Prerequisites

Almost essential
Firm: Basics

Frank Cowell: *Microeconomics*

The Firm: Optimisation

MICROECONOMICS
Principles and Analysis
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Overview...

Approaches to the firm's optimisation problem

- Firm: Optimisation
  - The setting
  - Stage 1: Cost Minimisation
  - Stage 2: Profit maximisation

The optimisation problem

- We want to set up and solve a standard optimisation problem.
- Let's make a quick list of its components.
- ... and look ahead to the way we will do it for the firm.

The optimisation problem

- Objectives **-*Profit maximisation?***
- Constraints **-*Technology; other***
- Method **- *2-stage optimisation***

Construct the objective function

- Use the information on prices…
  - w_i •price of input i
  - p •price of output
- …and on quantities…
  - z_i •amount of input i
  - q •amount of output
- …to build the objective function

How it's done

The firm's objective function

- Cost of inputs: $\sum_{i=1}^{m} w_i z_i$ •Summed over all *m* inputs
- Revenue: pq •Subtract Cost from Revenue to get
- Profits: $pq - \sum_{i=1}^{m} w_i z_i$

Optimisation: the standard approach

- Choose q and $\mathbf{z}$ to maximise

$$\Pi := pq - \sum_{i=1}^{m} w_i z_i$$

- ...subject to the production constraint...

$$q \leq \phi(\mathbf{z})$$

• Could also write this as $\mathbf{z} \in Z(q)$

- ..and some obvious constraints:

$$q \geq 0 \qquad \mathbf{z} \geq \mathbf{0}$$

•You can't have negative output or negative inputs

A standard optimisation method

- If ϕ is differentiable…
- Set up a Lagrangean to take care of the constraints $\mathcal{L}(\ldots)$
- Write down the First Order Conditions (FOC) $\frac{\partial}{\partial \mathbf{z}} \mathcal{L}(\ldots) = 0$ (necessity)
- Check out second-order conditions $\frac{\partial^2}{\partial \mathbf{z}^2} \mathcal{L}(\ldots)$ (sufficiency)
- Use FOC to characterise solution $\mathbf{z}^* = \ldots$

Uses of FOC

- First order conditions are crucial
- They are used over and over again in optimisation problems.
- For example:
  - Characterising efficiency.
  - Analysing “Black box” problems.
  - Describing the firm's reactions to its environment.
- More of that in the next presentation
- Right now a word of caution...

A word of warning

- We’ve just argued that using FOC is useful.
  - But sometimes it will yield ambiguous results.
  - Sometimes it is undefined.
  - Depends on the shape of the production function ϕ.
- You have to check whether it’s appropriate to apply the Lagrangean method
- You may need to use other ways of finding an optimum.
- Examples coming up…

A way forward

- We could just go ahead and solve the maximisation problem
- But it makes sense to break it down into two stages
  - The analysis is a bit easier
  - You see how to apply optimisation techniques
  - It gives some important concepts that we can re-use later
- The first stage is “minimise cost for a given output level”
  - If you have fixed the output level q…
  - …then profit max is equivalent to cost min.
- The second stage is “find the output level to maximise profits”
  - Follows the first stage naturally
  - Uses the results from the first stage.
- We deal with stage each in turn

Overview...

A fundamental multivariable problem with a brilliant solution

- Firm: Optimisation
  - The setting
  - **Stage 1: Cost Minimisation**
  - Stage 2: Profit maximisation

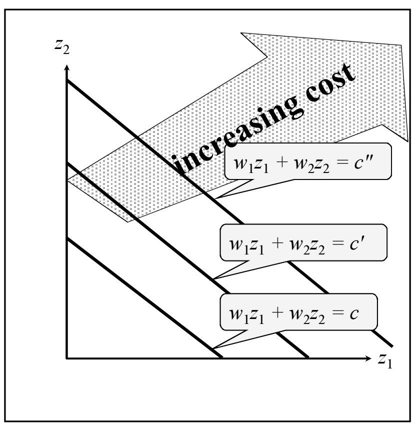
Stage 1 optimisation

- Pick a target output level q
- Take as given the market prices of inputs $\mathbf{w}$
- Maximise profits...
- ...by minimising costs $\sum_{i=1}^{m} w_i z_i$

A useful tool

- For a given set of input prices **w**...
- …the *isocost* is the set of points **z** in input space...
- ...that yield a given level of factor cost.
- These form a hyperplane (straight line)...
- ...because of the simple expression for factor-cost structure.

