Chapter 4

The Consumer

> *Consumer*: A person who is capable of choosing a president but incapable of choosing a bicycle without help from a government agency.
> – Herbert Stein, *Washington Bedtime Stories* (1979)

4.1 Introduction

It is now time to introduce the second of the principal economic actors in the economic system – the consumer. In a sense this is the heart of the microeconomics. Why else speak about "consumer sovereignty"? For what else, ultimately, is the economy's productive activity organised?

We will tackle the economic principles that apply to the analysis of the consumer in the following broad areas:

- Analysis of preferences.
- Consumer optimisation in perfect markets.
- Consumer's welfare.

This, of course, is just an introduction to the economics of individual consumers and households; in this chapter we concentrate on just the consumer in isolation. Issues such as the way consumers behave *en masse* in the market, the issues concerning the supply by households of factors such as labour and savings to the market and whether consumers "substitute" for the market by producing at home are deferred until chapter 5. The big topic of consumer behaviour under uncertainty forms a large part of chapter 8.

In developing the analysis we will see several points of analogy where we can compare the theory of consumer with the theory of the firm. This can make life much easier analytically and can give us several useful insights into economic problems in both fields of study.

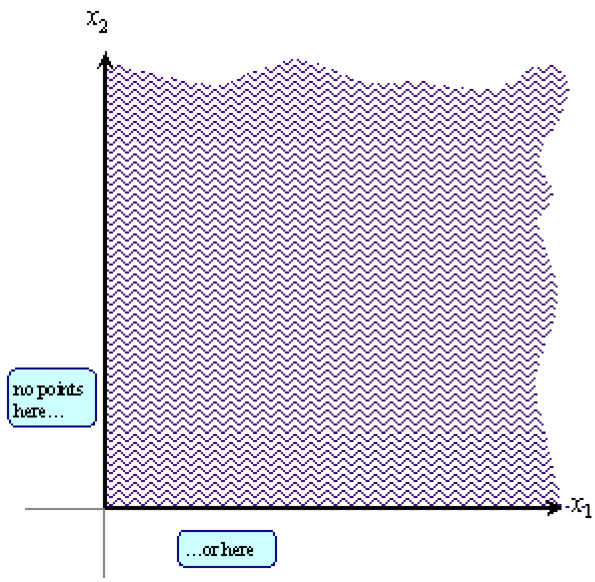

Figure 4.1: The consumption set: standard assumptions

4.2 The consumer's environment

As with the firm we begin by setting out the basic ingredients of the problem. First, a preliminary a word about who is doing the consuming. I shall sometimes refer to "the individual," sometimes to "the household" and sometimes – more vaguely – to "the consumer," as appropriate. The distinction does not matter as long as (a) if the consumer is a multiperson household, that household's membership is taken as given and (b) any multiperson household acts as though it were a single unit. However, in later work the distinction will indeed matter – see chapter 9.

Having set aside the issue of the consumer we need to characterise and discuss three ingredients of the basic optimisation problem:

- the commodity space;
- the market;
- motivation.

The commodity space.

We assume that there is a known list of n commodities where n is a finite, but perhaps huge, number. A consumption is just a list of commodities $\mathbf{x} :=$

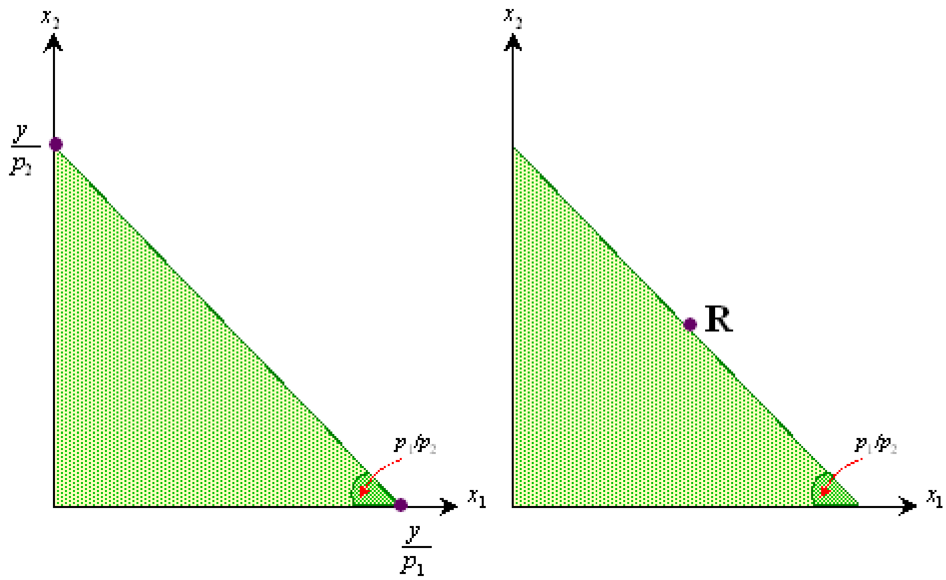

Figure 4.2: Two versions of the budget constraint

$(x_1, x_2, ..., x_n)$. We shall refer to the set of all *feasible* consumption bundles as X. In most cases we shall assume that X is identical to $\mathbb{R}^n_+$, the set of all non-negative n-vectors – see Figure 4.1; the implications of this are that a negative amount of any commodity makes no sense, that all commodities are divisible, and there is no physical upper bound to the amount of any one commodity that an individual could consume (that bound is going to be set by the budget, which we will come to in a moment).[1]

How do you draw the boundaries of goods classifications? This depends on the type of model you want to analyse. Very often you can get by with cases where you only have two or three commodities – and this is discussed further in chapter 5. Commodities could, in principle, be differentiated by space, time, or the state-of-the-world.

The market.

As in the case of the competitive firm, we assume that the consumer has access to a market in which the prices of all n goods are known: $\mathbf{p} := (p_1, p_2, ..., p_n)$. These prices will, in part, determine the individual's budget constraint.

However, to complete the description, there are two versions (at least) of this constraint which we may wish to consider using in our model of the consumer.

[1] How might one model indivisibilities in consumption? Describe the shape of the set X if good 1 is food, and good 2 is (indivisible) refrigerators.

These two versions are presented in Figure 4.2.

- In the left-hand version a fixed amount of money y is available to the consumer, who therefore finds himself constrained to purchase a bundle of goods $\mathbf{x}$ such that
$$p_1x_1 + p_2x_2 \leq y. \tag{4.1}$$
if all income y were spent on good 1 the person would be able to buy a quantity $x_1 = y/p_1$.

- In the right-hand version the person has an endowment of resources $\mathbf{R} := (R_1, R_2)$, and so his chosen bundle of goods must satisfy
$$p_1x_1 + p_2x_2 \leq p_1R + p_2R_2. \tag{4.2}$$

The two versions of the budget constraint look similar, but will induce different responses when prices change.[2]

Motivation.

This is not so easy to specify as in the case of the firm, because there is no overwhelmingly strong case for asserting that individuals or households maximise a particular type of objective function if, indeed anything at all. Households could conceivably behave in a frivolous fashion in the market (if *firms* behave in a frivolous fashion in the market then presumably they will go bust). But if they are maximising something, what is it that they are maximising? We will examine two approaches that have been attempted to this question, each of which has important economic applications. In the first we suppose that people make their choices in a way that reveals their own preferences. Secondly we consider a method of introspection.

4.3 Revealed preference

We shall tackle first the difficult problem of the consumer's motivation. To some extent it is possible to deduce a lot about a firm's objectives, technology and other constraints from external observation of how it acts. For example from data on prices and on firms' costs and revenue we could investigate whether firms' input and output decisions appear to be consistent with profit maximisation. Can the same sort of thing be done with regard to consumers?

The general approach presupposes that individuals' or households' actions in the market reflect the objectives that they were actually pursuing, which might be summarised as "what-you-see-is-what-they-wanted".

[2] (i) For each type of budget constraint sketch what will happen if the price of good 1 falls. (ii) Repeat this exercise for a rise in the price of good 2. (iii) Redraw the right-hand case for the situation in which the price at which one can buy a commodity is greater than the price at which one can sell the same commodity.

x_i	amount consumed of good i
$\mathbf{x}$	$(x_1, ..., x_n)$
X	the set of all $\mathbf{x}$
p_i	price of good i
$\mathbf{p}$	$(p_1, ..., p_n)$
y	income
$\succcurlyeq$	weak preference relation
$\triangleright$	revealed preference relation
U	utility function
υ	utility level

Table 4.1: The Consumer: Basic Notation

Definition 4.1 *A bundle* $\mathbf{x}$ *is* revealed preferred *to a bundle* $\mathbf{x}'$ *(written in symbols* $\mathbf{x} \triangleright \mathbf{x}'$*) if* $\mathbf{x}$ *is actually selected when* $\mathbf{x}'$ *was also available to the consumer.*

The idea is almost self-explanatory and is given operational content by the following axiom.

