Chapter 8

Uncertainty and Risk

The lottery is the one ray of hope in my otherwise unbearable life.
– Homer Simpson.

8.1 Introduction

All of the economic analysis so far has been based on the assumption of a certain world. Where we have touched on the issue of time it can effectively be collapsed into the present through discounting. Now we explicitly change that by incorporating uncertainty into the microeconomic model. This also gives us an opportunity to think more about the issue of time. We deal with a specific, perhaps rather narrow, concept of uncertainty that is, in a sense, exogenous. It is some external ingredient that has an impact upon individual agents' economic circumstances (it affects their income, their needs...) and also upon the agents' decisions (it affects their consumption plans, the pattern of their asset-holding...)

Although there are some radically new concepts to be introduced, the analysis can be firmly based on the principles that we have already established, particularly those used to give meaning to consumer choice. However, the approach will take us on to more general issues: by modelling uncertainty we can provide an insight into the definition of risk, attitudes to risk and a precise concept of risk aversion.

8.2 Consumption and uncertainty

We begin by looking at the way in which elementary consumer theory can be extended to allow for the fact that the future is only imperfectly known. To fix ideas, let us consider two examples of a simple consumer choice problem under uncertainty.

	"Budget day"	**"Election day"**
states of the world	fee does/ does not increase	Blue/Red wins
payoffs *(outcomes)*	–£20 or £0, depending on ω	capital gain/capital loss, depending on ω
prospects	states and outcomes seen from the morning	states and outcomes seen from the morning
ex ante/ex post	before/after 3pm	before/after the Election results

Table 8.1: Two simple decision problems under uncertainty

1. *Budget day.* You have a licence for your car which must be renewed annually and which still has some weeks before expiry. The government is announcing tax changes this afternoon which may affect the fee for your licence: if you renew the licence now, you pay the old fee, but you forfeit the unexpired portion of the licence; if you wait, you may have to renew the licence at a higher fee.
2. *Election day.* Two parties are contesting an election, and the result will be known at noon. In the morning you hold an asset whose value will be affected by the outcome of the election. If you do not sell the asset immediately your wealth will rise if the Red party wins, and drop if the Blue party wins.

The essential features in these two examples can be summarised in the accompanying box, and the following points are worth noting:

- The *states-of-the-world* indexed by ω act like labels on physically different goods.
- The set of all states-of-the-world Ω in each of the two examples is very simple – it contains only two elements. But in some interesting economic models may be (countably or uncountably) infinite.
- The *payoffs* in the two examples are scalars (monetary amounts); but in more general models it might be useful to represent the payoff as a consumption bundle – a vector of goods $\mathbf{x}$.
- Timing is crucial. Use the time-line Figure 8.1 as a simple parable; the left-hand side represents the "morning" during which decisions are made; the outcome of a decision is determined in the afternoon and will be influenced by the state-of-the-world ω. The dotted boundary represents the point at

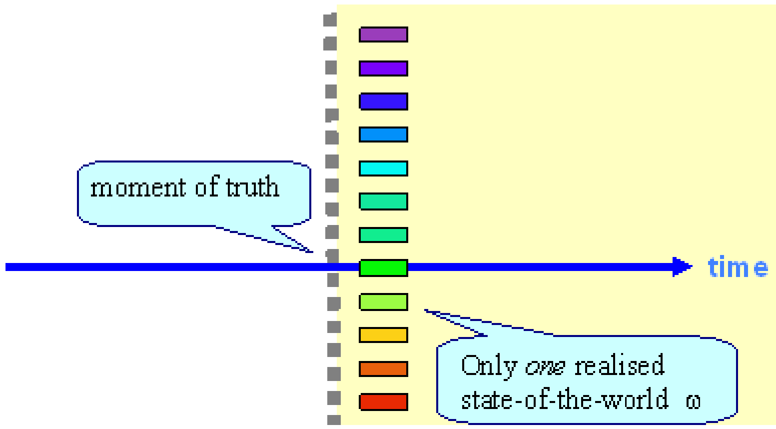

Figure 8.1: The *ex-ante/ex-post* distinction

which exactly one ω is realised out of a whole rainbow of possibilities. You must make your choice *ex ante*. It is too late to do it *ex post* – after the realisation of the event.

- The *prospects* could be treated like consumption vectors.

8.2.1 The nature of choice

It is evident that from these examples that the way we look at choice has changed somewhat from that analysed in chapter 4. In our earlier exposition of consumer theory actions by consumers were synonymous with consequences: you choose the action "buy x_1 units of commodity 1" and you get to consume x_1 units of commodity 1: it was effectively a model of instant gratification. We now have a more complex model of the satisfaction of wants. The consumer may choose to take some action (buy this or that, vote for him or her) but the consequence that follows is no longer instantaneous and predictable. The payoff – the consequence that directly affects the consumer – depends both on the action and on the outcome of some event.

To put these ideas on an analytical footing we will discuss the economic issues in stages: later we will examine a specific model of utility that appears to be well suited for representing choice under uncertainty and then consider how this model can be used to characterise attitudes to risk and the problem of choice under uncertainty. However, first we will see how far it is possible to get just by adapting the model of consumer choice that was used in chapter 4.

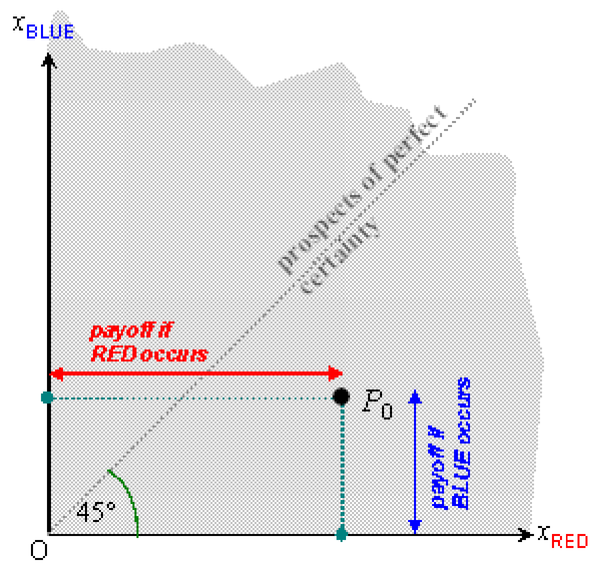

Figure 8.2: The state-space diagram: $\#\Omega = 2$

8.2.2 State-space diagram

As a simplified introduction take the case where there are just two possible states of the world, denoted by the labels RED and BLUE, and scalar payoffs; this means that the payoff in each state-of-the-world ω can be represented as the amount of a composite consumption good x_ω. Then consumption in each of the two states-of-the-world x_{RED} and x_{BLUE} can be measured along each of the two axes in Figure 8.2. These are *contingent goods*: that is x_{RED} and x_{BLUE} are quantities of consumption that are contingent on which state-of-the world is eventually realised. An individual prospect is represented as a vector of contingent goods such as that marked by the point P_0 and the set of all prospects is represented by the shaded area in Figure 8.2. If instead there were three states in Ω with scalar payoffs then a typical prospect would be such as P_0 in Figure 8.3. So the description of the environment in which individual choice is to be exercised is rather like that of ordinary consumption vectors – see page 71. However, the 45° ray in Figure 8.2 has a special significance: prospects along this line represent payoffs under complete certainty. It is arguable that such prospects are qualitatively different from anywhere else in the diagram and may accordingly be treated differently by consumers; there is no counterpart to this in conventional choice under certainty.

Now consider the representation of consumers' preferences – as viewed from the morning – in this uncertain world. To represent an individual's ranking of prospects we can use a weak preference relation of the form introduced in

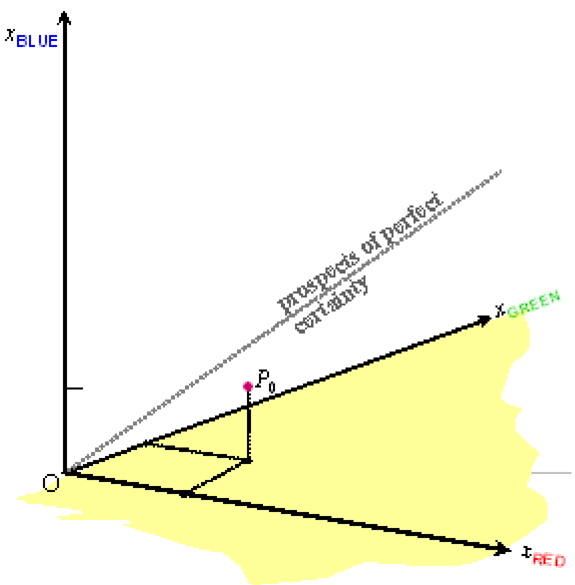

Figure 8.3: The state-space diagram: $\#\Omega = 3$

Definition 4.2. If we copy across the concepts used in the world of certainty from chapter 4 we might postulate indifference curves defined in the space of contingent goods – as in Figure 8.4. This of course will require the standard axioms of completeness, transitivity and continuity introduced in chapter 4 (see page 75). Other standard consumer axioms might also seem to be intuitively reasonable in the case of ranking prospects. An example of this is "greed" (Axiom 4.6 on page 78): prospect P_1 will, presumably, be preferred to P_0 in Figure 8.4.

But this may be moving ahead too quickly. Axioms 4.3 to 4.5 might seem fairly unexceptionable in the context where they were introduced – choice under perfect certainty – but some people might wish to question whether the continuity axiom is everywhere appropriate in the case of uncertain prospects. It may be that people who have a pathological concern for certainty have preferences that are discontinuous in the neighbourhood of the 45° ray: for such persons a complete map of indifference curves cannot be drawn.[1]

However, if the individual's preferences are such that you can draw indifference curves then you can get a very useful concept indeed: the *certainty equivalent* of any prospect P_0. This is point E with coordinates (ξ, ξ) in Figure 8.5; the amount ξ is simply the quantity of the consumption good, guaranteed with complete certainty, that the individual would accept as a straight swap for

[1] If the continuity axiom is violated in this way decribe the shape of the individual's prefernce map.

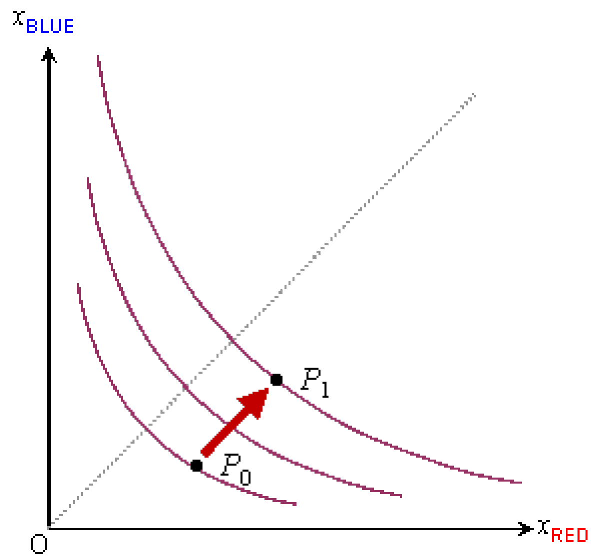

Figure 8.4: Preference contours in state-space

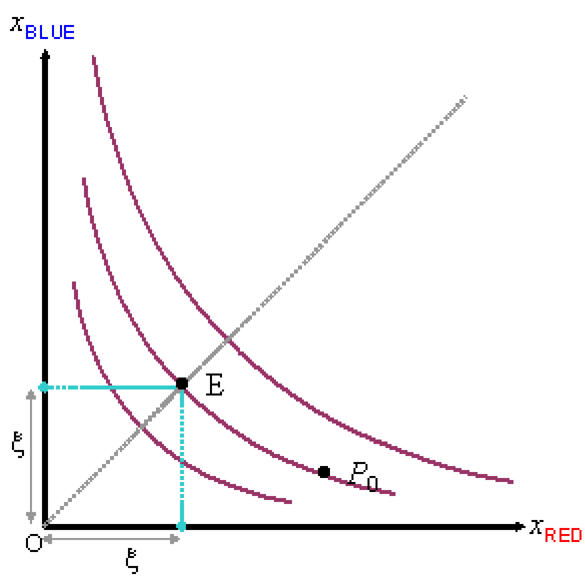

Figure 8.5: The certainty equivalent

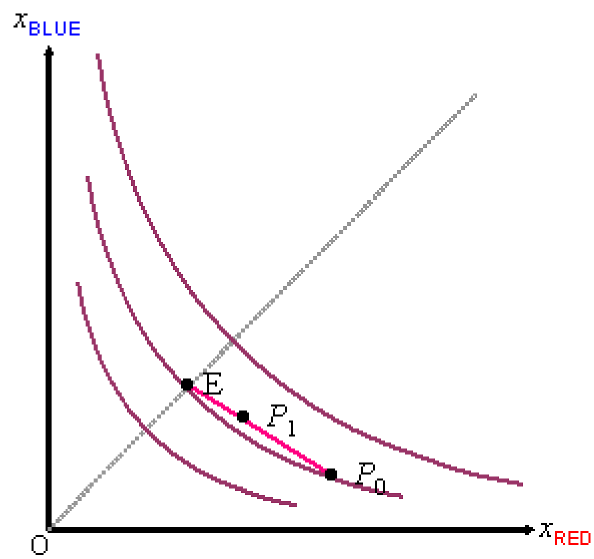

Figure 8.6: Quasiconcavity reinterpreted

the prospect P_0. It is clear that the existence of this quantity depends crucially on the continuity assumption.

Let us consider the concept of the certainty equivalent further. To do this, connect prospect P_0 and its certainty equivalent by a straight line, as shown in Figure 8.6. Observe that all points on this line are weakly preferred to P_0 if and only if the preference map is quasiconcave (you might find it useful to check the definition of quasiconcavity on page 506 in Appendix A). This suggests an intuitively appealing interpretation: if the individual always prefers a mixture of prospect P with its certainty equivalent to prospect P alone then one might claim that in some sense he or she has "risk averse" preferences. On this interpretation "risk aversion" implies, and is implied by, convex-to-the-origin indifference curves (I have used the quote marks around risk aversion because we have not defined what risk *is* yet).[2]

Now for another point of interpretation. Suppose RED becomes less likely to win (as perceived by the individual in the morning) – what would happen to the indifference curves? We would expect them to shift in the way illustrated in Figure 8.7. by replacing the existing light-coloured indifference curves with the heavy indifference curves The reasoning behind this is as follows. Take E as a given reference point on the 45° line – remember that it represents a payoff that is independent of the state of the world that will occur. Before the change the prospects represented by points E and P_0 are regarded as indifferent; however

[2]What would the curves look like for a risk-neutral person? For a risk-lover?