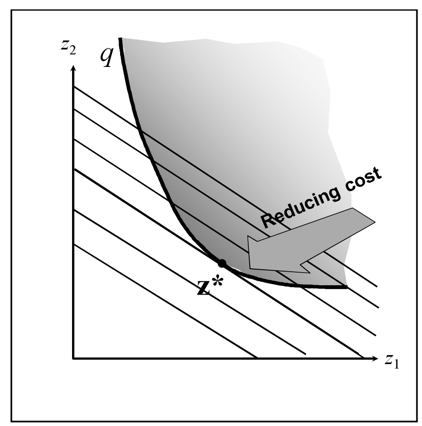
Iso-cost lines



- *Draw set of points where cost of input is c, a constant*
- *Repeat for a higher value of the constant*
- *Imposes direction on the diagram...*

Use this to derive optimum

Cost-minimisation



- *The firm minimises cost...*
- *Subject to output constraint*
- *Defines the stage 1 problem.*
- *Solution to the problem*

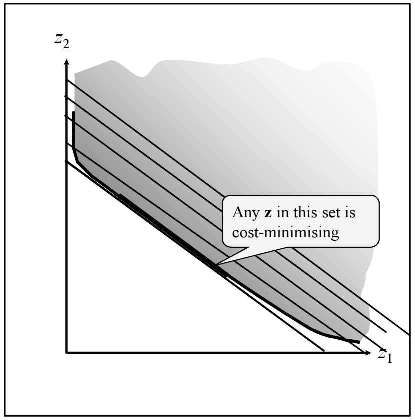
minimise

$$\sum_{i=1}^{m} w_i z_i$$

subject to $\phi(\mathbf{z}) \geq q$

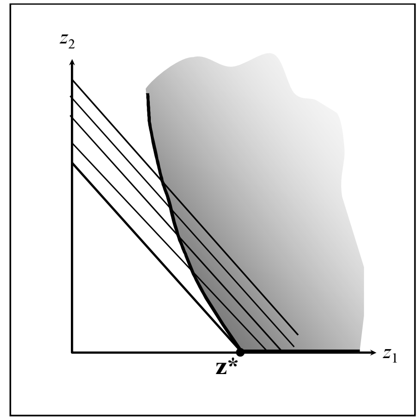
- *But the solution depends on the shape of the input-requirement set Z.*
- *What would happen in other cases?*

Convex, but not strictly convex Z



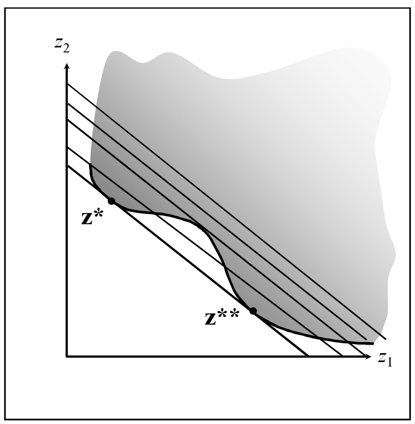
- An *interval* of solutions

Convex Z, touching axis



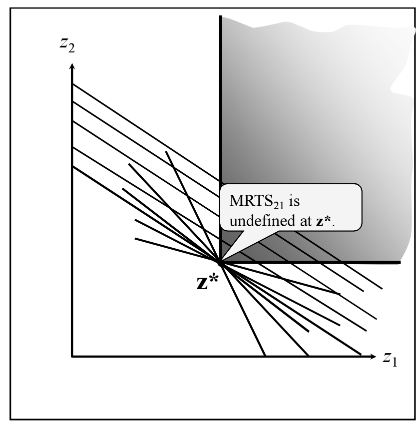
- Here $MRTS_{21} > w_1 / w_2$ at the solution.
- Input 2 is "too expensive" and so isn't used: $z_2^*=0$.