Axiom 4.1 (Axiom of rational choice) *The consumer always makes a choice, and selects the most preferred bundle that is available.*

This means that we can draw inferences about a person's preferences by observing the person's choices; it suggests that we might adopt the following simple – but very powerful – assumption.

Axiom 4.2 (Weak Axiom of Revealed Preference) *If* $\mathbf{x} \triangleright \mathbf{x}'$ *then* $\mathbf{x}' \not\triangleright \mathbf{x}$.

In the case where purchases are made in a free market this has a very simple interpretation. Suppose that at prices $\mathbf{p}$ the household could afford to buy either of two commodity bundles, $\mathbf{x}$ or $\mathbf{x}'$; assume that $\mathbf{x}$ is actually bought. Now imagine that prices change from $\mathbf{p}$ to $\mathbf{p}'$ (while income remains unchanged); if the household now selects $\mathbf{x}'$ then the weak axiom of revealed preference states that $\mathbf{x}$ cannot be affordable at the new prices $\mathbf{p}'$. Thus the axiom means that if

$$\sum_{i=1}^{n} p_i x_i \geq \sum_{i=1}^{n} p_i x_i' \tag{4.3}$$

then

$$\sum_{i=1}^{n} p_i' x_i > \sum_{i=1}^{n} p_i' x_i' \tag{4.4}$$

If you do not choose today something that you chose yesterday (when today's bundle was also available and affordable) it must be because now you cannot afford yesterday's bundle: see Figure 4.3.

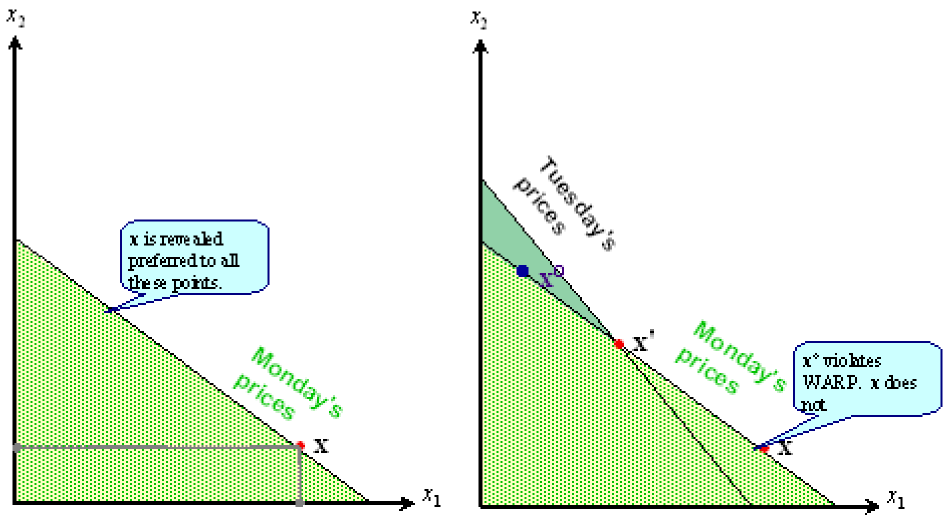

Figure 4.3: $\mathbf{x}$ is chosen Monday; $\mathbf{x}'$ is chosen Tuesday

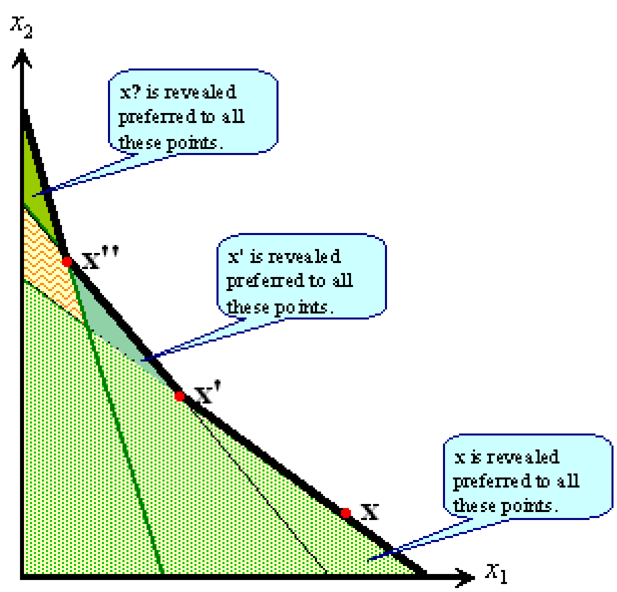

Figure 4.4: Extension of the revealed preference concept

You can get a long way in consumption theory with just this. Indeed with a little experimentation it seems as though we are almost sketching out the result of the kind of cost-minimisation experiment that we performed for the firm, in which we traced out a portion of a contour of the production function. Perhaps we might even suspect that we are on the threshold of discovering a counterpart to isoquants by the back door (we come to a discussion of "indifference curves" on page 77 below). For example, examine Figure 4.4: let $\mathbf{x} \rhd \mathbf{x}'$, and $\mathbf{x}' \rhd \mathbf{x}''$, and let $N(\mathbf{x})$ denote the set of points to which $\mathbf{x}$ is not revealed-preferred. Now consider the set of consumptions represented by the unshaded area: this is $N(\mathbf{x})$ $\cap\, N(\mathbf{x}') \cap N(\mathbf{x}'')$ and since $\mathbf{x}$ is revealed preferred to $\mathbf{x}'$ (which in turn is revealed preferred to $\mathbf{x}''$ we might think of this unshaded area as the set of points which are – directly or indirectly – revealed to be at least as good as $\mathbf{x}''$: the set is convex and the boundary does look a bit like the kind of contour we discussed in production theory. However, there are quite narrow limits to the extent that we can push the analysis. For example, it would be possible to have the following kind of behaviour: $\mathbf{x} \rhd \mathbf{x}'$, $\mathbf{x}' \rhd \mathbf{x}''$, $\mathbf{x}'' \rhd \mathbf{x}'''$ and yet also $\mathbf{x}''' \rhd \mathbf{x}$. To avoid this problem actually you need an additional axiom – the Strong Axiom of Revealed Preference which explicitly rules out cyclical preferences.

4.4 Preferences: axiomatic approach

In contrast to section 4.3 let us use the method of introspection. Instead of just drawing inferences from people's purchases, we approach the problem of specifying their preferences directly. We proceed by setting out a number of axioms which it might be reasonable to suppose that a consumer's preferences should satisfy. There is no special magic in any one axiom or set of axioms: they are just a way of trying to capture a structure that seems appropriate in the light of everyday experience. There is a variety of ways in which we might coherently axiomatise a model of consumer choice. Our fundamental concept is:

Definition 4.2 *The* weak preference relation $\succcurlyeq$ *is a binary relation on* X*. If* $\mathbf{x}, \mathbf{x}' \in X$ *then the statement "*$\mathbf{x} \succcurlyeq \mathbf{x}'$*" is to be read "*$\mathbf{x}$ *is at least as good as* $\mathbf{x}'$*".*

To make this concept useful we shall consider three basic axioms on preference.

Axiom 4.3 (Completeness) *For every* $\mathbf{x}, \mathbf{x}' \in X$*, either* $\mathbf{x} \succcurlyeq \mathbf{x}'$ *is true, or* $\mathbf{x}' \succcurlyeq \mathbf{x}$ *is true, or both statements are true.*

Axiom 4.4 (Transitivity) *For any* $\mathbf{x}, \mathbf{x}', \mathbf{x}'' \in X$*, if both* $\mathbf{x} \succcurlyeq \mathbf{x}'$ *and* $\mathbf{x}' \succcurlyeq \mathbf{x}''$*, then* $\mathbf{x} \succcurlyeq \mathbf{x}''$*.*

Axiom 4.5 (Continuity) [3] *For any* $\mathbf{x} \in X$*, the not-better-than-*$\mathbf{x}$ *set and the not-worse-than-*$\mathbf{x}$ *set are closed in* X*.*