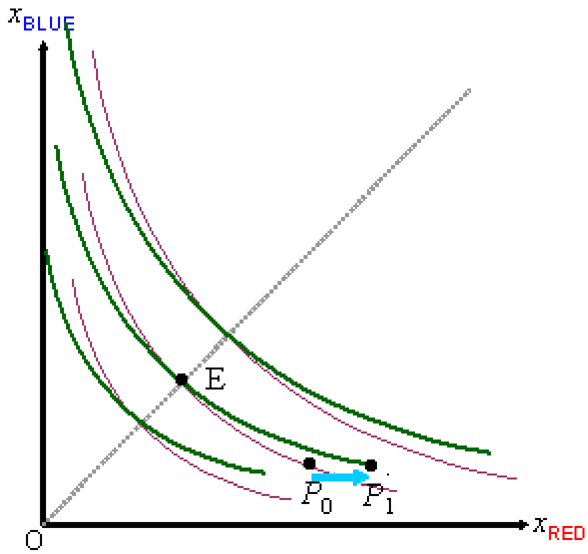

Figure 8.7: A change in perception

after the change it is P_1 – that implies a higher payoff under RED – that is regarded as being of "equal value" to point E.[3]

8.3 A model of preferences

So far we have extended the formal model of the consumer by reinterpreting the commodity space and reinterpreting preferences in this space. This reinterpretation of preference has included the first tentative steps toward a characterisation of risk including the way in which the preference map "should" change if the

[3] Consider a choice between the following two prospects:

$$P: \begin{cases} \$1\,000 & \text{with probability } 0.7 \\ \$100\,000 & \text{with probability } 0.3 \end{cases}$$

$$P': \begin{cases} \$1\,000 & \text{with probability } 0.2 \\ \$30\,000 & \text{with probability } 0.8 \end{cases}$$

Starting with Lichtenstein and Slovic (1983) a large number of experimental studies have shown the following behaviour

1. When a simple choice between P and P' is offered, many experimental subjects would choose P'.
2. When asked to make a dollar bid for the right to either prospect many of those who had chosen then put a higher bid on P than on P'.

This phenomenon is known as *preference reversal*. Which of the fundamental axioms appears to be violated?

	RED	**BLUE**	**GREEN**
P_{10}	1	6	10
$\hat{P}_{10}$	2	3	10

Table 8.2: Example for Independence Axiom

person's perception about the unknown future should change. It appears that we could – perhaps with some qualification – represent preferences over the space of contingent goods using a utility function as in Theorem 4.1 and the associated discussion on page 77.

However some might complain all this is a little vague: we have not specified exactly what risk is, nor have we attempted to move beyond an elementary two-state example. To make further progress, it is useful to impose more structure on preferences. By doing this we shall develop the basis for a standard model of preference in the face of uncertainty and show the way that this model depends on the use of a few powerful assumptions.

8.3.1 Key axioms

Let us suppose that all outcomes can be represented as vectors $\mathbf{x}$ which belong to $X \subset \mathbb{R}^n$. We shall introduce three more axioms.

Axiom 8.1 (State-irrelevance) *The state that is realised has no intrinsic value to the person.*

In other words, the colour of the state itself does not matter. The intuitive justification for this is that the objects of desire are just the vectors $\mathbf{x}$ and people do not care whether these materialise on a "red" day or a "blue" day; of course it means that one has to be careful about the way goods and their attributes are described: the desirability of an umbrella may well depend on whether it is a rainy or a sunny day.

Axiom 8.2 (Independence) *Let $P_{\mathbf{z}}$ and $\widehat{P}_{\mathbf{z}}$ be any two distinct prospects specified in such a way that the payoff in one particular state of the world is the same for both prospects: $\mathbf{x}_\omega = \widehat{\mathbf{x}}_\omega = \mathbf{z}$. Then, if prospect $P_{\mathbf{z}}$ is preferred to prospect $\widehat{P}_{\mathbf{z}}$ for* one *value of $\mathbf{z}$, $P_{\mathbf{z}}$ is preferred to $\widehat{P}_{\mathbf{z}}$ for* all *values of $\mathbf{z}$.*

To see what is involved, consider Table 8.2 in which the payoffs are scalar quantities. Suppose P_{10} is preferred to $\hat{P}_{10}$: would this still hold even if the payoff 10 (which always comes up under state GREEN) were to be replaced by the value 20? Look at the preference map depicted in Figure 8.8: each of the "slices" that have been drawn in shows a glimpse of the $(x_{\text{RED}}, x_{\text{BLUE}})$-contours for one given value of x_{GREEN}. The independence property also implies that the individual does not experience disappointment or regret – see Exercises 8.5 and 8.6.[4]

[4] Compare Exercises 8.5 and 8.6. What is the essential difference between regret and disappointment?

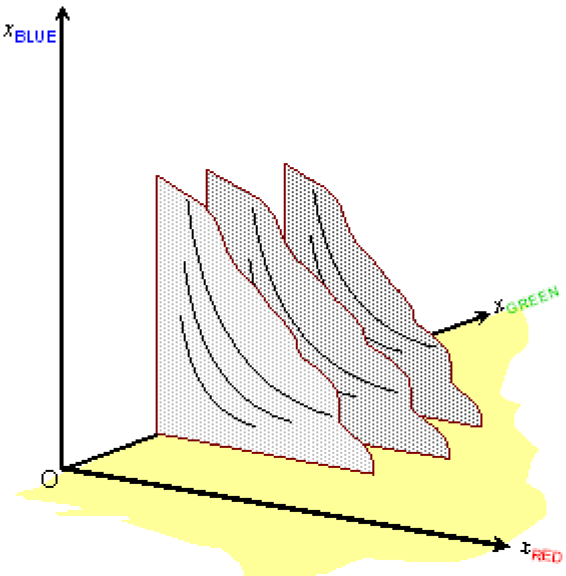

Figure 8.8: Independence axiom: illustration

Axiom 8.3 (Revealed Likelihood) *Let $\mathbf{x}^*$ and $\mathbf{x}$ be two payoffs such that under certainty $\mathbf{x}^*$ would be weakly preferred to $\mathbf{x}$. Let Ω_0 and Ω_1 be any two given subsets of the set of all states of the world Ω and suppose the individual weakly prefers the prospect*

$$P_0 = [\mathbf{x}^* \textit{ if } \omega \in \Omega_0; \mathbf{x} \textit{ if } \omega \notin \Omega_0]$$

to the prospect

$$P_1 = [\mathbf{x}^* \textit{ if } \omega \in \Omega_1; \mathbf{x} \textit{ if } \omega \notin \Omega_1]$$

for some *such $\mathbf{x}^*, \mathbf{x}$. Then he prefers P_0 to P_1 for* every *such $\mathbf{x}^*$, $\mathbf{x}$.*

Consider an example illustrating this property. Let the set of all states-of-the-world be given by

$$\Omega = \{\text{RED,ORANGE,YELLOW,GREEN,BLUE,INDIGO,VIOLET}\}.$$

Now, suppose we have a person who prefers one apple to one banana, and also prefers one cherry to one date. Consider two prospects P_0, P_1 which each have as payoffs an apple or a banana in the manner defined in Table 8.3:

Furthermore let us define two subsets of Ω, namely

$$\Omega_0 := \{\text{RED,ORANGE,YELLOW,GREEN,BLUE}\}$$

$$\Omega_1 := \{\text{GREEN,BLUE,INDIGO,VIOLET}\};$$

	RED	ORANGE	YELLOW	GREEN	BLUE	INDIGO	VIOLET
P_0	apple	apple	apple	apple	apple	banana	banana
P_1	banana	banana	banana	apple	apple	apple	apple

Table 8.3: Prospects with fruit

	RED	ORANGE	YELLOW	GREEN	BLUE	INDIGO	VIOLET
P_0'	cherry	cherry	cherry	cherry	cherry	date	date
P_1'	date	date	date	cherry	cherry	cherry	cherry

Table 8.4: Prospects with different fruit

we see that P_0 and P_1 then have the property described in the axiom. Suppose the individual prefers P_0 to P_1. Then the revealed-likelihood axiom requires that he also prefer P_0' to P_1', defined as in Table 8.4; it further implies that the above hold for any other arbitrary subsets Ω_0, Ω_1 of the set of all states-of-the-world.

The intuition is that the pairs (P_0, P_1) and (P_0', P_1') have in common the same pattern of subsets of the state-space where the "winner" comes up. By consistently choosing P_0 over P_1, P_0' over P_1', and so on, the person is revealing that he thinks that the subset of events Ω_0 is "more likely" than Ω_1. This assumption rules out so-called "ambiguity aversion" – see Exercise 8.7.

The three new assumptions then yield this important result, proved in Appendix C:

Theorem 8.1 (Expected utility) *Assume that preferences over the space of state-contingent goods can be represented by a utility function as in Theorem 4.1. If preferences also satisfy state-irrelevance, independence and revealed likelihood (axioms 8.1 – 8.3) then they can be represented in the form*

$$\sum_{\omega\in\Omega} \pi_\omega u(\mathbf{x}_\omega) \tag{8.1}$$

where the π_ω are real numbers and u is a real-valued function on X that is defined up to an increasing, affine transformation.

In honour of its origin the special form (8.1) is often known as a *von-Neumann-Morgenstern utility function.* As with the problem of aggregation discussed in chapter 5 (see page 112), once again the additional requirements imposed on the representation of preferences induce a set of restrictions on the class of admissible utility functions. It is difficult to overstate the importance of this result (and its alternate version in Theorem 8.4 below) for much of modern microeconomic analysis. Nevertheless, before we press on to its interpretation and some of its many applications, it is worth reminding ourselves that the additional structural axioms on which it rests may be subject to challenge as reasonable representations of people's preferences in the face of uncertainty. Specifically, experimental evidence has repeatedly rejected the independence

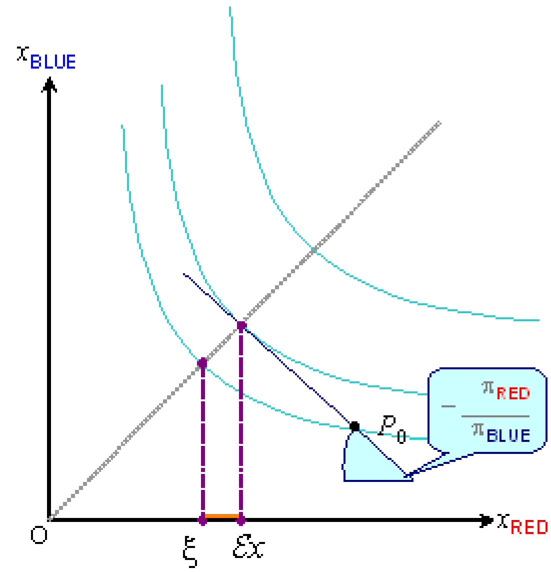

Figure 8.9: Contours of the Expected-Utility function

axiom as a representation of people's preferences in the face of choice under uncertainty.

8.3.2 Von-Neumann-Morgenstern utility

What does this special utility function look like? To scrutinise the properties of (8.1) and how they work we can extract a lot of information from the simple case of scalar payoffs – e.g. payoffs in money – as in section 8.2.2 above.

First the function u. Here we encounter a terminologically awkward corner. We should not really call u "the utility function" because the whole expression (8.1) is the person's utility; so u is sometimes known as the individual's *cardinal utility function* or *felicity function*; arguably neither term is a particularly happy choice of words. The last part of Theorem 8.1 means that the function u could be validly replaced by $\hat{u}$ defined by

$$\hat{u} := a + bu \tag{8.2}$$

where a is an arbitrary constant and $b > 0$: the scale and origin of u are unimportant. However, although these features of the function u are irrelevant, other features, such as its curvature, are important because they can be used to characterise the individual's attitude to risk: this is dealt with in section 8.4.

Now consider the set of weights $\{\pi_\omega : \omega \in \Omega\}$ in (8.1). If they are normalised so as to sum to 1,[5] then they are usually known as the *subjective probabilities*

[5] Show that, given the definition of u, this normalisation can always be done.

of the individual. Notice that the concept of probability has emerged naturally from the structural assumptions that we have introduced on personal preferences, rather than as an explicit construct. Furthermore, being "subjective," they could differ from one individual to another – one person might quite reasonably put a higher weight on the outcome "The red party will win the election" than another. We shall have much more to say about this and other aspects of probability later in this chapter.

In view of the subjective-probability interpretation of the πs the von-Neumann-Morgenstern utility function (8.1) can be interpreted as *expected utility*, and may more compactly be written $\mathcal{E}u(\mathbf{x})$. In the two-state, scalar payoff case that we used as an example earlier this would be written:

$$\pi_{\text{RED}} u(x_{\text{RED}}) + \pi_{\text{BLUE}} u(x_{\text{BLUE}}) \tag{8.3}$$

Using Figure 8.9 for the two-state case we can see the structure that (8.3) introduces to the problem:[6]

- The slope of the indifference curve where it crosses the 45° line is $(-)$ the ratio of the probabilities $\pi_{\text{RED}}/\pi_{\text{BLUE}}$.
- A corollary of this is that all the contours of the expected utility function must have the same slope at the point where they intersect the 45°-line.
- For any prospect such as point P_0 in Figure 8.9, if we draw a line with this slope through P_0, the point at which it cuts the 45°-line represents the *expected value* of the prospect P; the value of this is represented (on either axis) as $\mathcal{E}x$, where $\mathcal{E}$ is the usual expectations operator (see Definition A.28 on page 517).

8.3.3 The "felicity" function

Let us know interpret the function u in terms of individual attitudes. To fix ideas let us take the two-state case and suppose that payoffs are scalars; further assume that the individual assigns equal probability weight to the two states (this is not essential but it makes the diagram more tractable). Figure 8.10 illustrates three main possibilities for the shape of u.

- In the left-hand panel look at the diagonal line joining the points $(x_{\text{BLUE}}, u(x_{\text{BLUE}}))$ and $(x_{\text{RED}}, u(x_{\text{RED}}))$; halfway along this line we can read off the individual's expected utility (8.3); clearly this is strictly less than $u(\mathcal{E}x)$. So if u had this shape an individual would strictly prefer the expected value of the prospect (in this case $\pi_{\text{RED}}x_{\text{RED}} + \pi_{\text{BLUE}}x_{\text{BLUE}}$) to the prospect itself. It follows from this that the person would reject some "better-than-fair" gambles i.e. gambles where the expected payoff is higher than the stake money for the gamble.

