Exercise 2.8 *For any homothetic production function show that the cost function must be expressible in the form*

$$C(\mathbf{w}, q) = a(\mathbf{w})\, b(q).$$

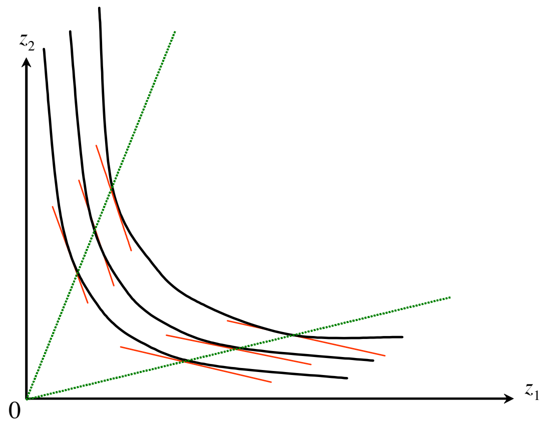

Figure 2.11: Homotheticity: expansion path

Outline Answer

From the definition of homotheticity, the isoquants must look like Figure 2.11; interpreting the tangents as isocost lines it is clear from the figure that the expansion paths are rays through the origin. So, if $H^i(\mathbf{w}, q)$ is the demand for input i conditional on output q, the optimal input ratio

$$\frac{H^i(\mathbf{w}, q)}{H^j(\mathbf{w}, q)}$$

must be independent of q and so we must have

$$\frac{H^i(\mathbf{w}, q)}{H^i(\mathbf{w}, q')} = \frac{H^j(\mathbf{w}, q)}{H^j(\mathbf{w}, q')}$$

for any q, q'. For this to true it is clear that the ratio $H^i(\mathbf{w}, q)/H^i(\mathbf{w}, q')$ must be independent of $\mathbf{w}$. Setting $q' = 1$ we therefore have

$$\frac{H^1(\mathbf{w}, q)}{H^1(\mathbf{w}, 1)} = \frac{H^2(\mathbf{w}, q)}{H^2(\mathbf{w}, 1)} = ... = \frac{H^m(\mathbf{w}, q)}{H^m(\mathbf{w}, 1)} = b(q)$$

and so

$$H^i(\mathbf{w}, q) = b(q) H^i(\mathbf{w}, 1).$$

Therefore the minimized cost is given by

$$\begin{aligned} C(\mathbf{w},q) &= \sum_{i=1}^{m} w_i H^i(\mathbf{w},q) \\ &= \sum_{i=1}^{m} w_i b(q) H^i(\mathbf{w},1) \\ &= b(q)\sum_{i=1}^{m} w_i H^i(\mathbf{w},1) \\ &= a(\mathbf{w})b(q) \end{aligned}$$

where $a(\mathbf{w}) = \sum_{i=1}^{m} w_i H^i(\mathbf{w},1)$.

Exercise 2.9 *Consider the production function*

$$q = \left[\alpha_1 z_1^{-1} + \alpha_2 z_2^{-1} + \alpha_3 z_3^{-1}\right]^{-1}$$

1. *Find the long-run cost function and sketch the long-run and short-run marginal and average cost curves and comment on their form.*
2. *Suppose input 3 is fixed in the short run. Repeat the analysis for the short-run case.*
3. *What is the elasticity of supply in the short and the long run?*

Outline Answer

1. The production function is clearly homogeneous of degree 1 in all inputs – i.e. in the long run we have constant returns to scale. But CRTS implies constant average cost. So

$$\text{LRMC} = \text{LRAC} = \text{constant}$$

Their graphs will be an identical straight line.

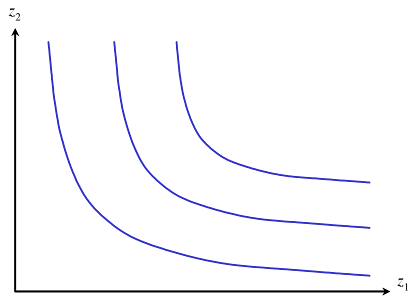

Figure 2.12: Isoquants do not touch the axes

2. In the short run $z_3 = \bar{z}_3$ so we can write the problem as the following Lagrangean

$$\hat{\mathcal{L}}(\mathbf{z}, \hat{\lambda}) = w_1 z_1 + w_2 z_2 + \hat{\lambda}\left[q - \left[\alpha_1 z_1^{-1} + \alpha_2 z_2^{-1} + \alpha_3 \bar{z}_3^{-1}\right]^{-1}\right]; \quad (2.13)$$