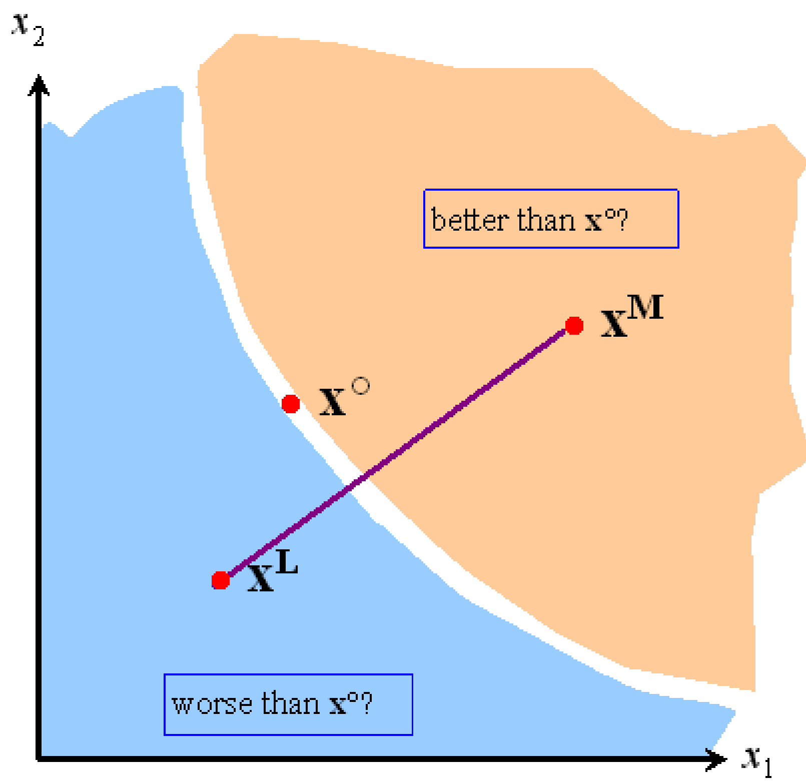

Figure 4.5: The continuity axiom

Completeness means that people do not shrug their shoulders helplessly when confronted with a choice; transitivity implies that (in a sense) they are consistent.[4] To see what the continuity axiom implies do the experiment illustrated in Figure 4.5. In a two-commodity diagram put some point $\mathbf{x}^\circ$ that represents positive amounts of both goods; plot any other point $\mathbf{x}^M$ that represents *m*ore of both goods, and some other point $\mathbf{x}^L$ that represents *l*ess of both goods (relative to $\mathbf{x}^\circ$); suppose the individual strictly prefers $\mathbf{x}^M$ to $\mathbf{x}^\circ$ and $\mathbf{x}^\circ$ to $\mathbf{x}^L$. Now consider points in the line $\left(\mathbf{x}^L, \mathbf{x}^M\right)$: clearly points "close" to $\mathbf{x}^M$ may reasonably

[3] What are the implications of dropping the continuity assumption?

[4] Each day I buy one piece of fruit for my lunch. On Monday apples and bananas are available, but no oranges: I buy an apple. On Tuesday bananas and oranges are available, but no apples: I buy a banana. On Wednesday apples and oranges are available (sorry we have no bananas): I buy an orange. Am I consistent?

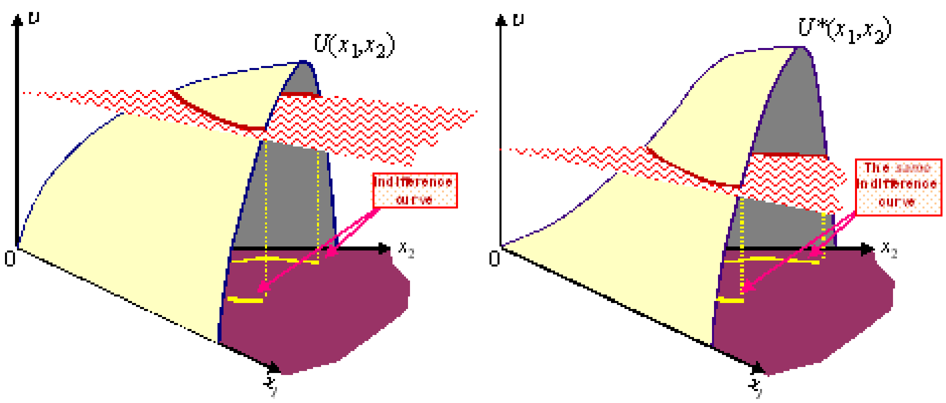

Figure 4.6: Two utility functions representing the same preferences

be considered to be better than $\mathbf{x}^\circ$ and points "close" to $\mathbf{x}^L$ worse than $\mathbf{x}^\circ$. But will there be a point in $\left(\mathbf{x}^L, \mathbf{x}^M\right)$ which is just indifferent to $\mathbf{x}^\circ$? If the continuity axiom holds then indeed this is always so. We can then draw an *indifference curve* through any point such as $\mathbf{x}^\circ$ (the set of points $\{\mathbf{x} : \mathbf{x} \in X; \mathbf{x} \sim \mathbf{x}^\circ\}$) and we have the following useful result (see Appendix C)::

Theorem 4.1 (Preference representation) *Given completeness, transitivity and continuity (axioms 4.3 – 4.5) there exists a continuous function U from X to the real line such that $U(\mathbf{x}) \geq U(\mathbf{x}')$ if and only if $\mathbf{x} \succcurlyeq \mathbf{x}'$, for all $\mathbf{x}, \mathbf{x}' \in X$.*[5]

The utility function makes life much easier in the analysis which follows. So, almost without exception, we shall assume that axioms 4.3 to 4.5 hold, and so we can then work with the notation U rather than the slightly clumsy-looking $\succcurlyeq$.

Notice that this function is just a way of ordering all the points in X in a very simple fashion; as such any strictly increasing transformation – any *cardinalisation* – of U would perform just as well. So, if you plotted the utility values for some particular function U as $\mathbf{x}$ ranged over X, you would get the same pattern of values if you plotted instead a function such such as U^2, U^θ or $\exp(U)$.[6] The utility function will tell you whether you are going uphill or

[5] Old George is a dipsomaniac. Friends speak in hushed tones about his lexicographic indifference map (this has nothing to do with his appointment in the University Library): he always strictly prefers the consumption bundle that has the greater amount of booze in it, regardless of the amount of other goods in the bundle; if two bundles contain the same amount of booze then he strictly prefers the bundle containing the greater amount of other goods. Sketch old George's preferences in a diagram. Which of the axioms used in Theorem 4.1 is violated by such an ordering?

[6] A consumer has a preference map represented by the utility function

$$U(\mathbf{x}) = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$$

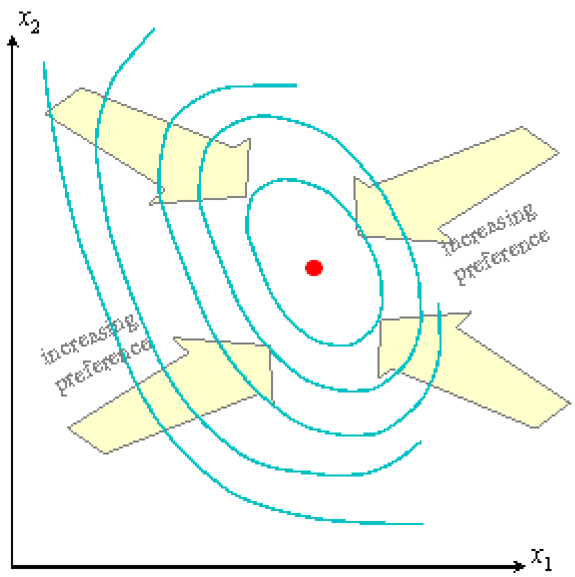

Figure 4.7: A bliss point

downhill in terms of preference, but it cannot tell you how fast you are going uphill, nor how high off the ground you are. Figure 4.6 shows two utility functions depicting exactly the same set of preferences: the (vertical) graph of utility against (x_1, x_2) may look different, but the functions project the same pattern of contours – the indifference curves – on to the commodity space. The preference contour map has the same shape even if the labels on the contours differ.

Axiom 4.6 (Greed) *If* $\mathbf{x} > \mathbf{x}'$ *(i.e.* $x_i \geq x_i'$ *for all* i *with strict inequality for at least one* i*) then* $U(\mathbf{x}) > U(\mathbf{x}')$*.*

This assumption implies that indifference curves can never be horizontal or vertical; furthermore they cannot bend round the wrong way as shown in

Can this also be represented by the following utility function?

$$\hat{U}(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i \log(x_i)$$

Can it also be represented by this utility function?

$$\tilde{U}(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i \log(x_i + 1)$$

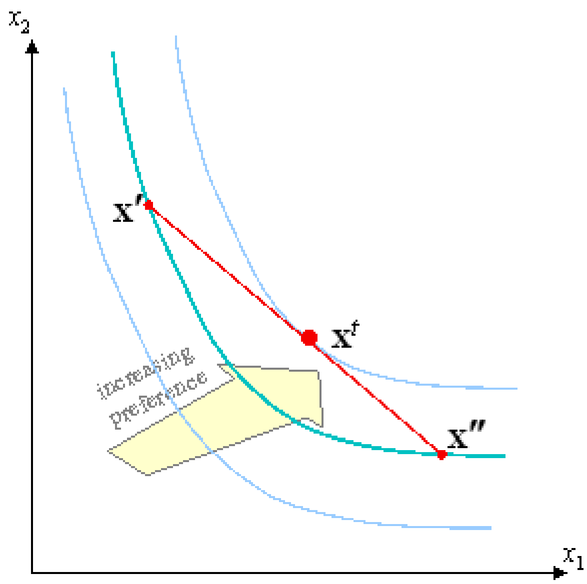

Figure 4.8: Strictly quasicconcave (concave-contoured) preferences

Figure 4.7.[7] In particular, as far as we are concerned, there is no such thing as economic bliss (the peak of the "mountain" in Figure 4.7). The final two assumptions concern the shape of the *indifference curves*, the contours of the function U.