[6] Explain why these results are true, using (8.3).

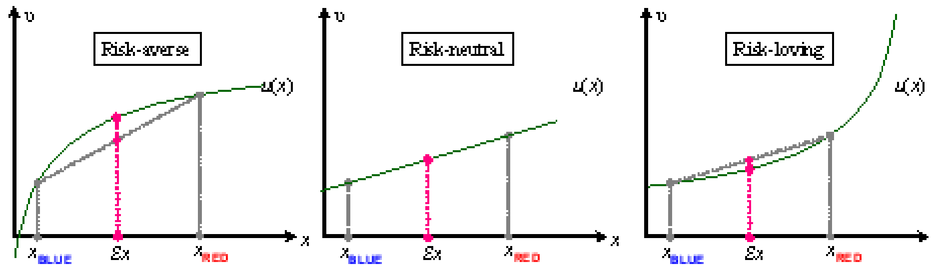

Figure 8.10: Attitudes to risk

- In the right-hand panel we see the opposite case; here the individual's expected utility is higher than $u(\mathcal{E}x)$ and so the person would accept some unfair gambles (where the expected payoff is strictly less than the stake money).[7]

- Finally the middle panel. Here the expected utility of the gamble just equals $u(\mathcal{E}x)$.

Clearly each of these cases is saying something important about the person's attitude to risk; let us investigate this further.

8.4 Risk aversion

We have already developed an intuitive approach to the concept of risk aversion. If the utility function U over contingent goods is quasiconcave (so that the indifference curves in the state-space diagram are convex to the origin) then we have argued that the person is risk averse – see page 183 above. However, we can now say more: if, in addition to quasiconcavity the utility function takes the von-Neumann-Morgenstern form (8.1) then the felicity function u must be concave.[8] This is precisely the case in the left-hand panel of Figure 8.10 and accords with the accompanying story explaining that the individual might reject some fair gambles, which is why the panel has been labelled "risk averse." By the same argument the second and third panels depict risk-neutral and risk-loving attitudes, respectively.[9] However, we can extract more information from the graph of the felicity function.

[7] Would a rational person buy lottery tickets?

[8] Prove this. Hint: use Figure 8.9 and extend the line through P_0 with slope $-\pi_{\text{RED}}/\pi_{\text{BLUE}}$ to cut the indifference curve again at a point P_1; then use the definition of quasiconcavity.

[9] Draw an example of a u-function similar to those in Figure 9 but where the individual is risk-loving for small risks and risk-averse for large risks.

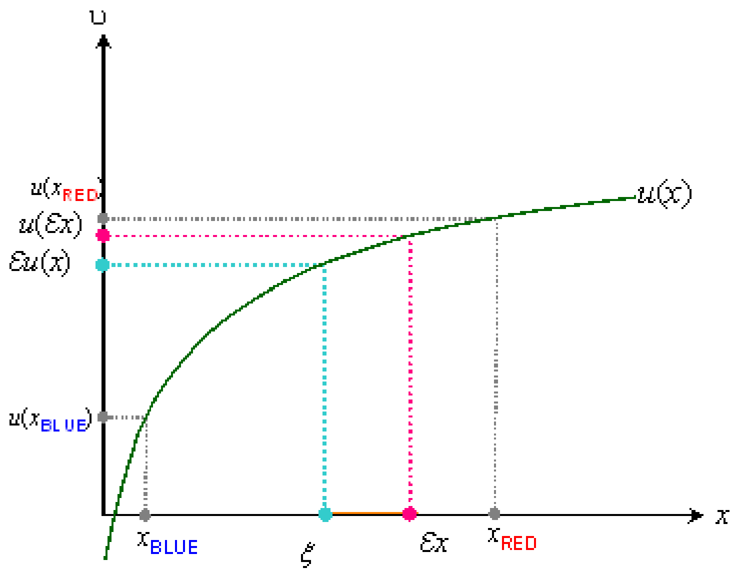

Figure 8.11: The "felicity" or "cardinal utility" function u.

8.4.1 Risk premium

We have already introduced the concept of the certainty equivalent in 8.2.2: as shown in Figure 8.5 this is the amount of perfectly certain income that you would be prepared to exchange for the random prospect lying on the same indifference curve. Now, using the von-Neumann-Morgenstern utility function, the certainty equivalent can be expressed using a very simple formula: it is implicitly determined as the number ξ that satisfies

$$u(\xi) = \mathcal{E}u(x). \tag{8.4}$$

Furthermore we can use the certainty-equivalent to define the *risk premium* as

$$\mathcal{E}x - \xi. \tag{8.5}$$

This is the amount of income that the risk-averse person would sacrifice in order to eliminate the risk associated with a particular prospect: it is illustrated on the horizontal axis of Figure 8.9,

Now we can also use the graph of the felicity function to illustrate both the certainty-equivalent and the risk premium – see Figure 8.11. In this figure $\pi_{\text{RED}} > \pi_{\text{BLUE}}$ and on the horizontal axis $\mathcal{E}x$ denotes the point $\pi_{\text{RED}} x_{\text{RED}} + \pi_{\text{BLUE}} x_{\text{BLUE}}$; on the vertical axis $\mathcal{E}u(x)$ denotes the point $\pi_{\text{RED}} u\left(x_{\text{RED}}\right) + \pi_{\text{BLUE}} u\left(x_{\text{BLUE}}\right)$. Use the curve to read off on the horizontal axis the income ξ that corresponds to $\mathcal{E}u(x)$ on the vertical axis. The distance between the two points ξ and $\mathcal{E}x$ on the horizontal axis is the risk premium.

But we can say more about the shape of the function u by characterising risk-aversion as a numerical index.

8.4.2 Indices of risk aversion

Why quantify risk-aversion? It is useful to be able to describe individuals' preferences in the face of uncertainty in a way that has intuitive appeal: a complex issue is made manageable through a readily interpretable index. However, it should not come as a surprise to know that there is more than one way of defining an index of risk aversion, although the good news is that the number of alternative approaches is small.

Assume that preferences conform to the standard von-Neumann-Morgenstern configuration. In the case where the payoff is a *scalar* (as in our diagrammatic examples above), we can define an index of risk aversion in a way that encapsulates information about the function u depicted in Figure 8.11. Use the subscript notation u_x and u_{xx} to denote the first and second derivatives of the felicity function u. Then we can introduce two useful definitions of risk aversion.

Absolute risk aversion

The first of the two risk-aversion concepts is just the normalised rate of decrease of marginal felicity:

Definition 8.1 *The index of* absolute risk aversion *is a function* α *given by*

$$\alpha(x) := -\frac{u_{xx}(x)}{u_x(x)}$$

We can also think of $\alpha(\cdot)$ as a sort of index of "curvature" of the function u; in general the value of $\alpha(x)$ may vary with the level of payoff x, although we will examine below the important special case where α is constant. The index α is positive for risk-averse preferences and zero for risk-neutral preferences (reason: follows immediately from the sign of $u_{xx}(\cdot)$). Furthermore α is independent of the scale and origin of the function u.[10]

This convenient representation enables us to express the risk premium in terms of the index of absolute risk aversion and the variance of the distribution of x:[11]

Theorem 8.2 (Risk premium and variance) *For small risks the risk premium is approximately* $\frac{1}{2}\alpha(x)\text{var}(x)$.

[10] Show why this property is true.

[11] Prove this. Hint, use a Taylor expansion around $\mathcal{E}x$ on the definition of the risk premium (see page 494).

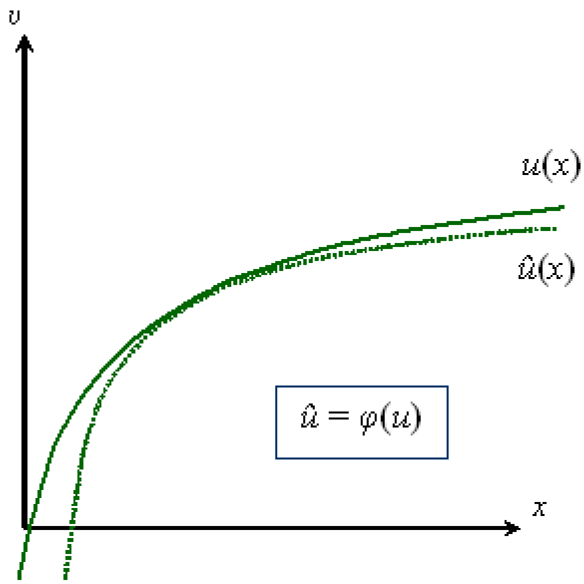

Figure 8.12: Concavity of u and risk aversion

Relative risk aversion

The second standard approach to the definition of risk aversion is this:

Definition 8.2 *The index of* relative risk aversion *is a function* ϱ *given by*

$$\varrho(x) := -x\frac{u_{xx}(x)}{u_x(x)}$$

Clearly this is just the "elasticity of marginal felicity". Again it is clear that $\varrho(x)$ must remain unchanged under changes in the scale and origin of the function u. Also, for risk-averse or risk-neutral preferences, increasing absolute risk aversion implies increasing relative risk aversion (but not vice versa).[12]

Comparisons of risk-attitudes

We have already seen in above (page 190) that a concave u-function can be interpreted as risk aversion everywhere, a convex u-function as risk preference everywhere. We can now be more precise about the association between concavity of u and risk aversion: if we apply a strictly concave transformation to u then either index of risk aversion must increase, as in the following theorem.[13]

[12] Show this by differentiating the expression in Definition 8.2.

[13] Prove this by using the result that the second derivative of a strictly concave function is negative.

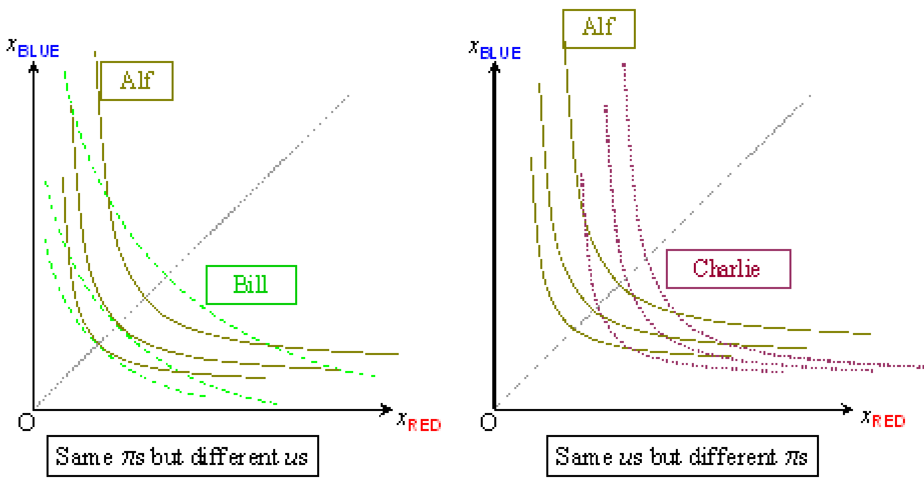

Figure 8.13: Differences in risk attitudes

Theorem 8.3 (Concavity and risk aversion) *Let u and $\widehat{u}$ be two felicity (cardinal utility) functions such that $\widehat{u}$ is a concave transformation of u. Then $\widehat{\alpha}(x) \geq \alpha(x)$ and $\widehat{\varrho}(x) \geq \varrho(x)$.*

So, the more "sharply curved" is the cardinal-utility or felicity function u, the higher is risk aversion (see Figure 8.12) on either interpretation. An immediate consequence of this is that the more concave is u the higher is the risk premium (8.5) on any given prospect.[14]

This gives us a convenient way of describing not only how an individual's attitude to risk might change, but also how one compare the risk attitudes of different people in terms of their risk aversion. Coupled with the notion of differences in subjective probabilities (page 188) we have quite a powerful method of comparing individuals' preferences. Examine Figure 8.13. On the left-hand side we find that Alf and Bill attach the same subjective probabilities to the two states RED and BLUE: for each of the two sets of indifference curves in the state-space diagram the slope where they intersect the 45° line is the same. But they have differing degrees of risk aversion – Alf's indifference curves are more sharply convex to the origin (his felicity function u will be more concave) than is the case for Bill. By contrast, on the right-hand side, Alf and Charlie exhibit the same degree of risk aversion (their indifference curves have the same "curvature" and their associated u-functions will be the same), but Charlie puts a higher probability weight on state RED than does Alf (look at the slopes where the indifference curves cross the 45° line).

[14] Show this using Jensen's inequality (see page 517 in Appendix A).

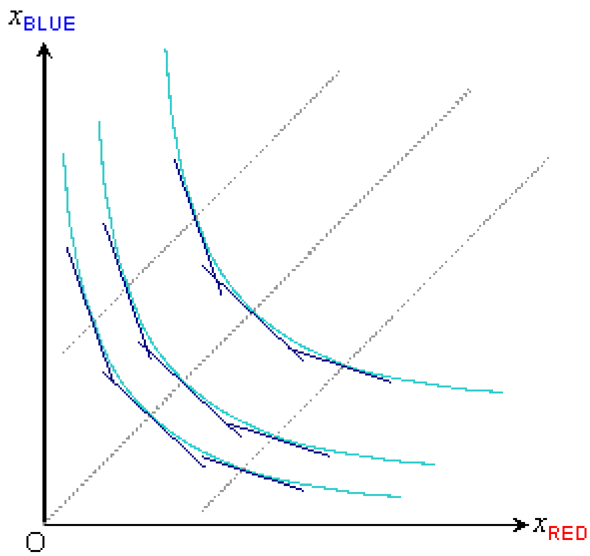

Figure 8.14: Indifference curves with constant absolute risk aversion

8.4.3 Special cases

The risk-aversion indices $\alpha(\cdot)$ and $\varrho(\cdot)$ along with the felicity function $u(\cdot)$ are quite general. However, for a lot of practical modelling it is useful to focus on a particular form of u. Among the many possibly fascinating special functional forms that might be considered it is clearly of interest to consider preferences where either $\alpha(x)$ or $\varrho(x)$ is constant for all x. In each case we get a particularly convenient formula for the felicity function u.