Non-convex Z



- There could be multiple solutions.
- But note that there's no solution point between z* and z**.

Non-smooth Z



- z* is unique cost-minimising point for q.
- True for all positive finite values of w_1, w_2

Cost-minimisation: strictly convex Z

- Minimise

$$\sum_{i=1}^{m} w_i z_i \;+\; \lambda[\boldsymbol{q} \leq \phi(\mathbf{z})]$$

Lagrange multiplier

- Because of strict convexity we have an interior solution.
- A set of m+1 First-Order Conditions

$$\left.\begin{aligned} \lambda^* \phi_1(\mathbf{z}^*) &= w_1 \\ \lambda^* \phi_2(\mathbf{z}^*) &= w_2 \\ \ldots \quad \ldots \quad &\ldots \\ \lambda^* \phi_m(\mathbf{z}^*) &= w_m \end{aligned}\right\}$$

one for each input

$$q = \phi(\mathbf{z}^*)$$

output constraint

- *Use the objective function*
- *...and output constraint*
- *...to build the Lagrangean*
- *Differentiate w.r.t. $z_1, \ldots, z_m$ and set equal to 0.*
- *... and w.r.t λ.*
- *Denote cost minimising values with a * .*

If isoquants can touch the axes...

- Minimise

$$\sum_{i=1}^{m} w_i z_i \;+\; \lambda[q - \phi(\mathbf{z})]$$

- Now there is the possibility of corner solutions.
- A set of m+1 First-Order Conditions

$$\left.\begin{aligned} \lambda^* \phi_1(\mathbf{z}^*) &\leq w_1 \\ \lambda^* \phi_2(\mathbf{z}^*) &\leq w_2 \\ \ldots \quad \ldots \quad &\ldots \\ \lambda^* \phi_m(\mathbf{z}^*) &\leq w_m \end{aligned}\right\}$$

Can get "<" if optimal value of this input is 0

$$q = \phi(\mathbf{z}^*)$$

Interpretation

From the FOC

- If both inputs i and j are used and MRTS is defined then...

$$\frac{\phi_i(\mathbf{z}^*)}{\phi_j(\mathbf{z}^*)} = \frac{w_i}{w_j}$$

- MRTS = input price ratio
- "implicit" price = market price

- If input i could be zero then...

$$\frac{\phi_i(\mathbf{z}^*)}{\phi_j(\mathbf{z}^*)} \leq \frac{w_i}{w_j}$$

- $\text{MRTS}_{ji} \leq$ input price ratio
- "implicit" price ≤ market price

Solution

The solution...

- Solving the FOC, you get a cost-minimising value for each input...

$$\mathbf{z}_i^* = H^i(\mathbf{w}, q)$$

- ...for the Lagrange multiplier

$$\lambda^* = \lambda^*(\mathbf{w}, q)$$

- ...and for the minimised value of cost itself.
- The *cost function* is defined as

$$C(\mathbf{w}, q) := \min_{\{\phi(\mathbf{z}) \geq q\}} \Sigma\, w_i z_i$$

vector of input prices

Specified output level

Interpreting the Lagrange multiplier

- The solution function:

$C(\mathbf{w}, q) = \Sigma_i w_i z_i^*$

$= \Sigma_i w_i z_i^* - \lambda^* [\phi(\mathbf{z}^*) - q]$

- Differentiate with respect to q:

$C_q(\mathbf{w}, q) = \Sigma_i w_i H^i_q(\mathbf{w}, q)$

$- \lambda^* [\Sigma_i \phi_i(\mathbf{z}^*)$

Vanishes because of FOC $\lambda^* \phi_i(\mathbf{x}^*) = w_i$

- Rearrange:

$C_q(\mathbf{w}, q) = \Sigma_i [w_i - \lambda^* \phi_i(\mathbf{z}^*)] H^i_q(\mathbf{w}, q) + \lambda^*$