Axiom 4.7 (Strict quasiconcavity) [8] *let* $\mathbf{x}', \mathbf{x}'' \in X$ *be such that* $U(\mathbf{x}') = U(\mathbf{x}'')$*; then for any number* t *such that* $0 < t < 1$ *it is true that* $U(t\mathbf{x}' + [1-t]\mathbf{x}'') > U(\mathbf{x}')$*.*

The immediate implication of this is that there can be no bumps or flat segments in the indifference curves – see Figure 4.8. Points $\mathbf{x}'$ and $\mathbf{x}''$ represent consumption vectors that yield the same level of utility. A point $\mathbf{x}^t$ on the line joining them, ($\mathbf{x}^t := t\mathbf{x}' + [1-t]\mathbf{x}''$) represents a "mix" of these two vectors. Clearly $\mathbf{x}^t$ must lie on a higher indifference curve. The deeper significance of this is that it presupposes that the consumer has a preference for mixtures of

[7] If the budget constraint actually passed "northwest" of the bliss point (so that the bliss point lay in the interior of the budget set) explain what the person would do.

[8] Notice that for a lot of results you can manage with the weaker requirements of *concave contours (quasiconcavity)*:

$$U(t\mathbf{x}' + [1-t]\mathbf{x}'') \geq U(\mathbf{x}'),$$

where $0 \leq t \leq 1$ and $U(\mathbf{x}') = U(\mathbf{x}'')$. For the results which follow, identify those that go through with this weaker assumption rather than strictly concave contours.

goods over extremes.[9]

Axiom 4.8 (Smoothness) *U is everywhere twice differentiable and its second partial derivatives commute – for any pair of goods i and j we have $U_{ij}(\mathbf{x}) = U_{ji}(\mathbf{x})$.*

This means that there can be no kinks in the indifference curves. Given that a person's preferences satisfy the smoothness requirement, an important tool then becomes available to us:

Definition 4.3 *The* marginal rate of substitution *of good i for good j (written MRS_{ij}) is $U_j(\mathbf{x})/U_i(\mathbf{x})$.*

Here, as before, $U_i(\mathbf{x})$ means $\partial U(\mathbf{x})/\partial \mathbf{x}_i$. A quick check confirms that MRS_{ij} is independent of the cardinal representation of U.[10] The marginal rate of substitution has an attractive intuition: MRS_{ij} is the person's marginal willingness to pay for commodity j, measured in terms of commodity i – a "subjective price ratio". This leaves us with a fairly general specification of the utility function. We will see later two cases where we might want to impose further restrictions on the class of admissible functions for use in representing a person's preferences

- aggregation over consumers – see chapter 5, page 112.
- analysis of uncertainty – see chapter 8, page 177.

4.5 Consumer optimisation: fixed income

There is more than one way of representing the optimisation problem that the consumer faces; perhaps the intuitively obvious way in which to do this involves finding a point on the highest utility contour within the appropriate constraint set – for example the kind of sets illustrated in Figure 4.2. In the case of a perfect market with exogenously fixed income y we have the standard problem of choosing a basket of goods $\mathbf{x}$ from the feasible set X so as to maximise utility subject to a budget constraint that is a simple generalisation of (4.1)

$$\sum_{i=1}^{n} p_i x_i \leq y \tag{4.5}$$

This is illustrated in the left-hand part of Figure 4.9: note the direction of increasing preference and the particular vector $\mathbf{x}^*$ which represents the optimum.

However we could also look at the optimisation problem in another way. Use the utility scale to fix a target utility level or standard of living (the units

[9] Every Friday I go out for a drink with the lads. I regard one pint of cider and one pint of beer as of equal utility; and one pint of either is strictly preferable to $\frac{1}{2}$ pint of both. Draw my indifference curves.

[10] Suppose that the utility function $\tilde{U}$ can be obtained from U by a differentiable monotonic transformation φ: i.e. $\tilde{U}(\mathbf{x}) = \varphi(U(\mathbf{x}))$ for all $\mathbf{x} \in X$. Prove this assertion about the MRS_{ij}.

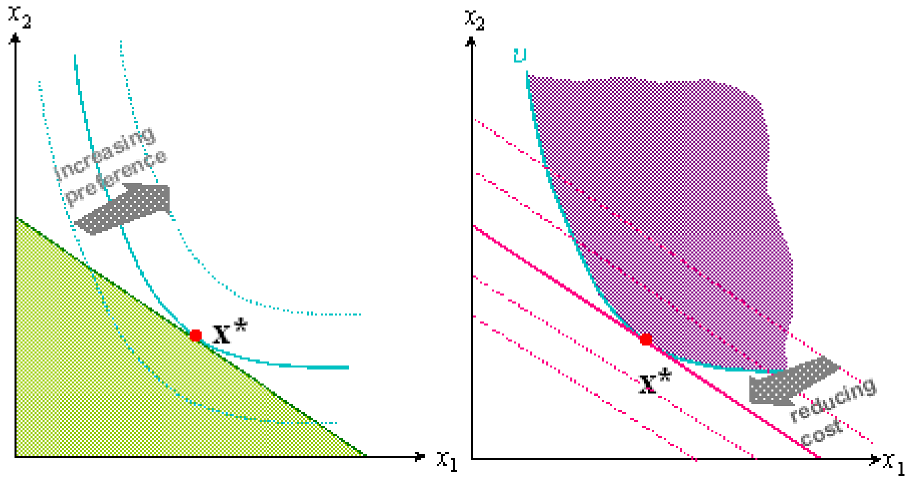

Figure 4.9: Two views of the consumer's optimisation problem

in which this is to be measured are arbitrary – dollars, tons, utils, quarks – since the cardinalisation of U is unimportant) and find the smallest budget that will enable the consumer to attain it. This yields an equivalent optimisation problem which may be regarded as the economic "dual" of the one we have just described. This involves minimising the budget $\sum_i p_i x_i$ subject to the non-negativity condition and a *utility constraint*

$$U(\mathbf{x}) \geq \upsilon \tag{4.6}$$

where υ is the exogenously specified utility level – see the right-hand part of Figure 4.9.

A glance at the two halves of Figure 4.9 reveals the utility-maximisation and the cost-minimisation problems are effectively equivalent, if the values of y and υ are appropriately specified. Obviously, in connecting these two problems, we would take υ as being the maximum utility obtainable under the first problem. So in the left-hand diagram we are saying "maximise the utility obtainable under a given budget:" we are maximising a quasiconcave function over a convex set – the budget set. In the right-hand diagram we are saying "minimise the cost of getting to any given utility level:" we are minimising a linear function over a convex set – the "better-than" set given by satisfying (4.6). We shall return to the "primal" problem of utility maximisation, but for the moment let us look at the solution to the problem depicted in the right-hand panel of Figure 4.9.

4.5.1 Cost-minimisation

Formally, the budget-minimising problem is one of minimising the Lagrangean

$$\mathcal{L}(\mathbf{x}, \lambda; \mathbf{p}, \upsilon) := \sum_{i=1}^{n} p_i x_i + \lambda[\upsilon - U(\mathbf{x})] \tag{4.7}$$

for some specified utility level υ, subject to the restrictions that $x_i \geq 0$ for every commodity i. Now inspection of (4.7) and comparison with the cost-minimisation problem for the firm reveals that the problem of cost-minimisation subject to a target utility level is formally equivalent to the firm's cost-minimisation problem subject to a target output level, where all input prices are given. So we may exploit all the results on the economic analysis of the firm that dealt with this problem. For this aspect of the problem can literally rub out and replace notation from our analysis the firm. For example we introduce the following counterpart:

Definition 4.4 *The consumer's* cost function *or* expenditure function *is a real valued function C of the price vector and the utility index such that:*

$$C(\mathbf{p}, \upsilon) := \min_{\{x_i \geq 0, U(\mathbf{x}) \geq \upsilon\}} \sum_{i=1}^{n} p_i x_i \tag{4.8}$$

The cost function (expenditure function) plays a key rôle in analysing the microeconomic behaviour of individuals and households, just as it did in the case of the firm. All of the properties of the function carry straight over from chapter 2, so we do not need to prove them again here. Just use Theorem 2.2 and replace the symbol $\mathbf{w}$ by $\mathbf{p}$, the symbol q by υ and z_i^* by x_i^*:

Theorem 4.2 (Properties of consumer's cost function) *the consumer's cost function $C(\mathbf{p}, \upsilon)$ is nondecreasing and continuous in $\mathbf{p}$,homogeneous of degree one in $\mathbf{p}$ and concave in $\mathbf{p}$. It is strictly increasing in υ and in at least one p_i. At every point where the differential is defined*

$$\frac{\partial C(\mathbf{p}, \upsilon)}{\partial p_i} = x_i^*. \tag{4.9}$$

the optimal demand for good i.

