Exercise 3.3 *A firm has the cost function*

$$F_0 + \frac{1}{2}aq_i^2$$

where q_i is the output of a single homogenous good and F_0 and a are positive numbers.

1. *Find the firm's supply relationship between output and price p; explain carefully what happens at the minimum-average-cost point $\underline{p} := \sqrt{2aF_0}$.*

2. *In a market of a thousand consumers the demand curve for the commodity is given by*
$$p = A - bq$$
where q is total quantity demanded and A and b are positive parameters. If the market is served by a single price-taking firm with the cost structure in part 1 explain why there is a unique equilibrium if $b \leq a\left[A/\underline{p} - 1\right]$ and no equilibrium otherwise.

3. *Now assume that there is a large number N of firms, each with the above cost function: find the relationship between average supply by the N firms and price and compare the answer with that of part 1. What happens as $N \to \infty$?*

4. *Assume that the size of the market is also increased by a factor N but that the demand per thousand consumers remains as in part 2 above. Show that as N gets large there will be a determinate market equilibrium price and output level.*

Outline Answer

1. Given the cost function
$$F_0 + \frac{1}{2}aq_i^2$$
marginal cost is aq_i and average cost is $F_0/q_i + \frac{1}{2}aq_i$. Marginal cost intersects average cost where
$$aq_i = F_0/q_i + \frac{1}{2}aq_i$$
i.e. where output is
$$\underline{q} := \sqrt{2F_0/a} \tag{3.9}$$
and marginal cost is
$$\underline{p} := \sqrt{2aF_0} \tag{3.10}$$
For $p > \underline{p}$ the supply curve is identical to the marginal cost curve $q_i = p/a$; for $p < \underline{p}$ the firm supplies 0 to the market; at $p = \underline{p}$ the firm supplies either 0 or $\underline{q}$. There is no price which will induce a supply in the interior of the interval $\left(0, \underline{q}\right)$. Summarising, firm i's optimal output is given by
$$q_i^* = S(p) := \begin{cases} p/a, \text{ if } p > \underline{p} \\ q \in \{0, \underline{q}\} \text{ if } p = \underline{p} \\ 0, \text{ if } p < \underline{p} \end{cases} \tag{3.11}$$

2. The equilibrium, if it exists, is found where supply=demand at a given price. This would imply

$$\begin{aligned}\frac{p}{a} &= \frac{A-p}{b}\\ p &= \frac{aA}{a+b}\end{aligned}$$

which would, in turn, imply an equilibrium quantity

$$q = \frac{A}{a+b}$$

but it can only be valid if $\frac{A}{a+b} \geq \underline{q}$. Noting that $\underline{q} = \underline{p}/a$ this condition is equivalent to $a\left[\frac{A}{\underline{p}} - 1\right] \geq b$.

3. If there are N such firms, each firm responds to price as in (3.11), and so the average output $\overline{q} := \frac{1}{N}\sum_{i=1}^{N} q_i^*$ is given by

$$\overline{q} = \begin{cases} p/a, \text{ if } p > \underline{p} \\ q \in J(\underline{q}) \text{ if } p = \underline{p} \\ 0, \text{ if } p < \underline{p} \end{cases} \quad (3.12)$$

where $J(\underline{q}) := \{\frac{i}{N}\underline{q} : i = 0, 1, ..., N\}$. As $N \to \infty$ the set $J(\underline{q})$ becomes dense in $[0, \underline{q}]$, and so we have the average supply relationship:

$$\overline{q} = \begin{cases} p/a, \text{ if } p > \underline{p} \\ q \in [0, \underline{q}] \text{ if } p = \underline{p} \\ 0, \text{ if } p < \underline{p} \end{cases} \quad (3.13)$$

4. Given that in the limit the average supply curve is continuous and of the piecewise linear form (3.13), and that the demand curve is a downward-sloping straight line, there must be a unique market equilibrium. The equilibrium will be found at $\left(\underline{p}, \frac{A-\underline{p}}{b}\right)$ which, using (3.10) is $\left(\sqrt{2aF_0}, \frac{A-\sqrt{2aF_0}}{b}\right)$. Using (3.9) this can be written $\left(\underline{p}, \beta\underline{q}\right)$ where

$$\beta := \frac{A-\underline{p}}{b\underline{p}/a}$$

In the equilibrium a proportion β of the firms produce $\underline{q}$ and $1-\beta$ of the firms produce 0.

Exercise 3.4 *A firm has a fixed cost F_0 and marginal costs*

$$c = a + bq$$

where q is output.