$C_q(\mathbf{w}, q) = \lambda^*$

At the optimum, either the constraint binds or the Lagrange multiplier is zero

Express demands in terms of $(\mathbf{w},q)$

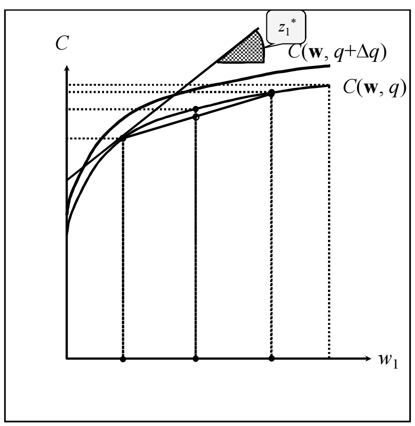
Lagrange multiplier in the stage 1 problem is just marginal cost

This result – extremely important in economics – is just an applications of a general "envelope" theorem.

The cost function is an amazingly useful concept

- Because it is a solution function...
- ...it automatically has very nice properties.
- These are true for *all* production functions.
- And they carry over to applications other than the firm.
- We'll investigate these graphically.

Properties of *C*

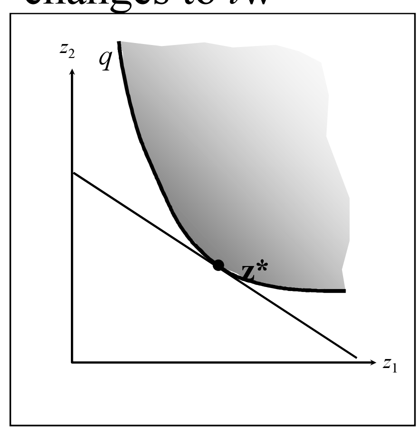


- *Draw cost as function of w_1*
- *Cost is non-decreasing in input prices .*
- *Cost is increasing in output.*
- *Cost is concave in input prices.*
- *Shephard's Lemma*

$C(t\mathbf{w}+[1-t]\mathbf{w}',q) \geq tC(\mathbf{w},q) + [1-t]C(\mathbf{w}',q)$

$\dfrac{\partial C(\mathbf{w},q)}{\partial w_j} = z_j^*$

What happens to cost if **w** changes to $t\mathbf{w}$



- *Find cost-minimising inputs for w, given q*
- *Find cost-minimising inputs for tw, given q*

- *So we have:*

$C(t\mathbf{w},q) = \Sigma_i t w_i z_i^* = t \Sigma_i w_i z_i^* = tC(\mathbf{w},q)$

- *The cost function is homogeneous of degree 1 in prices.*

Cost Function: 5 things to remember

- Non-decreasing in every input price.
  - Increasing in at least one input price.
- Increasing in output.
- Concave in prices.
- Homogeneous of degree 1 in prices.
- Shephard's Lemma.

Example

Production function: $q \leq z_1^{0.1} \; z_2^{0.4}$

Equivalent form: $\log q \leq 0.1 \log z_1 \; + \; 0.4 \log z_2$

Lagrangean: $w_1 z_1 + w_2 z_2 + \lambda \, [\log q - 0.1 \log z_1 - 0.4 \log z_2]$

FOCs for an interior solution:

$w_1 - 0.1 \, \lambda \, / \, z_1 = 0$

$w_2 - 0.4 \, \lambda \, / \, z_2 = 0$

$\log q = 0.1 \log z_1 \; + \; 0.4 \log z_2$

From the FOCs:

$\log q = 0.1 \log (0.1 \, \lambda \, / \, w_1) + \; 0.4 \log (0.4 \, \lambda \, / \, w_2 \,)$

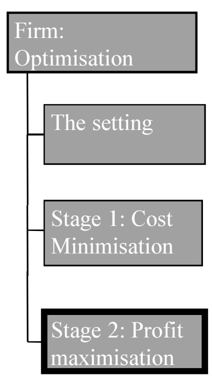
$\lambda = 0.1^{-0.2} \, 0.4^{-0.8} \, w_1^{0.2} \, w_2^{0.8} \, q^2$

Therefore, from this and the FOCs:

$w_1 z_1 + w_2 z_2 \; = \; 0.5\lambda \; = \; 1.649 \, w_1^{0.2} \, w_2^{0.8} \, q^2$

Overview...