Of course, if the "constraint-set" in this cost-minimisation problem, defined by (4.6), is appropriately shaped then we can borrow another result from the theory of the firm, and then introduce the household's counterpart to the firm's conditional demand functions:

Now for the rationale behind the usage of the letter H to denote certain kinds of demand function, for the firm and for the consumer – it is in honour of Sir John Hicks:

Definition 4.5 *The* compensated demand functions *or* Hicksian demand functions *for goods $i = 1, 2, ..., n$ constitute a set of real-valued functions H^i of prices and a utility level such that*

$$x_i^* = H^i(\mathbf{p}, \upsilon) \tag{4.10}$$

where $(x_1^*, x_2^*, ..., x_n^*)$ are the cost-minimising purchases for $\mathbf{p}$ and υ.

The basic result on demand functions carries over from the firm, with just some rebadging.

Theorem 4.3 (Existence of compensated demand functions) *If the utility function is strictly concave-contoured then compensated demand functions are always well-defined and continuous for all positive prices.*

We can also follow through the comparative statics arguments on the sign of the partial derivatives of the demand functions; again we only need to change the notation.

Theorem 4.4 (Properties of compensated demand functions) *(a) H_j^i, the effect of an increase in the price of good j on the compensated demand for good i, is equal to H_i^j the effect of an increase in the price of good i on the compensated demand for good j. (b) H_i^i, the effect of an increase in the price of good i on the compensated demand for good i must be non-positive. If the smoothness axiom holds, then H_i^i is strictly negative.*

The analogy between the two applications of cost-minimisation could be extended further.[11] However, the particular point of special interest in the case of the household is the close relationship between this problem and the "primal" problem of utility maximisation subject to a budget constraint. Some very useful results follow from this relationship, as we shall see.

4.5.2 Utility-maximisation

So let us now tackle problem 1 which we can set up, once again, as a standard Lagrangean:

$$\mathcal{L}(\mathbf{x}, \mu; \mathbf{p}, y) := U(\mathbf{x}) + \mu \left[y - \sum_{i=1}^{n} p_i x_i \right] \tag{4.11}$$

where μ is the Lagrange multiplier. The FOC for the maximum yield

$$U_i(\mathbf{x}^*) \leq \mu^* p_i \tag{4.12}$$

and also the boundary of the budget constraint:

$$\sum_{i=1}^{n} p_i x_i^* = y \tag{4.13}$$

[11] What is the equivalent of the "short run" in the case of the consumer?

Notice that not all goods may be demanded at the optimum; this observation enables us to distinguish between the two cases of equation (4.12):

1. If "<" holds in (4.12) for commodity i then we must have $x_i^* = 0$.[12]

2. Otherwise (case "=") we could have $x_i^* = 0$ or $x_i^* > 0$.

Again we find a very neat immediate consequence of the first-order condition (4.12): if cost-minimisation requires a positive amount of good i then for any other good j:[13]

$$\frac{U_j(\mathbf{x}^*)}{U_i(\mathbf{x}^*)} \leq \frac{p_j}{p_i} \tag{4.14}$$

with equality in (4.14) if commodity j is also purchased in positive amounts. So in the case where the cost-minimising amounts of both commodities are positive we have:

$$\boxed{\text{MRS}} = \boxed{\text{price ratio}}$$

Let us examine more closely the properties of the solution to the utility-maximisation problem. We have already established (by analogy with the case of the cost-minimising firm) the circumstances under which the household's compensated (or conditional) demands can be represented by a well-defined function of prices and utility. Now let us introduce the following result,[14] proved in Appendix C, and a new concept:

Theorem 4.5 (Existence of ordinary demand functions) *If the utility function is strictly concave-contoured then the* ordinary demand functions *for good i constitute a set of real-valued functions D^i of prices and income*

$$x_i^* = D^i(\mathbf{p}, y), \tag{4.15}$$

are well defined and continuous for all positive prices, where $(x_1^*, x_2^*, ..., x_n^*)$ are the utility-maximising commodity demands for $\mathbf{p}$ and y.

However, as the implicit definition of the set of demand functions suggests, we cannot just write out some likely-looking equation involving prices and income on the right-hand side and commodity quantities on the left-hand side and expect it to be a valid demand function. To see why, note two things:

---

[12] Draw a figure for the case where "<" holds in (4.14) .

[13] Interpret this condition using the idea of the MRS as "marginal willingess to pay" mentioned on page 80 (i) where one has "<" in (4.14) and (ii) where one has "=" in (4.14).

[14] Suppose that instead of the regular budget constraint (4.13) the consumer is faced with a quantity discount ongood 1 ("buy 5 items and get the 6th one free"). Draw the budget set and draw in an indifference curve to show that optimal commodity demand may be non-unique. What can be said about commodity deamdn as a function of price in this case?

1. Because of the budget constraint, binding at the optimum (equation 4.13), it must be true that the set of n functions (4.15) satisfy
$$\sum_{i=1}^{n} p_i D^i(\mathbf{p}, y) = y. \tag{4.16}$$

2. Again focus on the binding budget constraint (4.13). If all prices $\mathbf{p}$ and income y were simultaneously rescaled by some positive factor t (so the new prices and income are $t\mathbf{p}$ and ty) it is clear that the FOC remain unchanged and so the optimal values $(x_1^*, x_2^*, ..., x_n^*)$ remain unchanged. In other words
$$D^i(t\mathbf{p}, ty) = D^i(\mathbf{p}, y). \tag{4.17}$$

This enables us to establish:[15]

Theorem 4.6 (Properties of ordinary demand functions) *(a) the set of ordinary demand functions is subject to a linear restriction in that the sum of the demand for each good multiplied by its price must equal total income; (b) the ordinary demand functions are homogeneous of degree zero in all prices and income.*

Now let us look at the way in which the optimal commodity demands $\mathbf{x}^*$ respond to changes in the consumer's market environment. To do this use the fact that the utility-maximisation and cost-minimisation problems that we have described are two ways of approaching the same optimisation problem: since problems 1 and 2 are essentially the same, the solution quantities are the same. So:
$$H^i(\mathbf{p}, \upsilon) = D^i(\mathbf{p},y) \tag{4.18}$$
The two sides of this equation are just two ways of getting to the same answer (the optimised x_i^*) from different bits of information. Substituting the cost function into (4.18) we get:
$$H^i(\mathbf{p}, \upsilon) = D^i\left(\mathbf{p}, C(\mathbf{p}, \upsilon)\right). \tag{4.19}$$

Take equation (4.19) a stage further. If we differentiate it with respect to any price p_j we find:
$$H_j^i(\mathbf{p}, \upsilon) = D_j^i(\mathbf{p},y) + D_y^i(\mathbf{p}, y)C_j(\mathbf{p}, \upsilon) \tag{4.20}$$
Then use (4.20) to give the *Slutsky equation*:
$$D_j^i(\mathbf{p},y) = H_j^i(\mathbf{p}, \upsilon) - x_j^* D_y^i(\mathbf{p}, y) \tag{4.21}$$
The formula (4.21) may be written equivalently as
$$\frac{\partial x_i^*}{\partial p_j} = \left.\frac{dx_i^*}{dp_j}\right|_{\upsilon=\text{const}} - x_j^* \frac{\partial x_i^*}{\partial y} \tag{4.22}$$

[15] Prove this using the properties of the cost function established earlier.

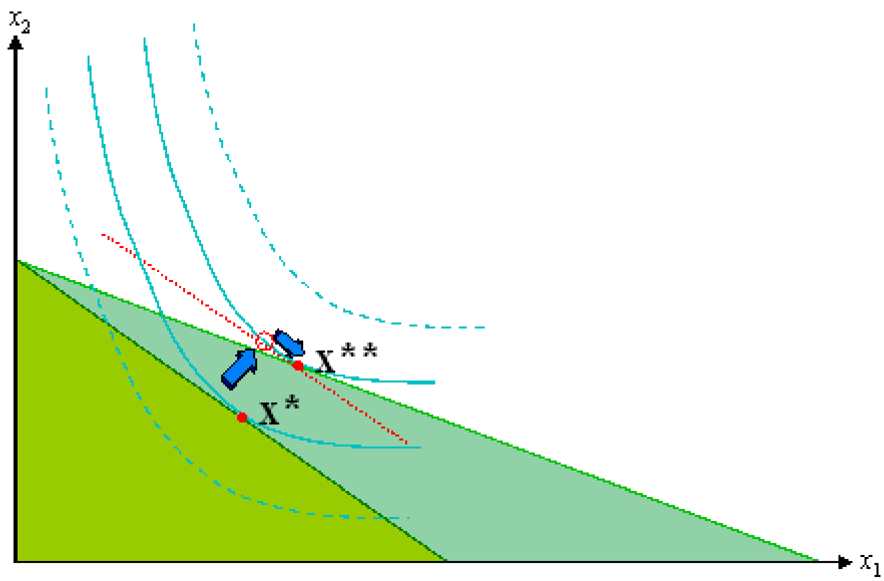

Figure 4.10: The effects of a price fall

and we may think of this decomposition formula (4.21 or 4.22) of the effect of a price change as follows:

$$\boxed{\begin{array}{c}\text{total}\\ \\ \text{effect}\end{array}} = \boxed{\begin{array}{c}\text{substitution}\\ \\ \text{effect}\end{array}} + \boxed{\begin{array}{c}\text{income}\\ \\ \text{effect}\end{array}}$$