Constant Absolute Risk Aversion In the case of constant *absolute* risk aversion the felicity function must take the form:[15]

$$u(x) = -\frac{1}{\alpha}e^{-\alpha x} \tag{8.6}$$

or some increasing affine transformation of this – see (8.2) above. Figure 8.14 illustrates the indifference curves in state space for the utility function (8.1) given a constant α: note that along any 45° line the MRS between consumption in the two states-of-the-world is constant.[16]

[15] Use Definition 8.1 to establish (8.6) if $\alpha(x)$ is everywhere a constant α.

[16] Suppose individual preferences satisfy (8.1) with u given by (8.6). Show how Figure 8.14 alters if (a) π_ω is changed, (b) α is changed.

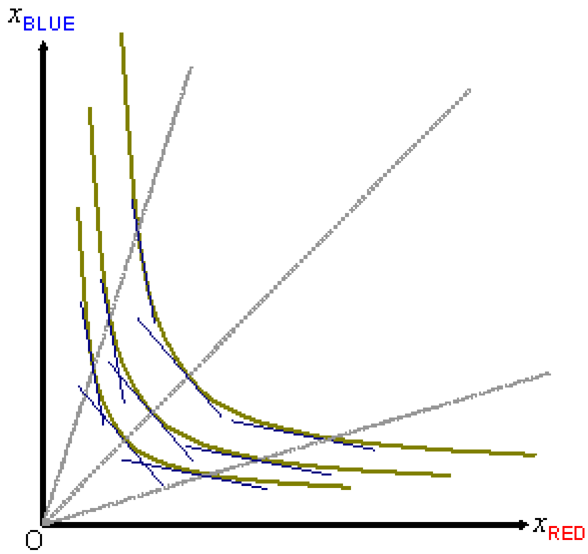

Figure 8.15: Indifference curves with constant relative risk aversion

Constant Relative Risk Aversion In the case of constant *relative* risk aversion the felicity function must take the form:[17]

$$u(x) = \frac{1}{1-\varrho}x^{1-\varrho} \tag{8.7}$$

illustrated in Figure 8.15[18] or some transformation of (8.7) of the form (8.2). Figure 8.14 illustrates the indifference curves in state space for the utility function (8.1) given a constant ϱ: in this case we see that the MRS is constant along any ray through the origin.

Other special cases are sometimes useful, in particular the case where u is a quadratic function – see Exercise 8.8.

Example 8.1 *How risk averse are people? Barsky et al. (1997) used survey questions from the Health and Retirement Survey – a panel survey of a nationally representative sample of the US population aged 51 to 61 in 1992 – to elicit information on risk aversion, subjective rate of time preference, and willingness to substitute intertemporally. The questions involved choice in hypothetical situations about willingness to gamble on lifetime income. Their principal evidence*

[17] Use Definition 8.2 to establish (8.7) if $\varrho(x)$ is everywhere a constant ϱ.

[18] Suppose individual preferences satisfy (8.1) with u given by (8.7). Show how Figure 8.15 alters if (a) π_ω is changed, (b) ϱ is changed.

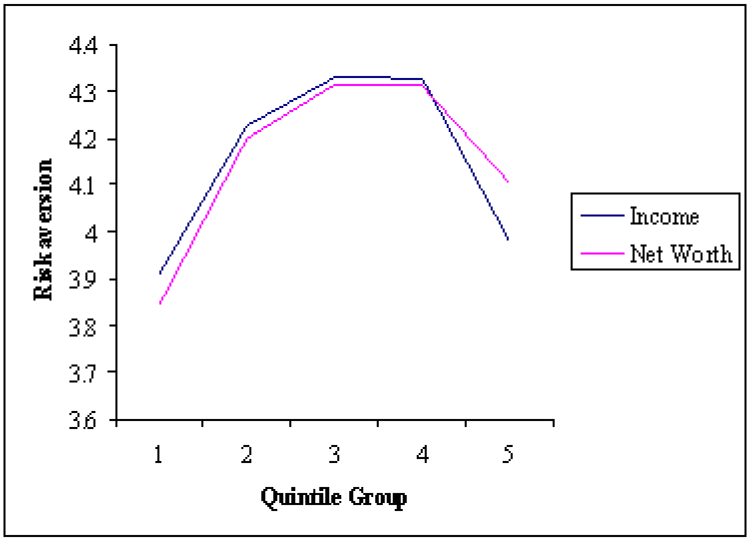

Figure 8.16: Estimates of ϱ by quintiles from Barsky et al. (1997)

concerns the degree of "relative risk tolerance" – the inverse of $\varrho(x)$ – by individuals at different points in the income distribution. The implications of these estimates for relative risk aversion by income and wealth groups group is shown in Figure 8.16.

8.5 Lotteries and preferences

sections 8.2 to 8.4 managed quite well without reference to probability, except as a concept derived from the structure of preferences in the face of the unknown future. This is quite a nice idea where there is no particular case for introducing an explicit probability model, but now we are going to change that. By an explicit probability model I mean that there is a well-defined concept of probability conforming to the usual axioms, and that the probability distribution is objectively knowable (section A.8 on page 515 reviews information on probability distributions). Where the probabilities come from – a coin-tossing, a spin of the roulette wheel – we do not enquire, but we just take them to be known entities.

We are going to consider the possibility that *probability distributions* are themselves the objects of choice. The motivation for this is easy to appreciate if we think of the individual making a choice amongst lotteries with a given set of prizes associated with the various possible states of the world: the prizes

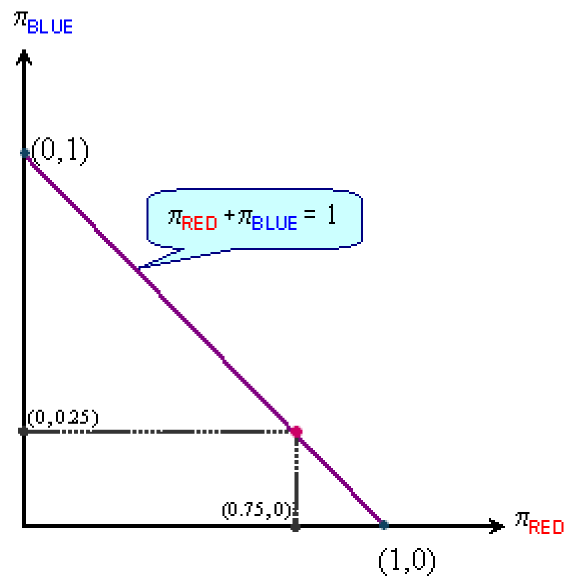

Figure 8.17: The probability diagram: $\#\Omega = 2$

are fixed but there are different probability vectors associated with different lotteries.

8.5.1 The probability space

To formalise this assume a finite set of states of the world ϖ as in (A.63): this is not essential, but it makes the exposition much easier. There is a payoff $\mathbf{x}_\omega$ and a probability π_ω associated with each state. We can imagine preferences being defined over the space of probability distributions, a typical member of which can be written as a ϖ-dimensional vector $\boldsymbol{\pi}$ given by (A.64)

$$\boldsymbol{\pi} := (\pi_{\text{RED}}, \pi_{\text{BLUE}}, \pi_{\text{GREEN}}, ...) \tag{8.8}$$

such that

$$\sum_{\omega \in \Omega} \pi_\omega = 1. \tag{8.9}$$

Figure 8.17 depicts the case $\varpi = 2$ where the set of points representing the lottery distributions is the 45° line from $(0, 1)$ to $(1, 0)$: the specific distribution $(0.75, 0.25)$ is depicted as a point on this line. Alternatively, for the case $\varpi = 3$, we can use Figure 8.18 where the set of points representing valid probability distributions is the shaded triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; the specific distribution $(0.5, 0.25, 0.25)$ is illustrated in the figure. (Figures 8.17 and 8.18 are essentially exactly the same as the normalised price diagrams, Figures

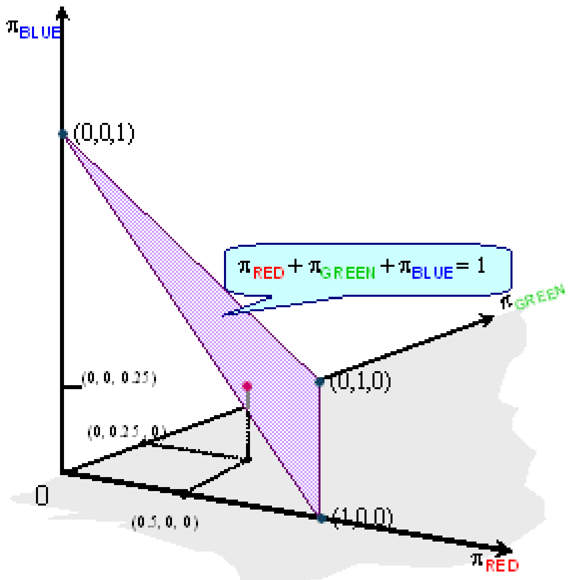

Figure 8.18: The probability diagram: $\#\Omega = 3$

7.8 and B.21) The $\varpi = 3$ case can be seen more clearly in Figure 8.19 where the probability triangle has been laid out flat.

8.5.2 Axiomatic approach

Now, suppose we consider an individual's preferences over the space of lotteries. Again we could try to introduce a "reasonable" axiomatisation for lotteries and then use this to characterise the structure of preference maps – a particular class of utility functions – that are to be regarded as suitable for problems of choice under uncertainty.

The three axioms that follow form the standard way of doing this axiomatisation. Here $\boldsymbol{\pi}^{\circ}, \boldsymbol{\pi}'$ and $\boldsymbol{\pi}''$ are lotteries with the same payoffs, each being ϖ-vectors of the form 8.8. The payoffs associated with the given set of prizes for each of the ϖ states-of-the-world is the ordered list of consumption vectors $[\mathbf{x}_{\text{RED}}, \mathbf{x}_{\text{BLUE}}, \mathbf{x}_{\text{GREEN}}, ...]$ and $(0, 1)$ is the set of numbers greater than zero but less than 1.

It is convenient to reintroduce the inelegant "weak preference" notation that was first used in chapter 4. Remember that the symbol "$\succcurlyeq$" should be read as "is at least as good as." Here are the basic axioms:

Axiom 8.4 (Transitivity over lotteries) *If* $\boldsymbol{\pi}^{\circ} \succcurlyeq \boldsymbol{\pi}'$ *and* $\boldsymbol{\pi}' \succcurlyeq \boldsymbol{\pi}''$ *then* $\boldsymbol{\pi}^{\circ} \succcurlyeq \boldsymbol{\pi}''$.

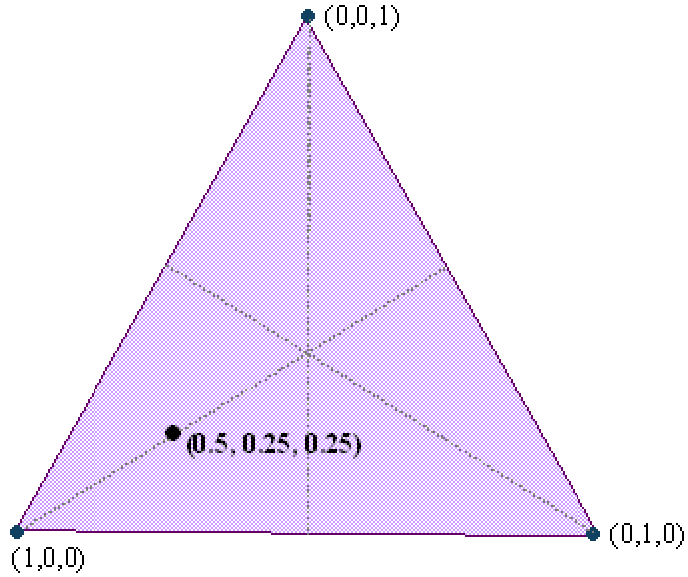

Figure 8.19: The probability diagram: $\#\Omega = 3$ (close-up)

Axiom 8.5 (Independence of lotteries) *If* $\pi^\circ \succcurlyeq \boldsymbol{\pi}'$ *and* $\lambda \in (0,1)$*, then*

$$\lambda\boldsymbol{\pi}^\circ + [1-\lambda]\,\boldsymbol{\pi}'' \succcurlyeq \lambda\boldsymbol{\pi}' + [1-\lambda]\,\boldsymbol{\pi}''.$$

Axiom 8.6 (Continuity over lotteries) *If* $\boldsymbol{\pi}^\circ \succ \boldsymbol{\pi}' \succ \boldsymbol{\pi}''$ *then there are numbers* $\lambda, \mu \in (0,1)$ *such that*

$$\lambda\boldsymbol{\pi}^\circ + [1-\lambda]\,\boldsymbol{\pi}'' \succ \boldsymbol{\pi}'$$

and

$$\boldsymbol{\pi}' \succ \mu\boldsymbol{\pi}^\circ + [1-\mu]\,\boldsymbol{\pi}''$$

Now for a very appealing result that obviously echoes Theorem 8.1 (for proof see Appendix C):

Theorem 8.4 (Lottery Preference Representation) *If axioms 8.4–8.6 hold then preferences can be represented as a* von-Neumann-Morgenstern utility function*:*

$$\sum_{\omega\in\Omega} \pi_\omega u\left(\mathbf{x}_\omega\right) \tag{8.10}$$

where u *is a real-valued function on* X *that is defined up to an increasing, affine transformation.*

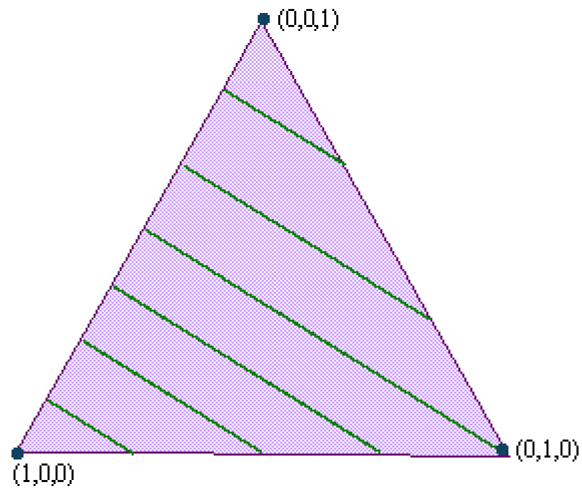

Figure 8.20: π-indifference curves

So with the set of three axioms over lotteries the individual's preference structure once again takes the expected utility form

$$\mathcal{E}u(\mathbf{x}).$$

Furthermore, it is clear that the utility function (8.10) can be rewritten as a simple "bilinear" form

$$\sum_{\omega \in \Omega} \pi_\omega \upsilon_\omega \tag{8.11}$$

where $\upsilon_\omega := u(\mathbf{x}_\omega)$ is the payoff in state-of-the-world ω, expressed in utility terms. We can see the objective function (8.11) in two equivalent ways:

1. As a weighted sum of payoffs (the payoffs are the utilities derived from consumption; the weights are the probabilities).