1. *If the firm were a price-taker, what is the lowest price at which it would be prepared to produce a positive amount of output? If the competitive price were above this level, find the amount of output q^* that the firm would produce.*

2. *If the firm is actually a monopolist and the inverse demand function is*

$$p = A - \frac{1}{2}Bq$$

(where $A > a$ and $B > 0$) find the expression for the firm's marginal revenue in terms of output. Illustrate the optimum in a diagram and show that the firm will produce

$$q^{**} := \frac{A - a}{b + B}$$

What is the price charged p^{**} and the marginal cost c^{**} at this output level? Compare q^{**} and q^*.

3. *The government decides to regulate the monopoly. The regulator has the power to control the price by setting a ceiling $p_{\max}$. Plot the average and marginal revenue curves that would then face the monopolist. Use these to show:*

   (a) If $p_{\max} > p^{**}$ the firm's output and price remain unchanged at q^{**} and p^{**}

   (b) If $p_{\max} < c^{**}$ the firm's output will fall below q^{**}.

   (c) Otherwise output will rise above q^{**}.

Outline Answer

1. Total costs are

$$F_0 + aq + \frac{1}{2}bq^2$$

So average costs are

$$\frac{F_0}{q} + a + \frac{1}{2}bq$$

which are a minimum at

$$\underline{q} = \sqrt{2\frac{F_0}{b}} \quad (3.14)$$

where average costs are

$$\sqrt{2bF_0} + a \quad (3.15)$$

Marginal and average costs are illustrated in Figure 3.1: notice that MC is linear and that AC has the typical U-shape if $F_0 > 0$. For a price above the level (3.15) the first-order condition for maximum profits is given by

$$p = a + bq$$

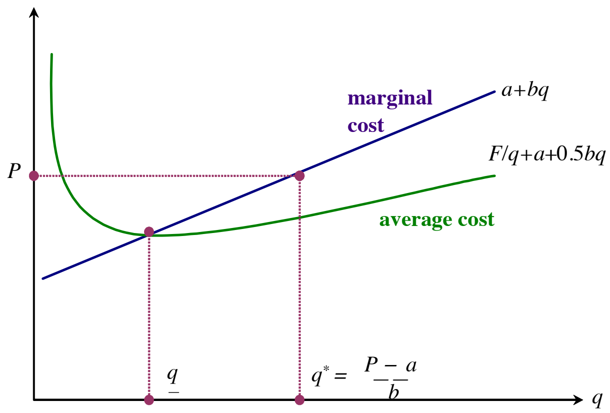

Figure 3.1: Perfect competition

from which we find

$$q^* := \frac{p-a}{b}$$

– see figure 3.1.

2. If the firm is a monopolist marginal revenue is

$$\frac{\partial}{\partial q}\left[Aq - \frac{1}{2}Bq^2\right] = A - Bq$$

Hence the first-order condition for the monopolist is

$$A - Bq = a + bq \tag{3.16}$$

from which the solution q^{**} follows. Substituting for q^{**} we also get

$$c^{**} = A - Bq^{**} = \frac{Ab + Ba}{B + b} \tag{3.17}$$

$$p^{**} = A - \frac{1}{2}Bq^{**} = c^{**} + \frac{1}{2}B\frac{A-a}{b+B} \tag{3.18}$$

– see figure 3.2.

3. Consider how the introduction of a price ceiling will affect average revenue. Clearly we now have

$$\text{AR}(q) = \left\{ \begin{array}{l} p_{\max} \text{ if } q \leq q_0 \\ A - \frac{1}{2}Bq \text{ if } q \geq q_0 \end{array} \right\} \tag{3.19}$$

where $q_0 := 2\left[A - p_{\max}\right]/B$: average revenue is a continuous function of q but has a kink at q_0. From this we may derive marginal revenue which is

$$\text{MR}(q) = \left\{ \begin{array}{l} p_{\max} \text{ if } q < q_0 \\ A - Bq \text{ if } q > q_0 \end{array} \right\} \tag{3.20}$$

