{right}, \text{M}) &= \text{RIGHT} \\ BR^a(\text{right}, \text{R}) &= \text{RIGHT} \end{aligned}$$

$$\begin{aligned} BR^b(\text{LEFT}, \text{L}) &= \text{left} \\ BR^b(\text{LEFT}, \text{M}) &= \text{left} \\ BR^b(\text{LEFT}, \text{R}) &= \text{left} \\ BR^b(\text{RIGHT}, \text{L}) &= \text{left}* \\ BR^b(\text{RIGHT}, \text{M}) &= \{\text{left}, \text{right}\} \\ BR^b(\text{RIGHT}, \text{R}) &= \text{left} \end{aligned}$$

$$\begin{aligned} BR^c(\text{LEFT}, \text{left}) &= \text{L} \\ BR^c(\text{LEFT}, \text{right}) &= \{\text{L}, \text{M}, \text{R}\} \\ BR^c(\text{RIGHT}, \text{left}) &= \text{L}* \\ BR^c(\text{RIGHT}, \text{right}) &= \text{R} \end{aligned}$$

it is clear that $(\text{RIGHT}, \text{left}, \text{L})$ is the unique Nash equilibrium. Everyone gets a payoff of 1 at the Nash equilibrium: total payoff is 3.

2. Charlie knows the coordination rule but not the outcome of the coin toss. The payoffs are now as in Table 10.7. Note that neither of the possible action combinations by Alf and Bill would have emerged under the Nash equilibrium in part 1. It is clear that now the expected payoff to Charlie of playing L is 1.5; the expected payoff of playing R is also 1.5. But the expected payoff of playing M is 2. So Charlie's best response is M Everybody gets a payoff of 2 with certainty: total payoff is 6.

3. Charlie now knows both the coordination rule and the outcome of the coin toss. From Table 10.7 it is clear that his best response is L if it is "heads" and R if it is "tails". Now he gets a payoff of 3 and the others get an equal chance of 0 or 1: total payoff is 4, less than that under part 2 but more than under part 1.

Exercise 10.7 *Consider a duopoly with identical firms. The cost function for firm f is*

$$C_0 + cq^f, f = 1, 2.$$

The inverse demand function is

$$\beta_0 - \beta q$$

where C_0, c, β_0 and β are all positive numbers and total output is given by $q = q^1 + q^2$.

1. *Find the isoprofit contour and the reaction function for firm 2.*
2. *Find the Cournot-Nash equilibrium for the industry and illustrate it in (q^1, q^2)-space.*
3. *Find the joint-profit maximising solution for the industry and illustrate it on the same diagram.*
4. *If firm 1 acts as leader and firm 2 as a follower find the Stackelberg solution.*
5. *Draw the set of payoff possibilities and plot the payoffs for cases 2-4 and for the case where there is a monopoly.*

Outline Answer

1. Firm 2's profits are given by

$$\begin{aligned} \Pi^2 &= pq^2 - \left[C_0 + cq^2\right] \\ &= \left[\beta_0 - \beta\left[q^1 + q^2\right]\right] q^2 - \left[C_0 + cq^2\right] \end{aligned}$$

   So it is clear that a typical isoprofit contour is given by the locus of (q^1, q^2) satisfying

$$\left[\beta_0 - c - \beta\left[q^1 + q^2\right]\right] q^2 = \text{constant}$$

   see Figure 10.5. The FOC for a maximum of Π^2 with respect to q^2 keeping q^1 constant is

$$\beta_0 - \beta\left[q^1 + 2q^2\right] - c = 0$$

   which yields the Cournot reaction function for firm 2

$$q^2 = \chi^2\left(q^1\right) = \frac{\beta_0 - c}{2\beta} - \frac{1}{2}q^1 \tag{10.7}$$

   – a straight line. Note that this relationship holds wherever firm 2 can make positive profits. See Figure 10.6 which shows the locus of points that maximise Π^2 for various given values of q^1.