or, using a transformation of the constraint to make the manipulation easier, we can use the Lagrangean

$$\mathcal{L}(\mathbf{z}, \lambda) = w_1 z_1 + w_2 z_2 + \lambda\left[\alpha_1 z_1^{-1} + \alpha_2 z_2^{-1} - k\right] \quad (2.14)$$

where λ is the Lagrange multiplier for the transformed constraint and

$$k := q^{-1} - \alpha_3 \bar{z}_3^{-1}. \quad (2.15)$$

Note that the isoquant is

$$z_2 = \frac{\alpha_2}{k - \alpha_1 z_1^{-1}}.$$

From the Figure 2.12 it is clear that the isoquants do not touch the axes and so we will have an interior solution. The first-order conditions are

$$w_i - \lambda \alpha_i z_i^{-2} = 0, \; i = 1, 2 \tag{2.16}$$

which imply

$$z_i = \sqrt{\frac{\lambda \alpha_i}{w_i}}, \; i = 1, 2 \tag{2.17}$$

To find the conditional demand function we need to solve for λ. Using the production function and equations (2.15), (2.17) we get

$$k = \sum_{j=1}^{2} \alpha_j \left[\frac{\lambda \alpha_j}{w_j}\right]^{-1/2} \tag{2.18}$$

from which we find

$$\sqrt{\lambda} = \frac{b}{k} \tag{2.19}$$

where

$$b := \sqrt{\alpha_1 w_1} + \sqrt{\alpha_2 w_2}.$$

Substituting from (2.19) into (2.17) we get minimised cost as

$$\tilde{C}(\mathbf{w}, q; \bar{z}_3) = \sum_{i=1}^{2} w_i z_i^* + w_3 \bar{z}_3 \tag{2.20}$$

$$= \frac{b^2}{k} + w_3 \bar{z}_3 \tag{2.21}$$

$$= \frac{qb^2}{1 - \alpha_3 \bar{z}_3^{-1} q} + w_3 \bar{z}_3. \tag{2.22}$$

Marginal cost is

$$\frac{b^2}{\left[1 - \alpha_3 \bar{z}_3^{-1} q\right]^2} \tag{2.23}$$

and average cost is

$$\frac{b^2}{1 - \alpha_3 \bar{z}_3^{-1} q} + \frac{w_3 \bar{z}_3}{q}. \tag{2.24}$$

Let $\underline{q}$ be the value of q for which MC=AC in (2.23) and (2.24) – at the minimum of AC in Figure 2.13 – and let $\underline{p}$ be the corresponding minimum value of AC. Then, using $p =$MC in (2.23) for $p \geq \underline{p}$ the short-run supply curve is given by $q^* = S(\mathbf{w}, p) = \begin{cases} 0 & \text{if } p < \underline{p} \\ 0 \text{ or } \underline{q} & \text{if } p = \underline{p} \\ q = \frac{\bar{z}_3}{\alpha_3}\left[1 - \frac{b}{\sqrt{p}}\right] & \text{if } p > \underline{p} \end{cases}$

3. Differentiating the last line in the previous formula we get

$$\frac{d \ln q}{d \ln p} = \frac{p}{q}\frac{dq}{dp} = \frac{1}{2}\frac{1}{\sqrt{p/b} - 1} > 0$$

Note that the elasticity decreases with b. In the long run the supply curve coincides with the MC,AC curves and so has infinite elasticity.

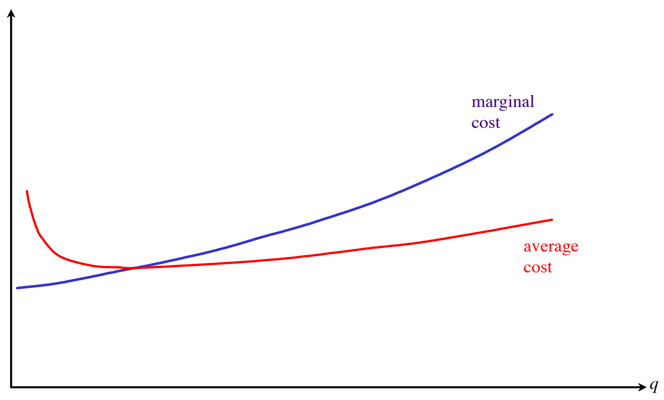

Figure 2.13: Short-run marginal and average cost

Exercise 2.10 *A competitive firm's output q is determined by*

$$q = z_1^{\alpha_1} z_2^{\alpha_2} ... z_m^{\alpha_m}$$

where z_i is its usage of input i and $\alpha_i > 0$ is a parameter $i = 1, 2, ..., m$. Assume that in the short run only k of the m inputs are variable.