2. As a weighted sum of probabilities (the weights are the scalar utility payoffs).

Version 1 is exactly what we already found from our first pass through the axiomatisation of preferences under uncertainty in section 8.3. Version 2 is perhaps the more natural when it is the probability distributions themselves that are the objects of choice.

π_ω	probability that state-of-the-world ω occurs
π_ω^h	subjective probability of ω according to h
u	felicity or cardinal utility function
β_j	holding of bonds of type j.
$r_{j\omega}$	rate-of-return on bonds of type j in state ω
$p_{i\omega}$	price of good i contingent on state ω
$\bar{y}$	initial wealth
y_ω	wealth in state ω

Table 8.5: Uncertainty and risk: notation

The linearity of the expression (8.11) implies that indifference curves must take the form illustrated[19] in Figure 8.20 and will exhibit the following properties:[20]

- The indifference curves must be parallel straight lines.
- If $\upsilon_{\text{RED}} > \upsilon_{\text{GREEN}} > \upsilon_{\text{BLUE}}$, the slope $\frac{d\pi_{\text{BLUE}}}{d\pi_{\text{RED}}}$ is positive.
- If υ_{BLUE} increases, then the slope also increases.

So we now have a second approach to the expected-utility representation individual's preferences under uncertainty. This alternative way of looking at the problem of uncertainty and choice is particularly useful when probabilities are well-defined and apparently knowable. It might seem that this is almost a niche study of rational choice in situations involving gaming machines, lotteries, horse-race betting and the like. But there is much more to it. We will find in chapter 10 that explicit randomisation is often appropriate as a device for the analysis and solution of certain types of economic problem: the range of potential application there is enormous.

8.6 Trade

Now that we have a fairly extensive view of how individuals' preferences uncertainty can be mapped we should try to put the analysis to work. To do this let us start by considering the logical extension of the exchange-economy analysis of chapter 7 to a world of uncertainty. We again make use of the timing convention introduced in Figure 8.1.

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[19] Another convenient way of representating the set of all probability distributions when $\varpi = 3$ can be constructed by plotting π_{RED} on the horizontal axis and π_{GREEN} on the vertical axis of a conventional two-dimensional diagram. (a) What shape will the set of all possible lotteries have in this diagrammatic representation? (b) How is π_{BLUE} to be determined in this diagram? (c) What shape will an expected-utility maximiser's indifference curves have in this diagram?

[20] In the case where $\varpi = 3$ show that these are true by using the fundamental property (8.9) and the bilnear form of utility (8.11).

8.6.1 Contingent goods: competitive equilibrium

If there are n physical commodities (anchovies, beef, champagne,...) and ϖ possible states-of-the-world (RED, BLUE,...) then, viewed from the morning, there are $n\varpi$ possible "contingent goods" (anchovies-under-RED, anchovies-under-BLUE, beef-under-RED,..., . It is possible that there are markets, open in the morning, in which titles to these contingent goods can be bought and sold. Then, using the principles established in chapter 7, one can then immediately establish the following:

Theorem 8.5 (Equilibrium in contingent goods) *If all individuals are risk-averse or risk-neutral then there market-clearing contingent-goods prices*

$$[p_{i\omega}], i = 1, ..., n, \omega \in \Omega \tag{8.12}$$

that will support an exchange equilibrium.[21]

If there is just one physical commodity ($n = 1$) and two states of the world the situation can be depicted as in Figure 8.21. In Alf has the endowment $(0, y_{\text{BLUE}})$ and Bill has the endowment $(y_{\text{RED}}, 0)$ where the size of the box is $y_{\text{RED}} \times y_{\text{BLUE}}$. Note that Alf's indifference curves all have the same slope where they intersect the 45° through the origin O^a; Bill's indifference curves all have the same slope where they intersect the 45° through the origin O^b; as drawn Alf and Bill have different subjective probabilities about the two events:

$$\frac{\pi^a_{\text{RED}}}{\pi^a_{\text{BLUE}}} > \frac{\pi^b_{\text{RED}}}{\pi^b_{\text{BLUE}}}$$

Equilibrium contingent-goods prices are shown as the line from the endowment point (top left-hand corner) to the equilibrium point on the contract curve.[22]

But the number of contingent goods $n\varpi$ may be huge, which suggests that it might be rather optimistic to expect all these markets to exist in practice. Could the scale of the problem be reduced somewhat?

8.6.2 Financial assets

Let us introduce "securities" – in other words financial assets. These securities are simply pieces of paper which say "the bearer is entitled to \$1 if state ω occurs". If person h has an endowment y^h of wealth in the morning, and if the price on the securities market (open in the morning) of an ω-security is σ_ω, then the following constraint holds:

$$\sum_{\omega\in\Omega} \sigma_\omega z^h_\omega \leq y^h$$

[21] Under what circumstances might it be possible to drop the assumption about risk aversion in this theorem?

[22] Redraw Figure 8.21 for two special cases: (a) where overall wealth in the economy is constant, independent of the state-of-the-world; (b) where Alf and Bill have the same subjective probabilities.

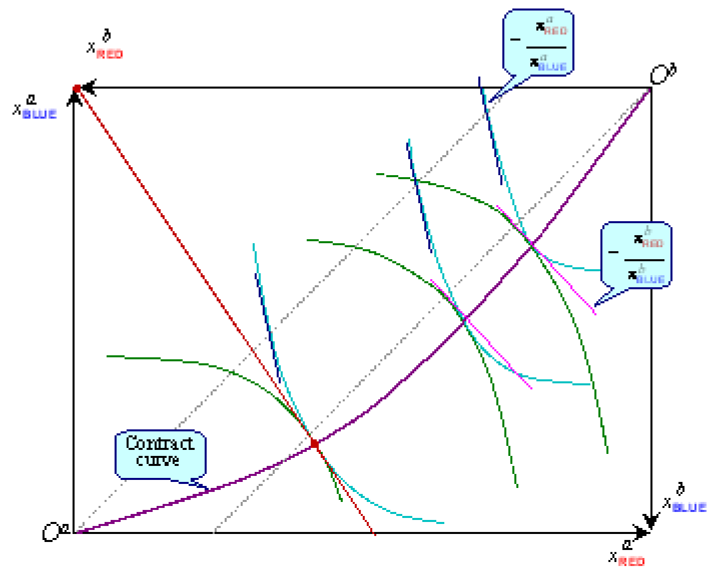

Figure 8.21: Contingent goods: equilibrium trade

where z^h_ω is the amount h buys of a ω-security. If the (morning) price of a claim on commodity i contingent on state ω is $p_{i\omega}$, and if $p_i|\omega$ is the (afternoon) price of commodity i given that state ω has actually occurred at lunch time, then equilibrium in the securities market, with all firms breaking even, requires:

$$\sigma_\omega p_i|\omega = p_{i\omega}$$

which, set out in plain language, says:

$$\boxed{\begin{array}{c}\text{price}\\ \text{of an } \omega\\ \text{security}\end{array}} \times \boxed{\begin{array}{c}\text{price of}\\ \text{champagne when}\\ \omega \text{ has occurred}\end{array}} = \boxed{\begin{array}{c}\text{contingent price}\\ \text{of champagne}\\ \text{given } \omega\end{array}}$$

There is in effect a two-stage budgeting process:

1. Choose the securities z^h_ω: this, along with the realisation of ω, determines income in the afternoon.

2. Given that state ω has occurred, choose the purchases $\mathbf{x}^h_\omega$ in the afternoon so as to maximise $u^h(\mathbf{x}^h_\omega)$.

This seems to reduce the scale of the problem by an order of magnitude, and to introduce a sensible separation of the optimisation problem.

But there is a catch. People have to do their financial shopping in the morning (lunchtime is too late). Now, when they are doing this, will they know what the $p_i|\omega$ would be for each commodity i in each possible state ω? This seems rather a demanding requirement, but they need to have this information in order to make sensible purchases of the securities z_ω^h in stage 1. Despite this logically awkward corner the two-stage simplification provides us a way of making the individual decision-maker's problem more tractable

8.7 Individual optimisation

In the light of the two-stage problem discussed in section 8.6.2 we can now extend the elementary modelling of the household's preferences and constraints to build in the essential characteristics of uncertainty. We will draw both upon standard consumer behaviour as presented in chapter 4, and the model of household production that was introduced in chapter 5. We shall develop further the idea of financial assets introduced in section 8.6.2 in order to focus upon the comparative statics of behaviour under risk.

To set the scene, consider a general version of the consumer's optimisation problem in an uncertain world. You have to go shopping for food, clothing and so on in the afternoon. The amount that you will have available to spend then may be stochastic (viewed from the morning), but that you can influence the probability distribution affecting your income by some choices that you make in the morning. These choices concern the disposition of your financial assets including the purchase of bonds and of insurance contracts.

Before we get down to the detail of the model let us again use Figure 8.1 to anchor the concepts that we need in developing the analysis. The timing of matters is in the following order

- The initial endowment is given. The person makes decisions on financial assets.
- The state-of-the-world ω is revealed: this and the financial decisions already made determine final wealth in state ω.
- Given final wealth the person determines consumptions using ex-post utility function and prices then ruling.

An explicit model of this is set out in section 8.7.2 below: first we will examine in more detail what the shape of the individual's attainable set is going to be in a typical problem of choice under uncertainty.

8.7.1 The attainable set

We need to consider the opportunities that may be open to the decision maker under uncertainty – the market environment and budget constraint. We have already introduced one aspect of this in that we have considered whether an individual would swap a given random prospect x for a certain payoff ξ: there may be some possibility of trading away undesirable risk. Is there, however, an analogue to the type of budget set we considered in chapters 4 and 5?

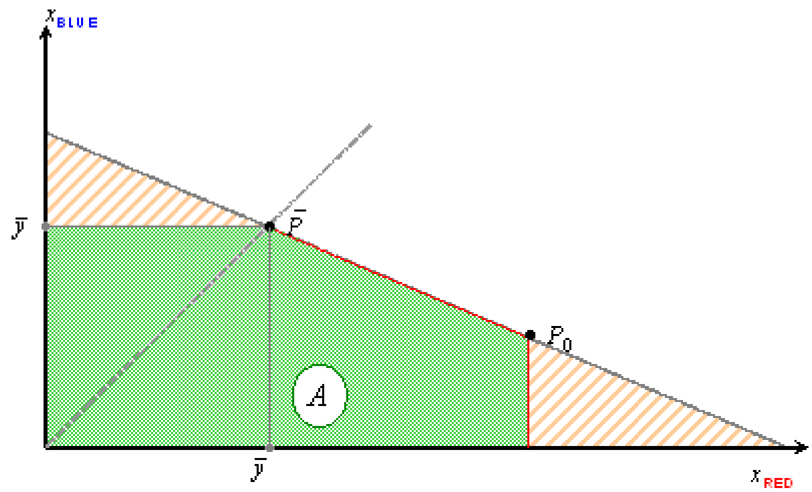

Figure 8.22: Attainable set: safe and risky assets

There are many ways that we might approach this question. However we will proceed by focusing on two key cases – where the individual's endowment is perfectly certain, and where it is stochastic – and then reasoning on a leading example of each case.

Determinate endowment: portfolio choice

Return to the two-state "RED/BLUE" examples above and examine Figure 8.22 which represents the attainable set for a simple portfolio composition problem. Imagine that an individual is endowed with an entitlement to a sum $\bar{y}$ (denominated in dollars) whichever state of the world is realised. We may think of this as money. He may use one or more of these dollars to purchase bonds in dollar units. For the moment, to keep things simple, there is only one type of bond: each bond has a yield of r° if state BLUE is realised, and r' if state RED is realised where we assume that

$$r' > 0 > r^\circ > -1$$

So if the individual purchases an amount β of bonds and holds the balance $\bar{y} - \beta$ in the form of money then the payoff in terms of ex-post wealth is either

$$y_{\text{RED}} = [\bar{y} - \beta] + \beta[1 + r']$$

or

$$y_{\text{BLUE}} = [\bar{y} - \beta] + \beta[1 + r^\circ]$$

In other words

$$(y_{\text{RED}}, y_{\text{BLUE}}) = (\bar{y} + \beta r', \bar{y} + \beta r^\circ) \tag{8.13}$$

By construction of the example, for all positive β we have $y_{\text{RED}} > \bar{y} > y_{\text{BLUE}}$. In Figure 8.22 the points $\bar{P}$ and P_0 represent, respectively the two cases where $\beta = 0$ and $\beta = \bar{y}$. Clearly the slope of the line joining $\bar{P}$ and P_0 is r'/r°, a negative number, and the coordinates of P_0 are

$$([1 + r']\bar{y},\ [1 + r^\circ]\bar{y})\,.$$

Given that he has access to such a bond market, any point on this line must lie in the feasible set; and assuming that free disposal of his monetary payoff is available in either case, the attainable set A must include all the points in the heavily shaded area shown in Figure 8.22. Are there any more such points? Perhaps.

First of all, consider points in the lightly shaded area above the line A. If one could "buy" a negative amount of bonds, then obviously the line the line from P_0 to $\bar{P}$ could be extended until it met the vertical axis. What this would mean is that the individual is now selling bonds to the market. Whether this is a practical proposition or not depends on other people's evaluation of him as to his "financial soundness": will he pay up if red materialises? With certain small transactions – betting on horse races among one's friends, for example – this may be quite reasonable. Otherwise one may have to offer an extremely large r' relative to r° to get anybody to buy one's bonds.

Secondly consider points in the area to the right of A. Why can't we just extend the line joining $\bar{P}$ and P_0 downwards until it meets the horizontal axis? In order to do this one would have to find someone ready to sell bonds "on credit" since one would then be buying an amount $\beta > \bar{y}$. Whoever extends this credit then has to bear the risk of the individual going bankrupt if blue is realised. So lenders might be found who would be prepared to advance him cash up to the point where he could purchase an amount $-\bar{y}/r^\circ$ of bonds. Again, we can probably imagine situations in which this is a plausible assumption, but it may seem reasonable to suppose that one may have to pay a very high premium for such a facility. Accordingly the feasible set might look like Figure 8.23, although for many purposes Figure 8.22 is the relevant shape.