2. By symmetry the reaction function for firm 1 is

$$q^1 = \frac{\beta_0 - c}{2\beta} - \frac{1}{2}q^2 \tag{10.8}$$

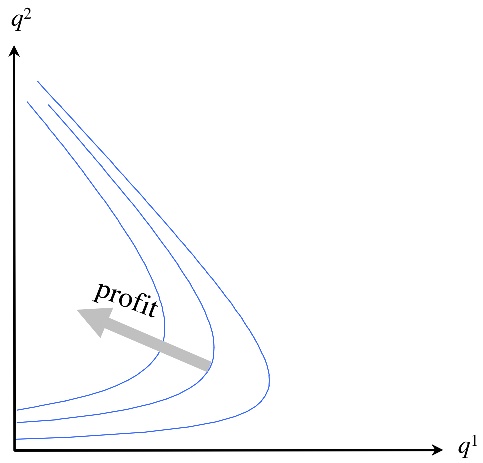

Figure 10.5: Iso-profit curves for firm 2

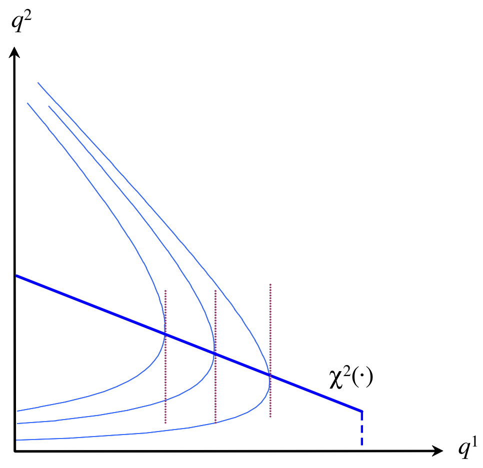

Figure 10.6: Reaction function for firm 2

The Cournot-Nash solution is where (10.7) and (10.8) hold simultaneously, i.e. where

$$q^1 = \frac{\beta_0 - c}{2\beta} - \frac{1}{2}\left[\frac{\beta_0 - c}{2\beta} - \frac{1}{2}q^1\right] \quad (10.9)$$

The solution is at $q^1 = q^2 = q_C$ where

$$q_C = \frac{\beta_0 - c}{3\beta} \quad (10.10)$$

– see Figure 10.7. The price is $\frac{2}{3}\beta_0 + \frac{1}{3}c$.

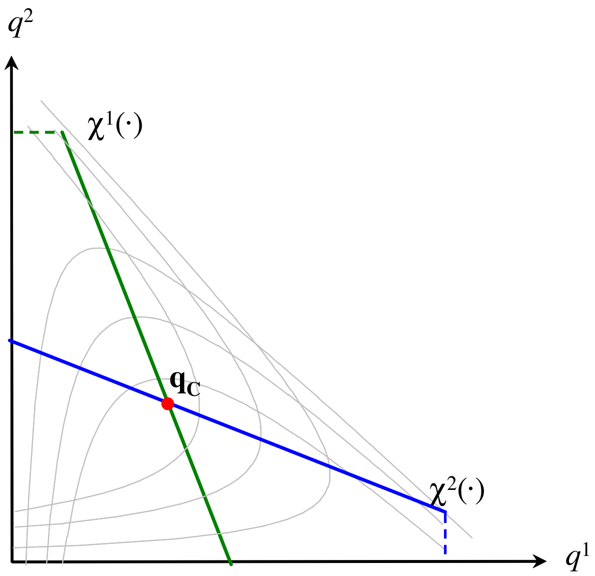

Figure 10.7: Cournot-Nash equilibrium

3. Writing $q = q^1 + q^2$, the two firms' joint profits are given by

$$\begin{aligned}\Pi^2 &= pq - [2C_0 + cq] \\ &= [\beta_0 - \beta q]\, q - [2C_0 + cq]\end{aligned}$$

The FOC for a maximum is

$$\beta_0 - c - 2\beta q = 0$$

which gives the collusive monopoly solution as

$$q_M = \frac{\beta_0 - c}{2\beta}. \quad (10.11)$$

with the corresponding price $\frac{1}{2}[\beta_0 + c]$. However, the break-down into outputs q^1 and q^2 is in principle undefined. Examine Figure 10.8. The points $(q_M, 0)$ and $(0, q_M)$ are the endpoints of the two reaction functions (each indicates the amount that one firm would produce if it knew that the other was producing zero). The solution lies somewhere on the line joining these two points. In particular the symmetric joint-profit maximising outcome (q_J, q_J) lies exactly at the midpoint where the isoprofit contour of firm 1 is tangent to the isoprofit contour of firm 2.