1. *Find the long-run average and marginal cost functions for this firm. Under what conditions will marginal cost rise with output?*

2. *Find the short-run marginal cost function.*

3. *Find the firm's short-run elasticity of supply. What would happen to this elasticity if k were reduced?*

Outline Answer

Write the production function in the equivalent form:

$$\log q = \sum_{i=1}^{m} \alpha_i \log z_i \tag{2.25}$$

The isoquant for the case $m = 2$ would take the form

$$z_2 = \left[q z_1^{-\alpha_1} \right]^{\frac{1}{\alpha_2}} \tag{2.26}$$

which does not touch the axis for finite (z_1, z_2).

1. The cost-minimisation problem can be represented as minimising the Lagrangean

$$\sum_{i=1}^{m} w_i z_i + \lambda \left[\log q - \sum_{i=1}^{m} \alpha_i \log z_i \right] \tag{2.27}$$

where w_i is the given price of input i, and λ is the Lagrange multiplier for the modified production constraint. Given that the isoquant does not touch the axis we must have an interior solution: first-order conditions are

$$w_i - \lambda \alpha_i z_i^{-1} = 0, \ i = 1, 2, .., m \tag{2.28}$$

which imply

$$z_i = \frac{\lambda \alpha_i}{w_i}, \ i = 1, 2, .., m \tag{2.29}$$

Now solve for λ. Using (2.25) and (2.29) we get

$$z_i^{\alpha_i} = \left[\frac{\lambda \alpha_i}{w_i} \right]^{\alpha_i}, \ i = 1, 2, .., m \tag{2.30}$$

$$q = \prod_{i=1}^{m} z_i^{\alpha_i} = \left[\frac{\lambda}{A} \right]^{\gamma} \prod_{i=1}^{m} w_i^{-\alpha_i} \tag{2.31}$$

where $\gamma := \sum_{j=1}^{m} \alpha_j$ and $A := [\prod_{i=1}^{m} \alpha_i^{\alpha_i}]^{-1/\gamma}$ are constants, from which we find

$$\begin{aligned} \lambda &= A\left[\frac{q}{\prod_{i=1}^{m} w_i^{-\alpha_i}}\right]^{1/\gamma} \\ &= A\left[q w_1^{\alpha_1} w_2^{\alpha_2} ... w_m^{\alpha_m}\right]^{1/\gamma}. \end{aligned} \tag{2.32}$$

Substituting from (2.32) into (2.29) we get the conditional demand function:

$$H^i(\mathbf{w}, q) = z_i^* = \frac{\alpha_i}{w_i} A\left[q w_1^{\alpha_1} w_2^{\alpha_2} ... w_m^{\alpha_m}\right]^{1/\gamma} \tag{2.33}$$

and minimised cost is

$$C(\mathbf{w}, q) = \sum_{i=1}^{m} w_i z_i^* = \gamma A\left[q w_1^{\alpha_1} w_2^{\alpha_2} ... w_m^{\alpha_m}\right]^{1/\gamma} \tag{2.34}$$

$$= \gamma B q^{1/\gamma} \tag{2.35}$$

where $B := A\left[w_1^{\alpha_1} w_2^{\alpha_2} ... w_m^{\alpha_m}\right]^{1/\gamma}$. It is clear from (2.35) that cost is increasing in q and increasing in w_i if $\alpha_i > 0$ (it is always nondecreasing in w_i). Differentiating (2.35) with respect to q marginal cost is

$$C_q(\mathbf{w}, q) = B q^{\frac{1-\gamma}{\gamma}} \tag{2.36}$$

Clearly marginal cost falls/stays constant/rises with q as $\gamma \gtreqless 1$.