There might be a rôle for many such financial assets – particularly if there were many possible states-of-the-world – in which case the attainable set A would have many vertices, a point to which we return in section 8.7.2.

Stochastic endowment: the insurance problem

Now consider a different problem using the same diagrammatic approach – see Figure 8.24. Suppose that the individual's endowment is itself stochastic – it equals if y_0 if RED is realised and $y_0 - L$ if BLUE is realised, where $0 < L < y_0$. As a simple example, state BLUE might be having one's house destroyed by fire and state RED is its not being destroyed, y_0 is the total value of your assets in the absence of a disaster and L is the monetary value of the loss. Let us suppose that fire insurance is available and interpret Figure 8.24. If full insurance coverage is available at a premium represented by

$$\kappa = y_0 - \bar{y} \tag{8.14}$$

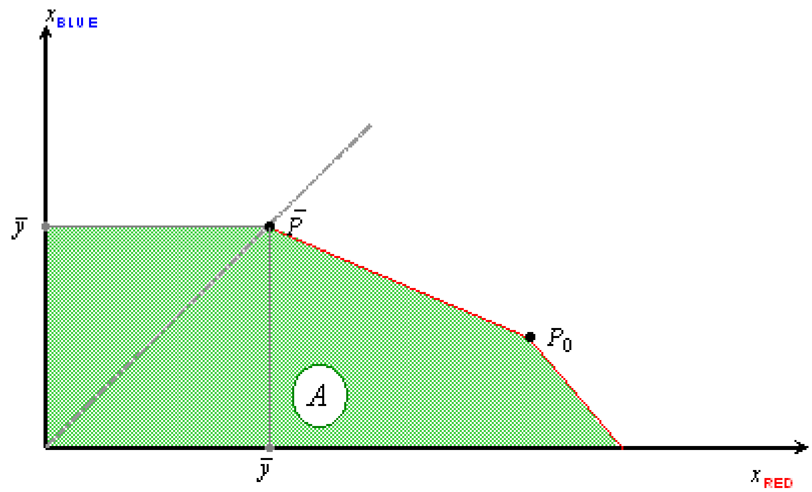

Figure 8.23: Attainable set: safe and risky assets (2)

then the outcome for such full insurance will be at point $\bar{P}$. If the individual may also purchase partial insurance at the same rates, then once again the whole of the line segment from $\bar{P}$ to P_0 – and hence the whole shaded pentagonal area – must lie in the attainable set A.

In this case too we can see that it may be that there are no further points available to the individual. Again consider the implications of enlarging the set A in the region above the horizontal line through point $\bar{P}$. At any point in this area the individual would in fact be better off if his house burned down than if it did not. The person has over-insured himself, a practice which is usually frowned upon. The reason that it is frowned upon is to be found in the concept of *moral hazard*. Moral hazard refers to the influence that the actions of the insured may have on the probability of certain events' occurrence. Up until now we have taken the probabilities – "objective" or "subjective" – attached to different events as exogenously given. But in practice the probability of a person's house burning down depends in part on his carelessness or otherwise. He may be more inclined to be careless if he knows that he has an insurance company to back him up if one day the house does burn down; furthermore the person may be inclined to be criminally negligent if he knows that he stands to *gain* by event BLUE being realised. So insurance companies usually prevent over-insurance and may indeed include an "excess clause" (otherwise known as "coinsurance") so that not even all of the shaded area is attainable.

Furthermore, for reasons similar to those of the portfolio selection example, it is unlikely that the points in the shaded area to the right of A could be included in the attainable set.

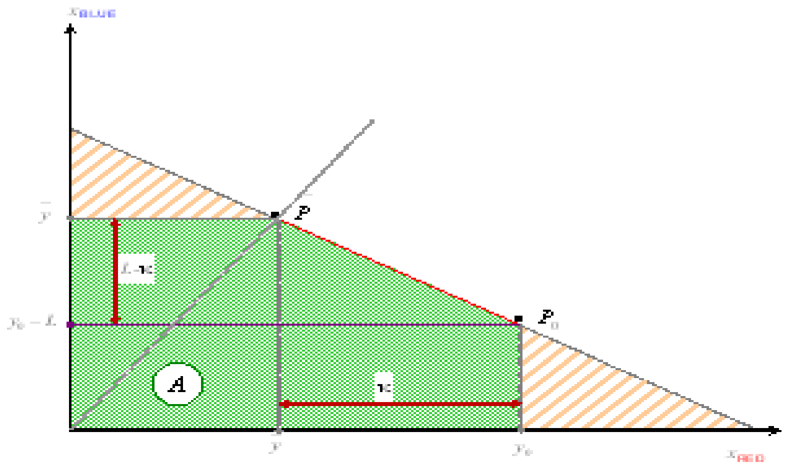

Figure 8.24: Attainable set: insurance

8.7.2 Components of the optimum

To set out the individual's optimisation problem let us assume that the person's opportunities are based on the model of section 8.7.1 with a determinate endowment. However, we will introduce one further consideration – the possibility of there being multiple financial assets in the form of "bonds." The person has a given amount of wealth $\bar{y}$ initially which he or she could invest in bonds of types $1, ..., m$. Denote by β_j the amount held of a type-j bond; then, under any particular state-of-the-world ω we can define income yielded by bond j as:

$$\boxed{\begin{array}{l}\text{return}\\ \text{on } j \text{ in}\\ \text{state } \omega\end{array}} \times \boxed{\begin{array}{l}\text{holding}\\ \text{of}\\ \text{bond } j\end{array}} = r_{j\omega}\beta_j$$

Then the value of of one's wealth after the financial decision becomes

$$y_\omega = \bar{y} + \sum_{j=1}^{m} r_{j\omega}\beta_j \tag{8.15}$$

– see equation (8.13). We could then further specify a standard consumer optimisation model conditional upon the realisation of a particular state-of-the-world ω

$$\max_{\mathbf{x}} U(\mathbf{x}) \text{ subject to } \sum_{i=1}^{n} p_i x_i \leq y_\omega$$

where I have written p_i as shorthand for $p_i|\omega$, the actual goods prices once state-of-the-world ω has been realised. and obtain a set of demand functions conditional upon ω:

$$x_i^* = D^i(\mathbf{p}, y_\omega).$$

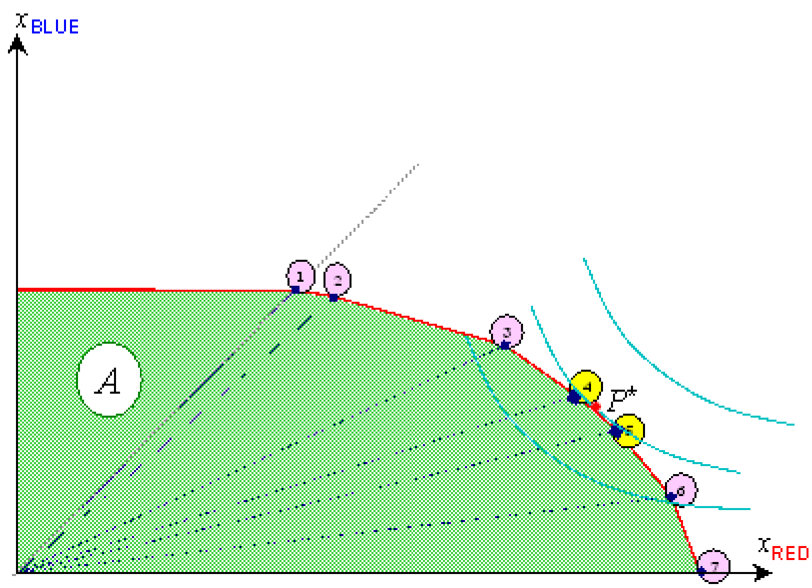

Figure 8.25: Consumer choice with a variety of financial assets

If we assume that goods' prices are known to be fixed then we may write the maximised utility in state ω as $u(y_\omega) := V(\mathbf{p}, y_\omega)$ where V is the conventional indirect utility function (Definition 4.6 page 88).

Suppose that there is a finite number of all possible states-of-the-world. Then clearly one also has to solve the problem:

$$\max_{\beta_1,\ldots,\beta_m} \sum_{\omega\in\Omega} \pi_\omega u\left(y_\omega\right)$$

subject to

$$y_\omega = \bar{y} + \sum_{j=1}^{m} r_{j\omega}\beta_j$$

But we have analysed this type of economic problem before. There is a close analogy with the general "household production" or "goods and characteristics" model discussed in chapter 5 (page 107). We just need to some translation of terminology; in the present case:

- $y_{\text{RED}}, y_{\text{BLUE}}, \ldots$ are the "consumption goods" from which one derives utility directly.
- $\beta_1, \ldots, \beta_m$ correspond to the "market goods" or "inputs" which are purchased by the household.

- Given uniform interest rates, one has a linear technology which transforms purchased assets into spendable income in each state of the world using (8.15).

This is illustrated in Figure 8.25 where the vertices of the attainable set A correspond to the various types of bond.[23] The slope of each facet is given by

$$\frac{r_{j+1,\text{BLUE}} - r_{j,\text{BLUE}}}{r_{j+1,\text{RED}} - r_{j,\text{RED}}}, \tag{8.16}$$

Drawing on the analysis of section 5.4 what can one say about the person's decisions regarding the purchase of financial assets in this set-up? Dominated financial assets will obviously be irrelevant to the optimal choice. There may be many zeros amongst the m assets – as illustrated in Figure 8.25 just two types of asset are purchased. As a result of this, when rates of return change one may get jumps in demand for financial assets as particular assets are brought into or dropped from the solution. One could also expect jumps in demand as initial wealth grows if the individual's indifference curves are not homothetic.[24] But can we say more about the way that this demand for financial assets will respond to changes in the distribution of rates of return?

8.7.3 The portfolio problem

We can say much more if we restrict attention to what happens on just one facet – i.e. if we rule out switching between facets of A as we change the the parameters of the model. Then, in the case where there are two states-of-the-world, the problem is effectively equivalent to that discussed earlier in section 8.7.1. However, although we will illustrate it for the two-state case using diagrams based on Figure 8.22 our approach will be more general in that we we will allow for arbitrarily many possible states of the world.

So we take a model in which there are just two assets: money and bonds. The person is endowed with a determinate amount of initial wealth $\bar{y}$. The rate of return on bonds is given by r a random variable with a known distribution having positive, finite mean; the density function of r is illustrated in Figure 8.26

If the person chooses to hold an amount β in the form of bonds, then wealth after the financial decision has been made is

$$y = \bar{y} + \beta r, \tag{8.17}$$

also a random variable – compare this with equations (8.13) and (8.15). Assume that the person's preferences are represented by a utility function of the form $\mathcal{E}u(y)$ where y is given by (8.17).

We can now set out the simplified optimisation problem:

$$\max_{\beta} \mathcal{E}u\left(\bar{y} + \beta r\right) \tag{8.18}$$

[23] As depicted bonds 1 and 7 are likely to be uninteresting – briefly explain what they are.
[24] Explain why you get jumps in the demand for bonds in this case.

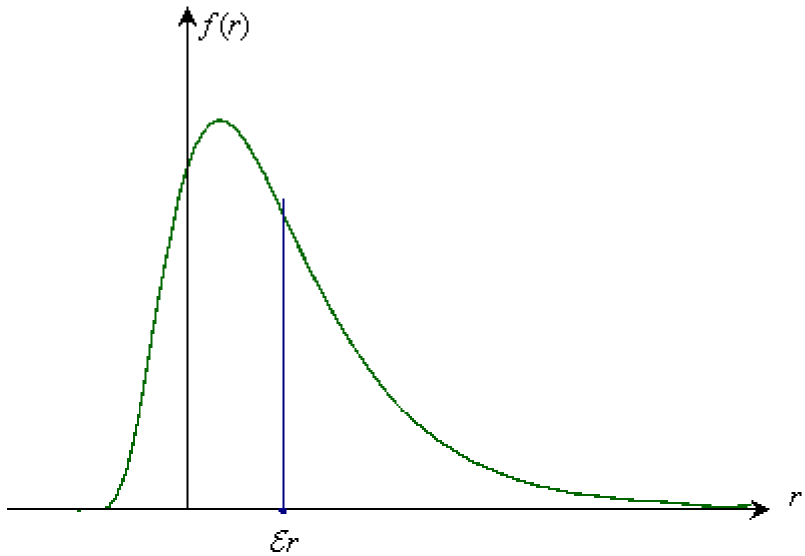

Figure 8.26: Distribution of returns

$$\text{subject to } 0 \leq \beta \leq \bar{y}$$

Letting u_y denote the first derivative of u with respect to y, the FOC condition for this maximisation problem is

$$\mathcal{E}\left(ru_y\left(\bar{y} + \beta r\right)\right) = 0 \tag{8.19}$$

for an interior solution – see Figure 8.27).[25]

Assuming the interior solution we can, in principle, solve equation (8.19) in order to derive the optimal purchases of bonds β^* which will be a function of the endowment of assets $\bar{y}$ and of the probability distribution of the rate of return r.

One clear-cut conclusion can easily be drawn from this approach. Consider what happens in the neighbourhood of point $\bar{P}$ in Figure 8.27; specifically consider the effect on the person's utility of a small increase in β away from 0:

$$\left.\frac{\partial \mathcal{E}\left(u\left(\bar{y} + \beta r\right)\right)}{\partial \beta}\right|_{\beta=0} = u_y\left(\bar{y}\right)\mathcal{E}r \tag{8.20}$$

So, given that $u_y\left(\bar{y}\right) > 0$, the impact of β on utility is positive if $\mathcal{E}r$ is positive. In other words:

[25] What would be the FOC corresponding to (8.19) for the two possible corner solutions (a) where the individual chooses to leave all resources in the riskless asset, (b) where the individual puts all resources into bonds?