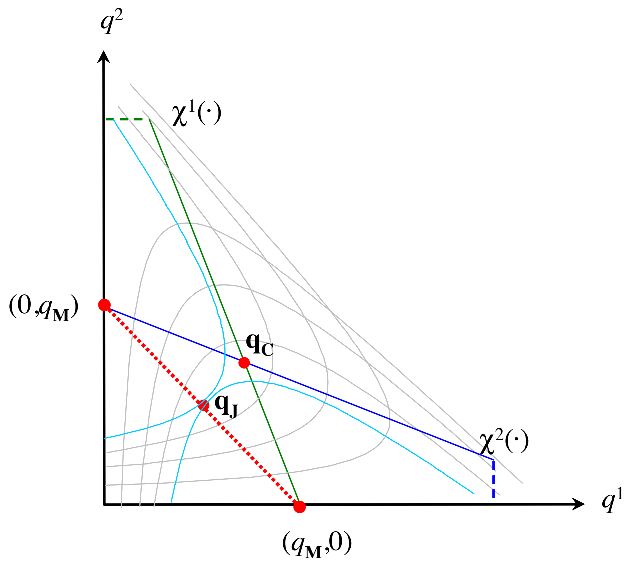

Figure 10.8: Joint-profit maximisation

4. If firm 1 is the leader and firm 2 is the follower then firm 1 can predict firm 2's output using the reaction function (10.7) and build this into its optimisation problem. The leader's profits are therefore given as

$$\left[\beta_0 - \beta\left[q^1 + \chi^2\left(q^1\right)\right]\right] q^1 - \left[C_0 + cq^1\right]$$

which, using (10.7), becomes

$$\begin{aligned} & \left[\beta_0 - \beta\left[q^1 + \frac{\beta_0 - c}{2\beta} - \frac{1}{2}q^1\right]\right] q^1 - \left[C_0 + cq^1\right] \\ = \; & \frac{1}{2}\left[\beta_0 - c - \beta q^1\right] q^1 - C_0 \end{aligned} \tag{10.12}$$

The FOC for the leader's problem is

$$\frac{1}{2}\left[\beta_0 - c\right] - \beta q^1 = 0$$

so that the leader's output is

$$q_S^1 = \frac{\beta_0 - c}{2\beta}$$

and, using (10.7), the follower's output must be

$$q_S^2 = \frac{\beta_0 - c}{4\beta}$$

	output	price	profit
Cournot	$\frac{\beta_0 - c}{3\beta}$	$\frac{1}{3}\beta_0 + \frac{2}{3}c.$	$\frac{[\beta_0 - c]^2}{9\beta} - C_0$
Joint profit max	$\frac{\beta_0 - c}{4\beta}$	$\frac{1}{2}[\beta_0 + c]$	$\frac{[\beta_0 - c]^2}{8\beta} - C_0$
Stackelberg leader	$\frac{\beta_0 - c}{2\beta}$	$\frac{1}{4}\beta_0 + \frac{3}{4}c$	$\frac{[\beta_0 - c]^2}{8\beta} - C_0$
Stackelberg follower	$\frac{\beta_0 - c}{4\beta}$	$\frac{1}{4}\beta_0 + \frac{3}{4}c$	$\frac{[\beta_0 - c]^2}{16\beta} - C_0$

Table 10.8: Outcomes of quantity competition – linear model

– see Figure 10.9. The price is $\frac{1}{4}\beta_0 + \frac{3}{4}c$.