2. In the short run inputs $1, ..., k$ $(k \leq m)$ remain variable and the remaining inputs are fixed. In the short-run the production function can be written as

$$\log q = \sum_{i=1}^{k} \alpha_i \log z_i + \log \theta_k \tag{2.37}$$

where

$$\theta_k := \exp\left(\sum_{i=k+1}^{m} \alpha_i \log \bar{z}_i\right) \tag{2.38}$$

and $\bar{z}_i$ is the arbitrary value at which input i is fixed; note that B is fixed in the short run. The general form of the Lagrangean (2.27) remains unchanged, but with q replaced by q/θ_k and m replaced by k. So the first-order conditions and their corollaries (2.28)-(2.32) are essentially as before, but γ and A are replaced by

$$\gamma_k := \sum_{j=1}^{k} \alpha_j \tag{2.39}$$

and $A_k := \left[\prod_{i=1}^{k} \alpha_i^{\alpha_i}\right]^{-1/\gamma_k}$. Hence short-run conditional demand is

$$\tilde{H}^i(\mathbf{w}, q; \bar{z}_{k+1}, ..., \bar{z}_m) = \frac{\alpha_i}{w_i} A_k \left[\frac{q}{\theta_k} w_1^{\alpha_1} w_2^{\alpha_2} ... w_k^{\alpha_k}\right]^{1/\gamma_k} \tag{2.40}$$

and minimised cost in the short run is

$$\begin{aligned}\tilde{C}(\mathbf{w}, q; \bar{z}_{k+1}, ..., \bar{z}_m) &= \sum_{i=1}^{k} w_i z_i^* + c_k \\ &= \gamma_k A_k \left[\frac{q}{\theta_k} w_1^{\alpha_1} w_2^{\alpha_2} ... w_k^{\alpha_k}\right]^{1/\gamma_k} + c_k \quad (2.41)\\ &= \gamma_k B_k q^{1/\gamma_k} + c_k \quad (2.42)\end{aligned}$$

where

$$c_k := \sum_{i=k+1}^{m} w_i \bar{z}_i \quad (2.43)$$

is the fixed-cost component in the short run and $B_k := A_k \left[w_1^{\alpha_1} w_2^{\alpha_2} ... w_k^{\alpha_k} / \theta_k\right]^{1/\gamma_k}$. Differentiating (2.42) we find that short-run marginal cost is

$$\tilde{C}_q(\mathbf{w}, q; \bar{z}_{k+1}, ..., \bar{z}_m) = B_k q^{\frac{1-\gamma_k}{\gamma_k}}$$

3. Using the "Marginal cost=price" condition we find

$$B_k q^{\frac{1-\gamma_k}{\gamma_k}} = p \quad (2.44)$$

where p is the price of output so that, rearranging (2.44) the supply function is

$$q = S(\mathbf{w}, p; \bar{z}_{k+1}, ..., \bar{z}_m) = \left[\frac{p}{B_k}\right]^{\frac{\gamma_k}{1-\gamma_k}} \quad (2.45)$$

wherever MC≥AC. The elasticity of (2.45) is given by

$$\frac{\partial \log S(\mathbf{w}, p; \bar{z}_{k+1}, ..., \bar{z}_m)}{\partial \log p} = \frac{\gamma_k}{1-\gamma_k} > 0 \quad (2.46)$$

It is clear from (2.39) that $\gamma_k \geq \gamma_{k-1} \geq \gamma_{k-2}$... and so the positive supply elasticity in (2.46) must fall as k falls.

Exercise 2.11 *A firm produces goods 1 and 2 using goods 3,...,5 as inputs. The production of one unit of good i ($i = 1, 2$) requires at least a_{ij} units of good j, ($j = 3, 4, 5$).*

1. *Assuming constant returns to scale, how much of resource j will be needed to produce q_1 units of commodity 1?*
2. *For given values of q_3, q_4, q_5 sketch the set of technologically feasible outputs of goods 1 and 2.*

Outline Answer

1. To produce q_1 units of commodity 1 $a_{1j}q_1$ units of resource j will be needed.
$$q_1a_{1i} + q_2a_{2i} \le R_i.$$
2. The feasibility constraint for resource j is therefore going to be
$$q_1a_{1j} + q_2a_{2j} \le R_j.$$
Taking into account all three resources, the feasible set is given as in Figure 2.14

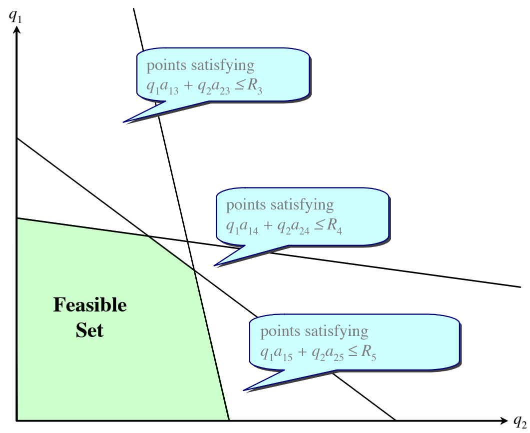

Figure 2.14: Feasible set

Exercise 2.12 *[see Exercise 2.4]*