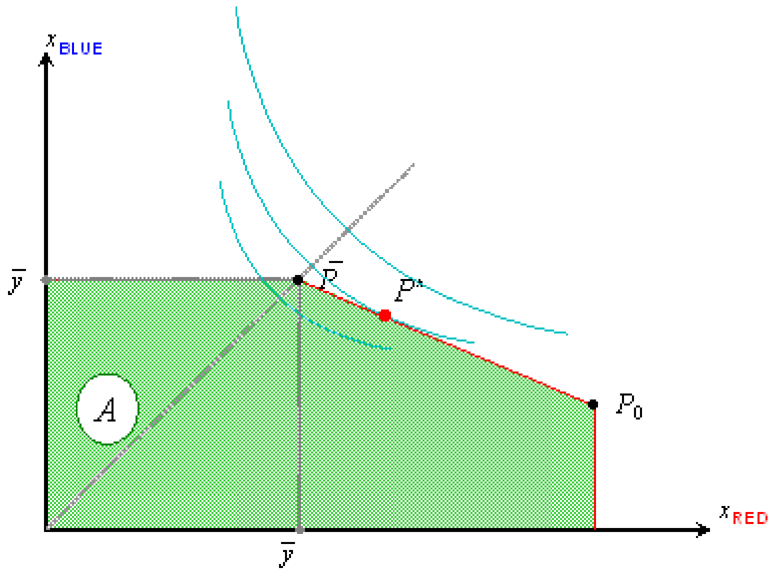

Figure 8.27: Consumer choice: safe and risky assets

Theorem 8.6 (Risk taking) *If the individual is nonsatiated and has a von-Neumann-Morgenstern utility function and if the expected return to risk-taking is positive then the individual will hold a positive amount of the risky asset.*

It would be interesting to know how this optimal demand for risky assets β^* changes in response to changes in the market environment by modelling the appropriate changes in the person's budget constraint. We can use the first-order condition (8.19) to look at a number of issues in comparative statics.

An increase in endowment.

Let us analyse the effect of a change in the person's assets by differentiating (8.19) with respect to $\bar{y}$:

$$\mathcal{E}\left(ru_{yy}\left(\bar{y}+\beta^* r\right)\left[1+r\frac{\partial\beta^*}{\partial\bar{y}}\right]\right)=0 \tag{8.21}$$

which implies

$$\frac{\partial\beta^*}{\partial\bar{y}}=\frac{-\mathcal{E}\left(ru_{yy}\left(\bar{y}+\beta^* r\right)\right)}{\mathcal{E}\left(r^2u_{yy}\left(\bar{y}+\beta^* r\right)\right)}. \tag{8.22}$$

The denominator of (8.22) is unambiguously negative, since u_{yy} is everywhere negative (the assumption of risk aversion) and r^2 is non-negative. However, the numerator could be positive or negative, since the risky asset could turn out to make a profit ($r > 0$) or a loss ($r < 0$). So it appears that the effect of wealth upon risk-taking is ambiguous.

In order to resolve this ambiguity, it is common to find the following additional assumption about preferences:

Axiom 8.7 (Decreasing absolute risk aversion) *$\alpha(x)$ decreases with x.*

This introduces a further restriction on the felicity function u.[26] But if we introduce decreasing absolute risk aversion along with the other standard assumptions, then we can show (see Appendix C):

Theorem 8.7 (Risk-taking and wealth) *If an individual has a von-Neumann-Morgenstern utility function with decreasing absolute risk aversion and holds a positive amount of the risky asset then the amount invested in the risky asset will increase as initial wealth increases.*

People whose risk aversion decreases with their endowment will buy more risky assets if their wealth increases. Notice that the result for any distribution of returns for which $\beta^* > 0$. This can be illustrated in Figure 8.28. The original equilibrium is at P^* and the lightly shaded area shows the increase in the attainable set when $\bar{y}$ increases to $\bar{y}+\Delta$. From (8.13) it is clear that if bond holdings were kept constant at β^* as $\bar{y}$ increased then point P^* would move out along a 45° line. However the indifference curves as drawn show decreasing absolute risk aversion (constant relative risk aversion) and the new equilibrium is at P^{**}, to the right of the 45° line through P^*: the holding of bonds must have increased.

A rightward shift of the distribution.

What happens to risk-taking if the returns on the risky asset change in an unambiguously favourable fashion? We can analyse this by supposing that the probability distribution of r is "translated" by adding the same determinate amount τ to every possible value of r; then we look at how β^* changes in response to small changes in τ, in the neighbourhood of $\tau = 0$.

Adding the amount τ to r as mentioned the FOC (8.19) becomes:

$$\mathcal{E}\left([r+\tau]\, u_y\left(\bar{y}+\beta^*[r+\tau]\right)\right) = 0 \tag{8.23}$$

Differentiate (8.23) with respect to τ:

$$\begin{aligned}&\mathcal{E}\left(u_y\left(\bar{y}+\beta^*[r+\tau]\right)\right)+\beta^*\mathcal{E}\left([r+\tau]\, u_{yy}\left(\bar{y}+\beta^*[r+\tau]\right)\right)\\&+\frac{\partial\beta^*}{\partial\tau}\mathcal{E}\left([r+\tau]^2 u_{yy}\left(\bar{y}+\beta^*[r+\tau]\right)\right)=0\end{aligned} \tag{8.24}$$

[26] This further restriction can be expressed as a condition on the third derivative of u: what is the condition?

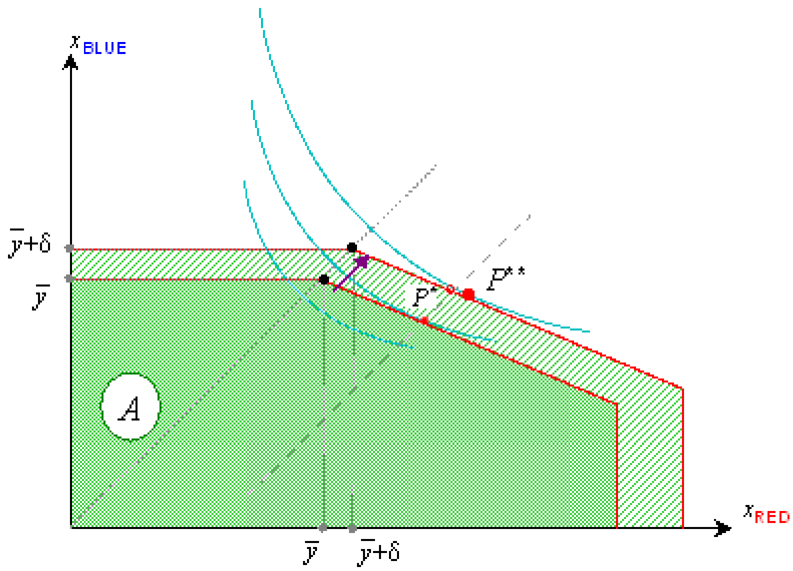

Figure 8.28: Effect of an increase in endowment

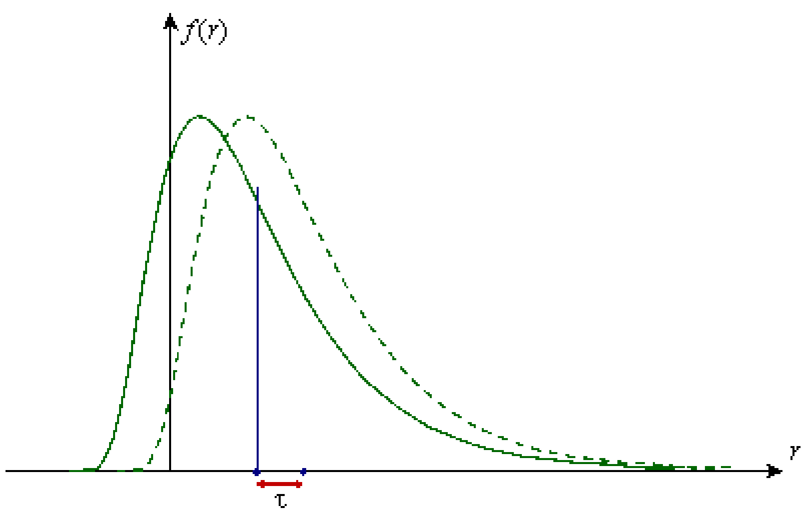

Figure 8.29: A rightward shift

Setting $\tau = 0$ we find

$$\mathcal{E}(u_y(y)) + \beta^* \mathcal{E}(r u_{yy}(y)) + \frac{\partial \beta^*}{\partial \tau} \mathcal{E}\left(r^2 u_{yy}(y)\right) = 0$$

where y is given by (8.17). So, in the neighbourhood of $\tau = 0$, we have

$$\frac{\partial \beta^*}{\partial \tau} = -\frac{\mathcal{E}(u_y(y))}{\mathcal{E}(r^2 u_{yy}(y))} - \beta^* \frac{\mathcal{E}(r u_{yy}(y))}{\mathcal{E}(r^2 u_{yy}(y))} \tag{8.25}$$

and, given (8.22), equation (8.25) becomes

$$\frac{\partial \beta^*}{\partial \tau} = -\frac{\mathcal{E}(u_y(y))}{\mathcal{E}(r^2 u_{yy}(y))} + \beta^* \frac{\partial \beta^*}{\partial \bar{y}} \tag{8.26}$$

From the way that (8.26) is written it is clear that if β^* increases with personal wealth $\bar{y}$, then it must also increase with this favourable shift in the distribution. Decreasing absolute risk aversion is a sufficient condition (although not a necessary condition) for this.

This is illustrated in Figure 8.30. The attainable set A expands in an unbalanced way: the point P_0 moves out along a 45° line, so that the boundary of A rotates through $\bar{P}$ as shown. Once again the dotted line through P^* is the locus that would be followed if the absolute amount of bonds bought β stayed constant: clearly the new equilibrium P^{**} must lie to the right of where this line intersects the new boundary of A (marked by "$\circ$").

An increased spread of the distribution.

We can handle this by supposing that the probability distribution of r is "scaled" by multiplying every possible value of r by a determinate constant t; then we look at how changes in t affect β^* in the neighbourhood of $t = 1$. The FOC becomes:

$$\mathcal{E}(t r u_y(\bar{y} + \beta^* t r)) = 0. \tag{8.27}$$

Differentiating this with respect to t we now find:[27]

$$\frac{t}{\beta^*} \frac{\partial \beta^*}{\partial t} = -1 \tag{8.28}$$

Equation (8.28) implies that the optimal purchase of bonds, β^*, is bound to decrease; the elasticity of bond purchases with respect to the scale factor t is -1. We do not need a special assumption about risk aversion in order to get this result.

[27] Fill in the missing lines from the differentiation and illustrate the outcome using a figure similar to Figure 8.30.

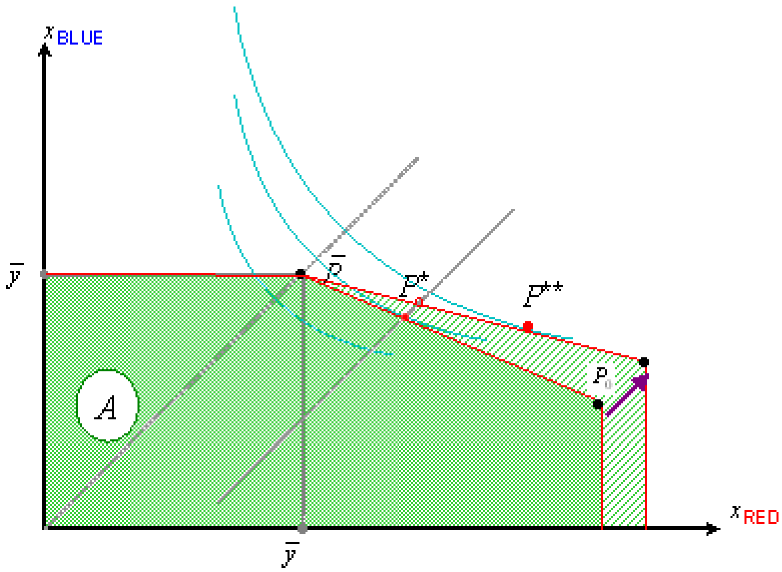

Figure 8.30: Effect of a rightward shift in the distribution

8.7.4 Insurance

From section 8.7.1 it appears that the economic problem of insurance can be treated in essentially the same way as the portfolio problem just discussed – i.e. as a trade-off between safe and risky assets that is determined by the person's own subjective probability of events, the nature of risk aversion and the returns to the risky asset. Some results can indeed just be copied across. In particular Theorem 8.6 shows that if the expected return to risk is positive then the individual will choose to hold a positive amount of the risky asset: let us see how this translates.

From Figure 8.24 and the accompanying discussion we can deduce the following. If the risk of loss is π_{BLUE} and the size of the loss is L then the expected payout equals the expected receipts for an insurance company if

$$\pi_{\text{BLUE}} L = \kappa \tag{8.29}$$

where κ, the premium, is given by (8.14). A quick check reveals that this is equivalent to

$$\frac{L-\kappa}{\kappa} = \frac{\pi_{\text{RED}}}{\pi_{\text{BLUE}}} \tag{8.30}$$

where the left-hand side is clearly the slope of the boundary of the attainable set A and the right-hand side is the slope of the indifference curve where it crosses the 45° line. So if the insurance premium is set such that the insurance company expects to break even (8.29) then the indifference curve is tangential

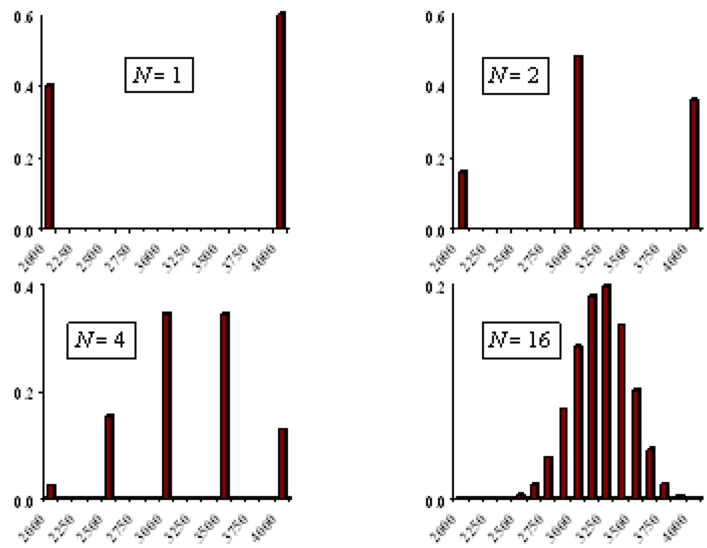

Figure 8.31: Effect of numbers

to the opportunity set at point $\bar{P}$: the person fully insures at the optimum. This means that if the terms of the insurance are unfair (replace the "=" by "=" in expressions 8.14 and 8.29) then the individual will take out less than full insurance or no insurance at all – i.e. equilibrium will be in either the interior of or at the right-hand end of the line joining $\bar{P}$ and P_0. This is the exact translation of the result concerning the positive expected return to risk.