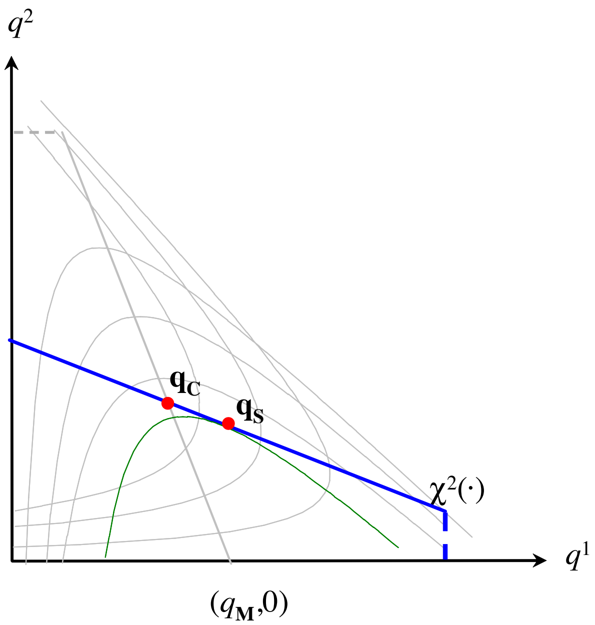

Figure 10.9: Firm 1 as Stackelberg leader

5. The outcomes of the various models are given in Table 10.8.and the possible payoffs are illustrated in Figure 10.10. Note that maximum total profit on the boundary of the triangle is exactly twice the entry in the"Joint profit max" row, namely $\frac{1}{4}[\beta_0 - c]^2/\beta - 2C_0$. This holds as long as there are two firms present – i.e. right up to a point arbitraily close to either of the end-points. But if one firm is closed down (so that the other becomes a monopolist) then its fixed costs are no longer incurred and the monopolist makes profit $\Pi_M := \frac{1}{4}[\beta_0 - c]^2/\beta - C_0$. In Figure 10.10 the point marked "∘" is where both firms are in operation but firm 1 is getting all of the joint profit and the point $(\Pi_M, 0)$ is the situation where firm 1 is operating on its own.

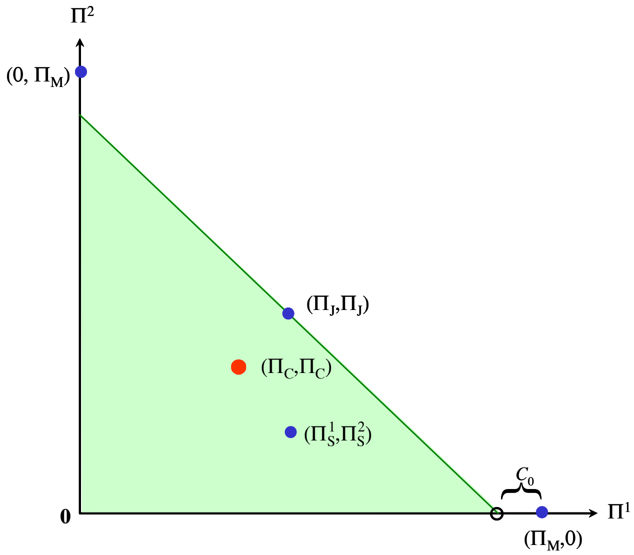

Figure 10.10: Possible payoffs

Exercise 10.8 *An oligopoly contains N identical firms. The cost function is convex in output. Show that if the firms act as Cournot competitors then as N increases the market price will approach the competitive price.*

Outline Answer

The assumption of convex costs will ensure that there is no minimum viable size of firm. Profits for a typical firm are given by

$$p\left(q^f + K\right) q^f - C\left(q^f\right) \tag{10.13}$$

where

$$K := \sum_{\substack{j=1 \\ j \neq f}}^{N} q^j$$

is the total output of all the other firms, which of course firm f takes to be constant under the Cournot assumption. Maximising this by choice of q^f gives the FOC for an interior solution

$$p_q\left(q^f + K\right) q^f + p\left(q^f + K\right) - C_q\left(q^f\right) = 0 \tag{10.14}$$

Given that all the firms are identical we may rewrite condition (10.14) as

$$p_q(q)\frac{q}{N} + p(q) - C_q = 0 \tag{10.15}$$

where q is industry output. This in turn can be rewritten as

$$p(q) = \frac{C_q}{1 + \frac{1}{\eta N}} \tag{10.16}$$

where

$$\eta := \frac{p(q)}{q p_q(q)}$$

is the elasticity of demand. The result follows immediately: as N becomes large (10.16) approaches

$$p(q) = C_q \tag{10.17}$$

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