This can be illustrated by Figure 4.10 where $\mathbf{x}^*$ denotes the original equilibrium: the equilibrium after good 1 has become cheaper is denoted by $\mathbf{x}^{**}$ on the higher indifference curve. Note the point on this indifference curve marked "$\circ$": this is constructed by increasing the budget at unchanged relative prices until the person can just reach the new indifference curve. Then the *income effect* – the change in consumption of each good that would occur if the person's real spending power alone increased – is given by the notional shift from $\mathbf{x}^*$ to "$\circ$". This effect could in principle be positive or negative: we will use the term *inferior good* to apply to any i for which the income effect is negative, and *normal good* for other case.[16] The notional shift from $\mathbf{x}^*$ to the new equilibrium $\mathbf{x}^{**}$ represents the effect on commodity demands that would arise if the relative price of good 1 were to fall while the budget was adjusted to keep the person on the same indifference curve. This is the *substitution effect*. (Since we are actually talking about infinitesimal changes in prices we could have equally well done this diagrammatic representation the other way round – i.e. first consider

[16] How might commodity grouping influence the income effect?

the substitution effect along the original indifference curve and then considered the notional income change involved in moving from one indifference curve to the other).

The substitution effect could be of either sign if $n > 2$ and j and i in (4.21) represent different goods (jelly and ice-cream let us say). We say that commodities i and j are *net substitutes* if $H^i_j > 0$ and *net complements* if the reverse inequality is true. Now, we know that $H^i_j = H^j_i$. So if jelly is a net substitute for ice-cream then ice-cream is a net substitute for jelly.[17]

Now let us take the "own-price" case; for example, let us look at the effect of the price of ice-cream on the demand for ice-cream. We get this by putting $j = i$ in (4.21). This gives us:

$$D^i_i(\mathbf{p},y) = H^i_i(\mathbf{p},\upsilon) - x^*_i D^i_y(\mathbf{p},y) \tag{4.23}$$

Once again we know from the the analysis of the firm that $H^i_i < 0$ for any smooth-contoured function (see page 33 in chapter 2). So the compensated demand curve (which just picks up the substitution effect) must be everywhere downward sloping. But what of the income effect? As we have seen, this could be of either sign (unlike the "output effect" in the own-price decomposition for a firm's input demand). So it is, strictly speaking, possible for the ordinary demand curve to slope upwards for some price and income combinations – the rare case of the *Giffen good*.[18] But if the income effect is positive or zero (a normal good) we may state the following fundamental result:

Theorem 4.7 (Own-price effect) *If a consumer's demand for a good never decreases when his income (alone) increases, then his demand for that good must definitely decrease when its price (alone) increases.*[19]

Notice throughout this discussion the difficulties caused by the presence of income effects. If all we ever had to consider were pure substitution effects – sliding around indifference curves – life would have been so much easier. However, as we shall see in other topics later in this book, income effects are nearly always a nuisance.

4.6 Welfare

Now look again at the solution to the consumer's optimisation problem this time in terms of the market environment in which the consumer finds himself. We will do this for the case where y is fixed although we could easily extend it to the endogenous – income case. To do this work out optimised utility in terms of $\mathbf{p}, y$:

[17](a) Why can we not say the same about gross substitutes and complements? (b) Explain why – in a two-good model – the goods must be net substitutes.

[18] Draw the income and substitution effects for a Giffen good.

[19] Prove this using the own-price version of the Slutsky equation (4.23).

Definition 4.6 *The* indirect utility function *is a real-valued function V of prices and income such that:*

$$V(\mathbf{p}, y) := \max_{\left\{\substack{x_i \geq 0, \\ \sum_{i=1}^{n} p_i x_i \leq y}\right\}} U(\mathbf{x}) \tag{4.24}$$

I should stress that this is not really new. As Figure 4.9 emphasises there are two fundamental, equivalent ways of viewing the consumer's optimisation problem and, just as (4.8) represents the solution to the problem as illustrated in the right-hand panel of the figure, so (4.24) represents the solution from the point of view of the left-hand panel. Because these are two aspects of the same problem we may write

$$y = C(\mathbf{p}, \upsilon) \tag{4.25}$$

and

$$\upsilon = V(\mathbf{p}, y) \tag{4.26}$$

where y is both the minimised cost in (4.8) and the constraint income in (4.24), 1while υ is both the constraint utility in (4.8) and the maximal utility in (4.24).

In view of this close relationship the function V must have properties that are similar to C. Specifically we find:

- V's derivatives with respect to prices satisfy $V_i(\mathbf{p}, y) \leq 0$ [20]
- $V_y(\mathbf{p}, y) = \mu^*$, the optimal value of the Lagrange multiplier which appears in (4.12).[21]
- A further derivative property can be found by substituting the cost function from (4.25) into (4.26) we have

  $$V(\mathbf{p}, C(\mathbf{p}, \upsilon)) = \upsilon\ ; \tag{4.27}$$

  then, differentiating (4.27) with respect to p_i and rearranging, we get:

  $$x_i^* = -\frac{V_i(\mathbf{p}, y)}{V_y(\mathbf{p}, y)}, \tag{4.28}$$

  a result known as *Roy's Identity*.[22]
- V is homogeneous of degree zero in all prices and income[23] and quasiconvex in prices (see Appendix C).

We can use (4.24) in a straightforward fashion to measure the welfare change induced by, say, an exogenous change in prices. To fix ideas let us suppose that

[20] Explain why V_i may be zero for some, but not all goods i.
[21] Use your answer to Chapter 2's footnote 16 (pages 25 and 525) to explain why this is so.
[22] Use (4.27) to derive (4.28).
[23] Show this using the properties of the cost function.

the price of commodity 1 falls while other prices and income y remain unchanged – the story that we saw briefly in Figure 4.10. Denote the price vector before the fall as $\mathbf{p}$, and that after the fall as $\mathbf{p}'$. Define the utility level υ as in (4.26) and υ' thus:

$$\upsilon' := V(\mathbf{p}', y) \tag{4.29}$$

This price fall is good news if the consumer was actually buying the commodity whose price has fallen. So we know that υ' is greater than υ: but how much greater?

One approach to this question is to take prices at their new values $\mathbf{p}'$, and then to compute that change in income which would bring the consumer back from υ' to υ. This is what we mean by the *compensating variation* of the price change $\mathbf{p} \to \mathbf{p}'$. More formally it is an amount of income CV such that

$$\upsilon = V(\mathbf{p}', y - \mathrm{CV}) \tag{4.30}$$

– compare this with (4.26) and (4.29). Now equation (4.27) suggests that we could write this same concept using the cost function C instead of the indirect utility function V. Doing so, we get:[24]

$$\mathrm{CV}(\mathbf{p} \to \mathbf{p}') := C(\mathbf{p}, \upsilon) - C(\mathbf{p}', \upsilon) \tag{4.31}$$

This suggests yet another way in which we could represent the CV. Consider Figure 4.11 which depicts the compensated demand curve for good 1 at the original utility level υ: the amount demanded at prices $\mathbf{p}$ and utility level υ is x_1^*. Now remember that Shephard's Lemma tells us that the derivative of the cost function C with respect to the price p_1 is $x_1^* = H^1(\mathbf{p}, \upsilon)$: this means that we can write the CV of a price fall of commodity 1 to the new value p_1' as the following integral:

$$\mathrm{CV}(\mathbf{p} \to \mathbf{p}') := \int_{p_1'}^{p_1} H^1(\rho, p_2, ..., p_n, \upsilon)\, d\rho \tag{4.32}$$

So the CV of the price fall that we have been discussing is just the shaded area trapped between the compensated demand curve and the axis.

It is worth repeating that equations (4.30)-(4.32) all contain the same concept, just dressed up in different guises. However, there are alternative ways in which we can attempt to calibrate the effect of a price fall in monetary terms. For example, take the prices at their original values $\mathbf{p}$, and then compute that change in income which would have brought the consumer from υ to υ'; this is known as the *equivalent variation* of the price change $\mathbf{p} \to \mathbf{p}'$. Formally we define this as an amount of income EV such that

$$\upsilon' = V(\mathbf{p}, y + \mathrm{EV}) \tag{4.33}$$

or, in terms of the cost function:

$$\mathrm{EV}(\mathbf{p} \to \mathbf{p}') := C(\mathbf{p}, \upsilon') - C(\mathbf{p}', \upsilon') \tag{4.34}$$

[24] Use (4.27) to fill in the one line that enables you to get (4.31) from equation (4.30).