Other results will work in the same way. For example, if an individual with decreasing risk aversion chooses to be partially insured then, if his wealth grows, the amount of his insurance coverage cannot increase (see exercise 8.12). However, this type of analysis assumes that an insurance market exists for this type of risk – but under what circumstances would such a market exist?

First, there is a necessary condition of large numbers in the market to permit the pooling of risks. Take a simple example of an economy consisting of clones. Each clone faces an identical independent risk on his wealth, and evaluates the risk with identical subjective probability: \$2,000 with probability 0.4 and \$4,000 with probability 0.6. Let us suppose the clones assemble themselves and agree to pool their wealth and share equally the combined realised payoff. Clearly the mathematical expectation is \$3,200. Now consider Figure 8.31. As the economy is replicated to 2, 4, 8, 16, ... persons, we can see that the distribution of payoffs to the individual soon becomes symmetric and concentrated about the expected value. In the limit, of course, the probability of any payoff different from the expected value becomes infinitesimal. If the insurance company is owned by a large number of "small" individuals – that is if the shares in the profits and losses from insurance are reasonably diffuse – then the risks are not only pooled but also *spread*. Under such circumstances there may be the basis for effective competition in both the demand side and the supply side of a market for insurance.

So it appears that with a very large number of agents each one ought to be able to "buy insurance" against the risk on his income at an actuarially fair price corresponding to the probabilities given above. But the example also reveals some obvious pitfalls:

- There is the "moral hazard" problem as described above. We must assume that no person can have direct influence on the probability of any particular payoff being realised.

- Each agent must be "small" in the limit.

- The *risks must be independent* – the results will not work if all the agents' risks are closely correlated.

- The payoffs must be *tradeable* amongst the individuals in the form of some transferable commodity. Obviously there are some risks which people confront where the payoffs cannot be thus transferred, and where losses cannot be compensated for in money.

For any of these reasons a market may simply not exist; more on this and related problems in chapter 11.

8.8 Summary

The basic approach to decision-making under uncertainty can be analysed as a straightforward extension of consumer theory, by considering a class of utility functions that are additively separable over the states of the world. Furthermore the analysis of market equilibrium and of individual portfolio behaviour in the face of risk follow on immediately from the core analysis of previous chapters once we have appropriately modelled preferences and opportunities.

It is evident that at the core of the approach is the concept of expected utility – see the two Theorems 8.1 and 8.4. But why do the job twice over? Our first approach to the subject showed that the special structure of utility function follows naturally from a coherent representation of preferences over a space of "contingent goods" without a specific construct of probability; the second approach shows what happens when one treats probability distributions – lotteries – as the focus of the choice problem. The first approach provides an essential link to the standard analysis of decision making treated in chapters 2 to 7; we shall find this second approach is essential in providing the basis for the analysis of games in chapter 10.

8.9 Reading notes

On the foundations of expected-utility analysis in terms of choices over lotteries refer to von Neumann and Morgenstern (1944) and Friedman and Savage (1948) and for early, penetrating critiques see Allais (1953) and Ellsberg (1961). The von Neumann-Morgenstern approach in some ways builds on the classic contribution by Bernouilli (1954) originally published in 1738. A useful survey is to be found in Machina (1987). The consumner-theoretic approach to uncertainty is developed in Deaton and Muellbauer (1980).

The concept of risk aversion developed in Arrow (1970) and Pratt (1964). On the discussion of conditions such as increasing risk aversion see Menezes et al. (1980).

8.10 Exercises

8.1 *Suppose you have to pay \$2 for a ticket to enter a competition. The prize is \$19 and the probability that you win is* $\frac{1}{3}$. *You have an expected utility function with* $u(x) = \log x$ *and your current wealth is \$10.*

1. *What is the certainty equivalent of this competition?*
2. *What is the risk premium?*
3. *Should you enter the competition?*

8.2 *You are sending a package worth 10 000€. You estimate that there is a 0.1 percent chance that the package will be lost or destroyed in transit. An insurance company offers you insurance against this eventuality for a premium of 15€. If you are risk-neutral, should you buy insurance?*

8.3 *Consider the following definition of risk aversion. Let* $P := \{(x_\omega, \pi_\omega) : \omega \in \Omega\}$ *be a random prospect, where* x_ω *is the payoff in state* ω *and* π_ω *is the (subjective) probability of state* ω *, and let* $\mathcal{E}x := \sum_{\omega\in\Omega} \pi_\omega x_\omega$*, the mean of the prospect, and let* $P_\lambda := \{(\lambda x_\omega + [1-\lambda]\mathcal{E}x, \pi_\omega) : \omega \in \Omega\}$ *be a "mixture" of the original prospect with the mean. Define an individual as risk averse if he always prefers* P_λ *to* P *for* $0 < \lambda < 1$.

1. *Illustrate this concept on a diagram similar to Figure 8.6 and contrast it with the concept of risk aversion mentioned on page 183.*
2. *Show that this definition of risk aversion need not imply convex-to-the-origin indifference curves. (Rothschild and Stiglitz 1970)*

8.4 *This is an example of the* Allais paradox *(Allais 1953). Suppose you are asked to choose between two lotteries. In one case the choice is between* P_1 *and*

P_2,and in the other case the choice offered is between $P_3$and P_4, as specified below:

$$
\begin{aligned}
&P_1: \quad \$1,000,000 \quad \textit{with probability 1}\\
&P_2: \left\{\begin{array}{cl} \$5,000,000 & \textit{with probability 0.1}\\ \$1,000,000 & \textit{with probability 0.89}\\ \$0 & \textit{with probability 0.01}\end{array}\right.\\
&P_3: \left\{\begin{array}{cl} \$5,000,000 & \textit{with probability 0.1}\\ \$0 & \textit{with probability 0.9}\end{array}\right.\\
&P_4: \left\{\begin{array}{cl} \$1,000,000 & \textit{with probability 0.11}\\ \$0 & \textit{with probability 0.89}\end{array}\right.
\end{aligned}
$$

It is often the case that people prefer P_1 to P_2 and then also prefer P_3 to P_4. Show that these preferences violate the independence axiom.

8.5 *This is an example to illustrate disappointment (Bell 1988, Machina 1989) Suppose the payoffs are as follows*

x'' *weekend for two in your favourite holiday location*

x' *book of photographs of the same location*

x° *fish-and-chip supper*

Your preferences under certainty are $x'' \succ x' \succ x^\circ$. Now consider the following two prospects

$$
\begin{aligned}
&P_1: \left\{\begin{array}{ll} x'' & \textit{with probability } 0.99\\ x' & \textit{with probability } 0\\ x^\circ & \textit{with probability } 0.01\end{array}\right.\\
&P_2: \left\{\begin{array}{ll} x'' & \textit{with probability } 0.99\\ x' & \textit{with probability } 0.01\\ x^\circ & \textit{with probability } 0\end{array}\right.
\end{aligned}
$$

Suppose a person expresses a preference for P_1 over P_2. Briefly explain why this might be the case in practice. Which axiom in section 8.3 is violated by such preferences?

8.6 *An example to illustrate regret. Let*

$$P := \{(x_\omega, \pi_\omega) : \omega \in \Omega\}$$

$$P' := \{(x'_\omega, \pi_\omega) : \omega \in \Omega\}$$

be two prospects available to an individual. Define the expected regret if the person chooses P rather than P' as

$$\sum_{\omega\in\Omega} \pi_\omega \max\{x'_\omega - x_\omega, 0\} \tag{8.31}$$

Now consider the choices amongst prospects presented in Exercise 8.4. Show that if a person is concerned to minimise expected regret as measured by (8.31), then it is reasonable that the person select P_1 when P_2 is also available and then also select P_3 when P_4 is available (Bell 1982, Loomes and Sugden 1982).

8.7 *An example of the Ellsberg paradox (Ellsberg 1961). There are two urns marked Left and Right each of which contains 100 balls. You know that in Urn L there exactly 49 white balls and the rest are black and that in Urn R there are black and white balls, but in unknown proportions. Consider the following two experiments:*

1. *One ball is to be drawn from each of L and R. The person must choose between L and R before the draw is made. If the ball drawn from the chosen urn is black there is a prize of $1000, otherwise nothing.*

2. *Again one ball is to be drawn from each of L and R; again the person must choose between L and R before the draw. Now if the ball drawn from the chosen urn is white there is a prize of $1000, otherwise nothing.*
   You observe a person choose Urn L in both experiments. Show that this violates Axiom 8.3.

8.8 *An individual faces a prospect with a monetary payoff represented by a random variable x that is distributed over the bounded interval of the real line $[\underline{a}, \overline{a}]$. He has a utility function $\mathcal{E}v(x)$ where*

$$u(x) = a_0 + a_1 x - \frac{1}{2} a_2 x^2$$

and a_0, a_1, a_2 are all positive numbers.

1. *Show that the individual's utility function can also be written as $\varphi(\mathcal{E}x, var(x))$. Sketch the indifference curves in a diagram with $\mathcal{E}x$ and $var(x)$ on the axes, and discuss the effect on the indifference map altering (i) the parameter a_1, (ii) the parameter a_2.*

2. *For the model to make sense, what value must $\overline{a}$ have? [Hint: examine the first derivative of u.]*

3. *Show that increases both absolute and relative risk aversion increase with x .*

8.9 *A person lives for 1 or 2 periods. If he lives for both periods has a utility function given by (5.13) where the parameter δ is the pure rate of time preference. The probability of survival to period 2 is γ, and the person's utility in period 2 if he does not survive is 0.*

1. *Show that if the person's preferences in the face of uncertainty are represented by the functional form in (8.1) then the person's utility can be written as*

$$u(x_1) + \delta' u(x_2) \,. \tag{8.32}$$

   What is the value of the parameter δ'?

2. *What is the appropriate form of the utility function if the person could live for an indefinite number of periods, the rate of time preference is the same for any adjacent pair of periods, and the probability of survival to the next period given survival to the current period remains constant?*

8.10 *A person has an objective function $\mathcal{E}u(y)$ where u is an increasing, strictly concave, twice-differentiable function, and y is the monetary value of his final wealth after tax. He has an initial stock of assets K which he may keep either in the form of bonds, where they earn a return at a stochastic rate r, or in the form of cash where they earn a return of zero. Assume that $\mathcal{E}r > 0$ and that* $\Pr\{r < 0\} > 0$.

1. *If he invests an amount β in bonds ($0 < \beta < K$) and is taxed at rate t on his* income, *write down the expression for his disposable final wealth y, assuming full loss offset of the tax.*
2. *Find the first-order condition which determines his optimal bond portfolio β^*.*
3. *Examine the way in which a small increase in t will affect β^*.*
4. *What would be the effect of basing the tax on the person's* wealth *rather than income?*

8.11 *An individual taxpayer has an income y that he should report to the tax authority. Tax is payable at a constant proportionate rate t. The taxpayer reports x where $0 \leq x \leq y$ and is aware that the tax authority audits some tax returns. Assume that the probability that the taxpayer's report is audited is π, that when an audit is carried out the true taxable income becomes public knowledge and that, if $x < y$, the taxpayer must pay both the underpaid tax and a surcharge of s times the underpaid tax.*

1. *If the taxpayer chooses $x < y$, show that disposable income c in the two possible states-of-the-world is given by*

$$\begin{aligned} c_{\text{NOAUDIT}} &= y - tx, \\ c_{\text{AUDIT}} &= [1 - t - st]\, y + stx. \end{aligned}$$

2. *Assume that the individual chooses x so as to maximise the utility function*

$$[1 - \pi]\, u\left(c_{\text{NOAUDIT}}\right) + \pi u\left(c_{\text{AUDIT}}\right).$$

   where u is increasing and strictly concave.

   (a) *Write down the FOC for an interior maximum.*

   (b) *Show that if $1 - \pi - \pi s > 0$ then the individual will definitely under-report income.*

3. *If the optimal income report* x^* *satisfies* $0 < x^* < y$*:*
   (a) *Show that if the surcharge is raised then under-reported income will decrease.*
   (b) *If true income increases will under-reported income increase or decrease?*

8.12 *A risk-averse person has wealth* y_0 *and faces a risk of loss* $L < y_0$ *with probability* π*. An insurance company offers cover of the loss at a premium* $\kappa > \pi L$*. It is possible to take out partial cover on a pro-rata basis, so that an amount* tL *of the loss can be covered at cost* $t\kappa$ *where* $0 < t < 1$*.*

1. *Explain why the person will not choose full insurance*
2. *Find the conditions that will determine* t^**, the optimal value of* t*.*
3. *Show how* t *will change as* y_0 *increases if all other parameters remain unchanged.*

8.13 *Consider a competitive, price-taking firm that confronts one of the following two situations:*

- *"uncertainty": price* p *is a random variable with expectation* $\overline{p}$*.*
- *"certainty": price is fixed at* $\overline{p}$*.*

It has a cost function $C(q)$ *where* q *is output and it seeks to maximise the expected utility of profit.*

1. *Suppose that the firm must choose the level of output before the particular realisation of* p *is announced. Set up the firm's optimisation problem and derive the first- and second-order conditions for a maximum. Show that, if the firm is risk averse, then increasing marginal cost is not a necessary condition for a maximum, and that it strictly prefers "certainty" to "uncertainty". Show that if the firm is risk neutral then the firm is indifferent as between "certainty" and "uncertainty".*
2. *Now suppose that the firm can select* q *after the realisation of* p *is announced, and that marginal cost is strictly increasing. Using the firm's competitive supply function write down profit as a function of* p *and show that this profit function is convex. Hence show that a risk-neutral firm would strictly prefer "uncertainty" to "certainty".*

8.14 *Every year Alf sells apples from his orchard. Although the market price of apples remains constant (and equal to 1), the output of Alf's orchard is variable yielding an amount* R_1, R_2 *in good and poor years respectively; the probability of good and poor years is known to be* $1-\pi$ *and* π *respectively. A buyer, Bill offers Alf a contract for his apple crop which stipulates a down payment (irrespective of whether the year is good or poor) and a bonus if the year turns out to be good.*

1. *Assuming Alf is risk averse, use a diagram similar to Figure 8.21 to sketch the set of such contracts which he would be prepared to accept. Assuming that Bill is also risk averse, sketch his indifference curves in the same diagram.*

2. *Assuming that Bill knows the shape of Alf's acceptance set, illustrate the optimum contract on the diagram. Write down the first-order conditions for this in terms of Alf's and Bill's utility functions.*

8.15 *In exercise 8.14, what would be the effect on the contract if (i) Bill were risk neutral; (ii) Alf risk neutral?*