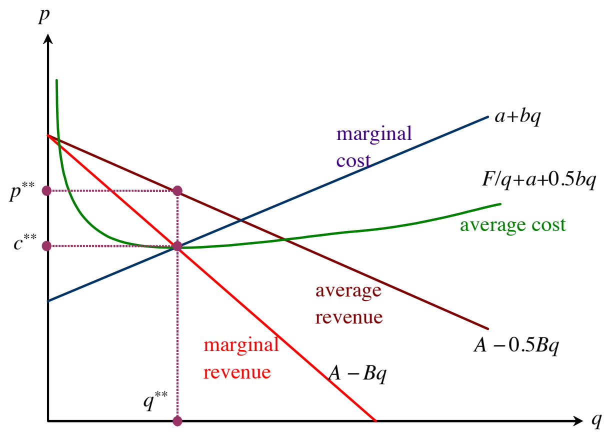

Figure 3.2: Unregulated monopoly

– notice that there is a discontinuity exactly at q_0. The modified curves (3.19) and (3.20) are shown in Figure 3.3: notice that they coincide in the flat section to the left of q_0. Clearly the outcome depends crucially on whether MC intersects (modified) MR (a) to the left of q_0, (b) to the right of q_0, (c) in the discontinuity exactly at q_0. Case (c) is illustrated, and it is clear that output will have risen from q^{**} to q_0. The other cases can easily be found by appropriately shifting the curves on Figure 3.3 .

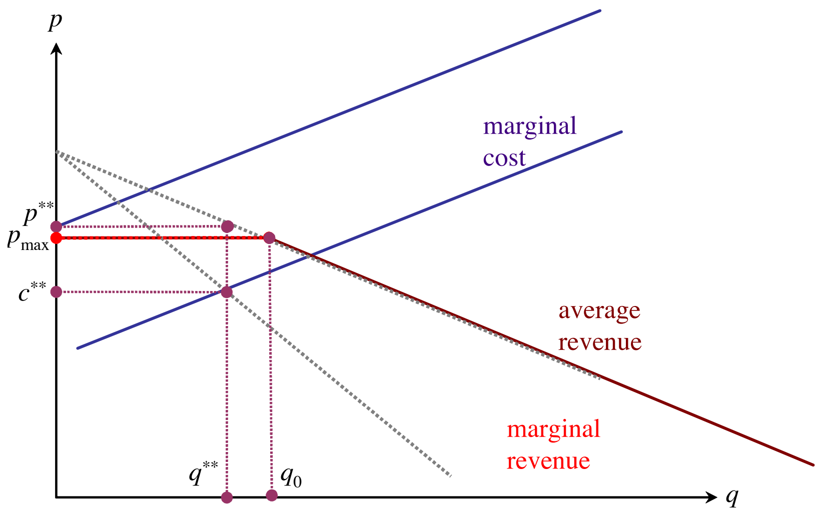

Figure 3.3: Regulated Monopoly

Exercise 3.5 *A monopolist has the cost function*

$$C(q) = 100 + 6q + \frac{1}{2}[q]^2$$

1. *If the demand function is given by*

   $$q = 24 - \frac{1}{4}p$$

   calculate the output-price combination which maximises profits.

2. *Assume that it becomes possible to sell in a separate second market with demand determined by*

   $$q = 84 - \frac{3}{4}p.$$

   Calculate the prices which will be set in the two markets and the change in total output and profits from case 1.

3. *Now suppose that the firm still has access to both markets, but is prevented from discriminating between them. What will be the result?*

Outline Answer

1. Maximizing the simple monopolist's profits

   $$\Pi_0 = (96 - 4q)q - \left(100 + 6q + \frac{q^2}{2}\right)$$

   with respect to q yields optimum output of q_0=10. Hence $p_0 = 56$ and $\Pi_0 = 350$.

2. Now let the monopolist sell $q_1$in market 1 for price p_1 and q_2 in market 2 for price p_2 .The new problem is to choose q_1, q_2 so as to maximise the function

   $$\Pi_{12} = (96 - 4q_1)q_1 + (112 - \frac{4}{3}q_2)q_2 - \left(100 + 6q_1 + 6q_2 + \frac{(q_1 + q_2)^2}{2}\right).$$

   First-order conditions yield

   $$\begin{aligned} 9q_1 + q_2 &= 90 \\ q_1 + \frac{11}{3}q_2 &= 106. \end{aligned}$$

   Solving we find $q_1 = 7$, $q_2 = 27$ and hence $p_1 = 68$, $p_2 = 76$ and $\Pi_{12} = 1646$.

3. If we abandon discrimination, a uniform price $\widehat{p}$ must be charged. If $\widehat{p} > 112$ nothing is sold to either market. If $112 > \widehat{p} > 96$ only market 2 is served. If $96 > \widehat{p}$ both market are served and the demand curve is $\widehat{q} = 108 - \widehat{p}$. Clearly this is the relevant region. Maximising simple monopoly profits we find $\widehat{q} = 34$, $\widehat{p} = 74$ and $\widehat{\Pi} = 1634$.

   Hence the total output is identical to that under discrimination, $p_1 < \widehat{p} < p_2$ and $\Pi_{12} > \widehat{\Pi}$. These results are quite general.