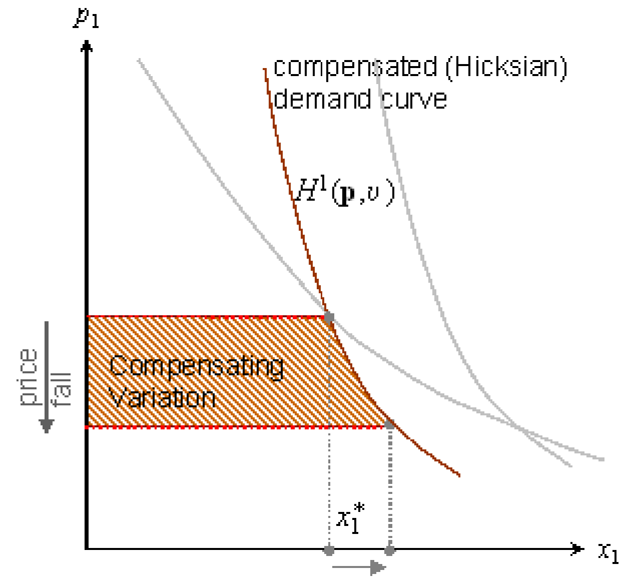

Figure 4.11: Compensated demand and the value of a price fall

– see Figure 4.12.

We can see that CV and EV will be positive if and only if the change $\mathbf{p} \to \mathbf{p}'$ increases welfare – the two numbers always have the same sign as the welfare change.[25] We also see that, by definition:

$$\text{CV}(\mathbf{p} \to \mathbf{p}') = -\text{EV}(\mathbf{p}' \to \mathbf{p}) \tag{4.35}$$

The CV and the EV represent two different ways of assessing the value of the fall: the former takes as a reference point the original utility level; the latter takes as a reference the terminal utility level. Clearly either has a claim to our attention, as may other utility levels, for that matter.

At this point we ought to mention another method of trying to evaluate a price change that is often found convenient for empirical work. This is the concept of *consumer's surplus* (CS):

$$\text{CS}(\mathbf{p} \to \mathbf{p}') := \int_{p_1'}^{p_1} D^1\left(\rho, p_2, ..., p_n, y\right) d\rho \tag{4.36}$$

which is just the area under the ordinary demand curve – compare (4.36) with (4.32).

The relationship amongst these three concepts is illustrated in Figure 4.13. Here we have modified Figure 4.11 by putting in the compensated demand curve

[25] Use (4.26)-(4.34) to explain in words why this is so.

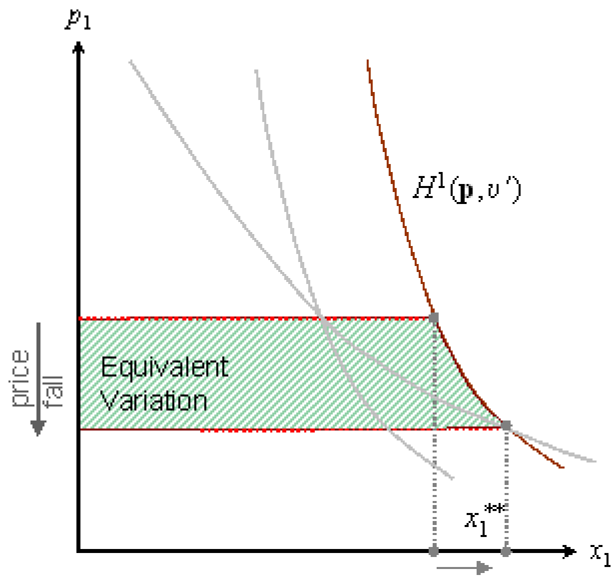

Figure 4.12: Compensated demand and the value of a price fall (2)

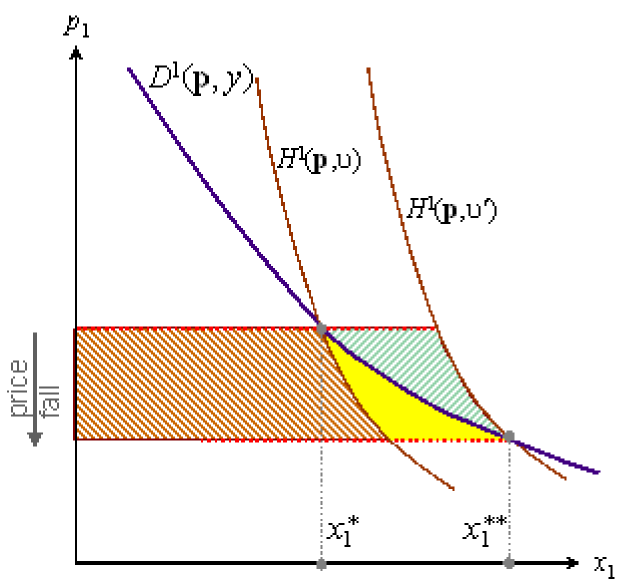

Figure 4.13: Three ways of measuring the benefits of a price fall

for the situation after the price fall (when the amount consumed is x_1^{**}) and the ordinary demand curve: this has been drawn for the case of a normal good (where the ordinary demand curve D^1 is not steeper than the compensated demand curve H^1).[26] From Figure 4.13 we can see that the CV for the price fall is the smallest shaded area; by the same argument, using equation (4.34), the EV is the largest shaded area (made up of the three shaded components); and using (4.36) the CS is the intermediate area consisting of the area CV plus the triangular shape next to it.

So we can see that for normal goods the following must hold:

$$\boxed{\text{CV} \quad \leq \quad \text{CS} \quad \leq \quad \text{EV}}$$

and for the case of inferior goods we just replace "$\leq$" by "$>$". Of course all three concepts coincide if the income effect for the good in question is zero.

4.6.1 An application: price indices

We can use the analysis that we have just developed as a basis for specifying a number of practical tools. Suppose we wanted a general index of changes in the cost of living. This could be done by measuring the proportionate change in the cost that a "representative consumer" would face in achieving a particular reference level of utility as a result of the change in price from $\mathbf{p}$ to $\mathbf{p}'$. The analysis suggests we could do this using either the "base year" utility level υ (utility before the price change – the CV concept) or the "current-year" utility level υ' (utility after the price change – the EV concept) which would give us two cost-of-living indices:

$$I_{\text{CV}} = \frac{C(\mathbf{p}',\upsilon)}{C(\mathbf{p},\upsilon),} \tag{4.37}$$

$$I_{\text{EV}} = \frac{C(\mathbf{p}',\upsilon')}{C(\mathbf{p},\upsilon').} \tag{4.38}$$

These are exact price indices in that no empirical approximations have to be used. However in general each term in (4.37) and (4.38) requires a complete evaluation of the cost function, which can be cumbersome, and unless preferences happen to be such that the cost function can be rewritten neatly like this:

$$C(\mathbf{p},\upsilon) = a(\mathbf{p})b(\upsilon) \tag{4.39}$$

then the indices in (4.37) and (4.38) will depend on the particular reference level utility, which is very inconvenient (more on this in Exercise 4.10).

What is often done in practice is to adopt an expedient by using either of two corresponding approximation indices – the *Laspeyres* and the *Paasche* indices

[26] Use the Slutsky decomposition to explain why this property of the slopes of the two curves must be true.

– which are given by:

$$I_{\mathrm{L}} = \frac{\sum_{i=1}^{n} p_i' x_i}{\sum_{i=1}^{n} p_i x_i} \tag{4.40}$$

$$I_{\mathrm{P}} = \frac{\sum_{i=1}^{n} p_i' x_i'}{\sum_{i=1}^{n} p_i x_i'} \tag{4.41}$$

For example the Retail Price Index (RPI) in the UK is a Laspeyres index (Central Statistical Office 1991). These indices are easier to compute, since you just work out two "weighted averages", in the case of I_{L} using the base-year quantities as weights, and in the case of I_{P} using the final-year quantities. But unfortunately they are biased since examination of (4.37)-(4.41) reveals that $I_{\mathrm{L}} \geq I_{\mathrm{CV}}$ and $I_{\mathrm{P}} \leq I_{\mathrm{EV}}$.[27] So the RPI will overestimate the rise in the cost of living if the appropriate basis for evaluating welfare changes is the CV concept.

4.7 Summary

The optimisation problem has many features that are similar to the optimisation problem of the firm, and many of the properties of demand functions follow immediately from results that we obtained for the firm. A central difficulty with this subfield of microeconomics is that an important part of the problem – the consumer's motivation – lies outside the realm of direct observation and must in effect be "invented" by the model-builder. The consumer's objectives can be modelled either on the basis of indirect observation – market behaviour – or on an *a priori* basis.

If we introduce a set of assumptions about the structure of preferences that enable representation by a well-behaved utility function, considerable progress can be made. We can then formulate the economic problem of the consumer in a way that is very similar to that of the firm that we analysed in chapter 2; the cost function finds a natural reinterpretation for the consumer and makes it easy to derive some basic comparative-static results. Extending the logic of the cost-function approach also provides a coherent normative basis for assessing the impact of price and income changes upon the welfare of consumers. This core model of the consumer also provides the basis for dealing with some of the more difficult questions concerning the relationship between the consumer and the market as we will see in chapter 5.