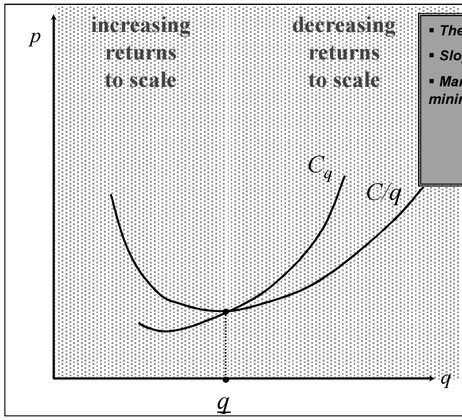
…using the results of stage 1



Stage 2 optimisation

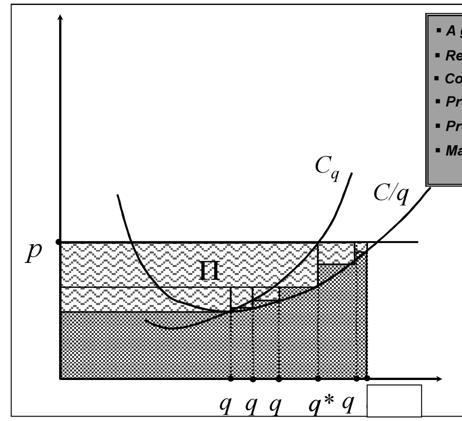
- Take the cost-minimisation problem as solved.
- Take output price p as given.
  - Use minimised costs $C(\mathbf{w},q)$.
  - Set up a 1-variable maximisation problem.
- Choose q to maximise profits.
- First analyse the components of the solution graphically.
  - Tie-in with properties of the firm introduced in the previous presentation.
- Then we come back to the formal solution.

Average and marginal cost



- The average cost curve.
- Slope of AC depends on RTS.
- Marginal cost cuts AC at its minimum

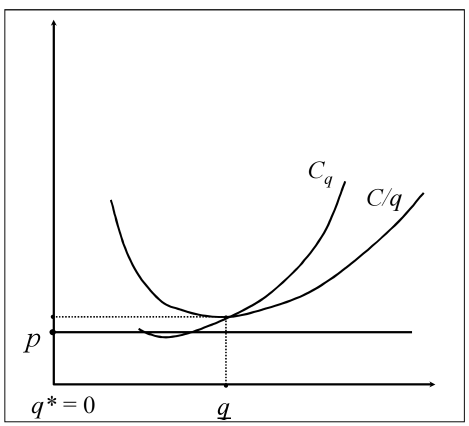
Revenue and profits



- A given market price p.
- Revenue if output is q.
- Cost if output is q.
- Profits if output is q.
- Profits vary with q.
- Maximum profits

- price = marginal cost

What happens if price is low...



- price < marginal cost

Profit maximisation

- Objective is to choose q to max:

  $pq - C(\mathbf{w}, q)$ — *"Revenue minus minimised cost"*

- From the First-Order Conditions if $q^* > 0$:

  $p = C_q(\mathbf{w}, q^*)$ — *"Price equals marginal cost"*

  $p \geq \frac{C(\mathbf{w}, q^*)}{q^*}$ — *"Price covers average cost"*

- In general:

  $p \leq C_q(\mathbf{w}, q^*)$

  $pq^* \geq C(\mathbf{w}, q^*)$

  covers both the cases: $q^* > 0$ *and* $q^* = 0$

Example (continued)

Production function: $q \leq z_1^{0.1}\ z_2^{0.4}$

Resulting cost function: $C(\mathbf{w}, q) = 1.649\, w_1^{0.2} w_2^{0.8} q^2$

Profits:

$pq - C(\mathbf{w}, q) = pq - A\, q^2$

where $A := 1.649\, w_1^{0.2} w_2^{0.8}$

FOC:

$p - 2\, Aq = 0$

Result:

$q = p \,/\, 2A.$

$= 0.3031\, w_1^{-0.2} w_2^{-0.8} p$

Summary

- *Key point:* Profit maximisation can be viewed in two stages:
  - Review Stage 1: choose inputs to minimise cost
  - Review Stage 2: choose output to maximise profit
- *What next?* Use these to predict firm's reactions