4.8 Reading notes

On the fundamentals of consumer theory see Deaton and Muellbauer (1980), chapters 2 and 7.

[27] Use the definition of the cost function and to prove these assertions. Explain the conditions under which there will be exact equality rather than an inequality.

The pioneering work on revealed-preference analysis is due to Samuelson (1938, 1948) and Houthakker (1950); for a thorough overview see Suzumura (1983), chapter 2. The representation theorem 4.1 is due to Debreu (1954); for a comprehensive treatment of axiomatic models of preference see Fishburn (1970). On indifference curve analysis the classic reference is Hicks (1946). There are several neat treatments of the Slutsky equation – see for example Cook (1972). The indirect utility function was developed in Roy (1947), the concept of consumer's surplus is attributable to Dupuit (1844) and the relationship of this concept to compensating and equivalent variation is in Hicks (1956). For a discussion of the use of consumer's surplus as an appropriate welfare concept see Willig (1976).

4.9 Exercises

4.1 *You observe a consumer in two situations: with an income of \$100 he buys 5 units of good 1 at a price of \$10 per unit and 10 units of good 2 at a price of \$5 per unit. With an income of \$175 he buys 3 units of good 1 at a price of \$15 per unit and 13 units of good 2 at a price of \$10 per unit. Do the actions of this consumer conform to the basic axioms of consumer behaviour?*

4.2 *Draw the indifference curves for the following four types of preferences:*

$$\begin{array}{lcl} \text{Type A} & : & \alpha \log x_1 + [1-\alpha] \log x_2 \\ \text{Type B} & : & \beta x_1 + x_2 \\ \text{Type C} & : & \gamma [x_1]^2 + [x_2]^2 \\ \text{Type D} & : & \min \{\delta x_1, x_2\}. \end{array}$$

where x_1, x_2 denote respectively consumption of goods 1 and 2 and $\alpha, \beta, \gamma, \delta$ are strictly positive parameters with $\alpha < 1$. What is the consumer's cost function in each case?

4.3 *Suppose a person has the* Cobb-Douglas *utility function*

$$\sum_{i=1}^{n} a_i \log(x_i)$$

where x_i is the quantity consumed of good i, and $a_1, ..., a_n$ are non-negative parameters such that $\sum_{j=1}^{n} a_j = 1$. If he has a given income y, and faces prices $p_1, ..., p_n$, find the ordinary demand functions. What is special about the expenditure on each commodity under this set of preferences?

4.4 *The elasticity of demand for domestic heating oil is -0.5, and for gasoline is -1.5. The price of both sorts of fuel is 60¢ per litre: included in this price is an excise tax of 48¢ per litre. The government wants to reduce energy consumption in the economy and to increase its tax revenue. Can it do this (a) by taxing domestic heating oil? (b) by taxing gasoline?*

4.5 *Define the uncompensated and compensated price elasticities as*

$$\varepsilon_{ij} := \frac{p_j}{x_i^*}\frac{\partial D^i(\mathbf{p},y)}{\partial p_j}, \varepsilon_{ij}^* := \frac{p_j}{x_i^*}\frac{\partial H^i(\mathbf{p},\upsilon)}{\partial p_j}$$

and the income elasticity

$$\varepsilon_{iy} := \frac{y}{x_i^*}\frac{\partial D^i(\mathbf{p},y)}{\partial y}.$$

Show how equations (4.20) and (4.21) can be expressed in terms of these elasticities and the expenditure share of each commodity in the total budget.

4.6 *You are planning a study of consumer demand. You have a data set which gives the expenditure of individual consumers on each of n goods. It is suggested to you that an appropriate model for consumer expenditure is the* Linear Expenditure System*:(Stone 1954)*

$$e_i = \xi_i p_i + \alpha_i \left[y - \sum_{j=1}^{n} p_j \xi_j \right]$$

where p_i is the price of good i, e_i is the consumer's expenditure on good i, y is the consumer's income, and $\alpha_1, ..., \alpha_n$, $\xi_1, ..., \xi_n$ are non-negative parameters such that $\sum_{j=1}^{n} \alpha_j = 1$.

1. *Find the effect on x_i, the demand for good i, of a change in the consumer's income and of an (uncompensated) change in any price p_j.*
2. *Find the substitution effect of a change in price p_j on the demand for good i.*
3. *Explain how you could check that this demand system is consistent with utility-maximisation and suggest the type of utility function which would yield the demand functions implied by the above formula for consumer expenditure. [Hint: compare this with Exercise 4.3]*

4.7 *Suppose a consumer has a two-period utility function of the form labelled type A in Exercise 4.2. where x_i is the amount of consumption in period i. The consumer's resources consist just of inherited assets A in period 1, which is partly spent on consumption in period 1 and the remainder invested in an asset paying a rate of interest r.*

1. *Interpret the parameter α in this case.*
2. *Obtain the optimal allocation of (x_1, x_2)*
3. *Explain how consumption varies with A, r and α.*
4. *Comment on your results and examine the "income" and "substitution" effects of the interest rate on consumption.*

4.8 *Suppose a consumer is rationed in his consumption of commodity 1, so that his consumption is constrained thus $x_1 \leq a$. Discuss the properties of the demand functions for commodities $2, ..., n$ of a consumer for whom the rationing constraint is binding. [Hint: use the analogous set of results from section 2.4].*

4.9 *A person has preferences represented by the utility function*

$$U(\mathbf{x}) = \sum_{i=1}^{n} \log x_i$$

where x_i is the quantity consumed of good i and $n > 3$.

1. *Assuming that the person has a fixed money income y and can buy commodity i at price p_i find the ordinary and compensated demand elasticities for good 1 with respect to p_j, $j = 1, ..., n$.*

2. *Suppose the consumer is legally precommitted to buying an amount A_n of commodity n where $p_n A_n < y$. Assuming that there are no additional constraints on the choices of the other goods find the ordinary and compensated elasticities for good 1 with respect to p_j, $j = 1, ...n$. Compare your answer to part 1.*

3. *Suppose the consumer is now legally precommitted to buying an amount A_k of commodity k, $k = n-r, ..., n$ where $0 < r < n-2$ and $\sum_{k=n-r}^{n} p_k A_k < y$. Use the above argument to explain what will happen to the elasticity of good 1 with respect to p_j as r increases. Comment on the result.*

4.10 *Show that if the utility function is homothetic, then $I_{\mathrm{CV}} = I_{\mathrm{EV}}$ [Hint: use the result established in Exercise 2.6.]*

4.11 *Suppose an individual has Cobb-Douglas preferences given by those in Exercise 4.2.*

1. *Write down the consumer's cost function and demand functions.*

2. *The republic of San Serrife is about to join the European Union. As a consequence the price of milk will rise to eight times its pre-entry value. but the price of wine will fall by fifty per cent. Use the compensating variation to evaluate the impact on consumers' welfare of these price changes.*

3. *San Serrife economists have estimated consumer demand in the republic and have concluded that it is closely approximated by the demand system derived in part 1. They further estimate that the people of San Serrife spend more than three times as much on wine as on milk. They conclude that entry to the European Union is in the interests of San Serrife. Are they right?*

4.12 *In a two-commodity world a consumer's preferences are represented by the utility function*

$$U(x_1, x_2) = \alpha x_1^{\frac{1}{2}} + x_2$$

where (x_1, x_2) *represent the quantities consumed of the two goods and* α *is a non-negative parameter.*

1. *If the consumer's income* y *is fixed in money terms find the demand functions for both goods, the cost (expenditure) function and the indirect utility function.*

2. *Show that, if both commodities are consumed in positive amounts, the compensating variation for a change in the price of good 1* $p_1 \to p_1'$ *is given by*
$$\frac{\alpha^2 p_2^2}{4}\left[\frac{1}{p_1'} - \frac{1}{p_1}\right]$$

3. *In this case, why is the compensating variation equal to the equivalent variation and consumer's surplus?*

4.13 *Take the model of Exercise 4.12. Commodity 1 is produced by a monopolist with fixed cost* C_0 *and constant marginal cost of production* c. *Assume that the price of commodity 2 is fixed at 1 and that* $c > \alpha^2/4y$.

1. *Is the firm a "natural monopoly"? (page 64)*

2. *If there are* N *identical consumers in the market find the monopolist's demand curve and hence the monopolist's equilibrium output and price* p_1^*.

3. *Use the solution to Exercise 4.12 to show the aggregate loss of welfare* $L(p_1)$ *of all consumers' having to accept a price* $p_1 > c$ *rather than being able to buy good 1 at marginal cost* c. *Evaluate this loss at the monopolist's equilibrium price.*

4. *The government decides to regulate the monopoly. Suppose the government pays the monopolist a performance bonus* B *conditional on the price it charges where*
$$B = K - L(p_1)$$
and K *is a constant. Express this bonus in terms of output. Find the monopolist's new optimum output and price* p_1^{**}. *Briefly comment on the solution.*