Problem Set 3 2016

1. Suppose that we have a farm that produces oranges next to a farm that produces honey. The total cost functions of the two firms are $C_0(q_o, q_h) = \frac{{q_o}^2}{100} - q_h$ and $C_h(q_h) = \frac{{q_h}^2}{100}$ respectively. The two firms operate as price takers in perfectly competitive markets with $p_o = 4$ and $p_h = 2$.
   - i) How much honey and how many oranges are being produced if the two firms operate independently of one another?
   - ii) Suppose that the two firms merge. Find the optimal amounts of honey and oranges in this new setting and compare it with the Pareto Optimal outcome.s

2. We have two agents, with utility functions over a numeraire good m_i, a private good x_i and an externality h. That is, $u_1(m_1, x_1, h) = m_1 + 10\ln(1 + x_1) + 5h - h^2$and $u_2(m_1, x_1, h) = m_2 + 5\ln(1 + x_2) - h^2$.
   - i) Derive the agents' indirect utility functions depending on their wealth level w and on the price of the private good p (hint: you need to maximize each agent's utility function subject to their budget constraint for given prices).
   - ii) Given that p and w will not change in our partial equilibrium setting, derive the utility functions of the agents depending only on the level of the externality and find the externality level that agent 1 will choose to generate.
   - iii) Find the Pareto Optimal amount of externality.
   - iv) Suppose that agent 2 is given the right to an externality-free environment so that agent 1 must pay a price p_h to agent 2 for each unit of externality that she generates. Also, assume that this price is formed in a competitive market. Derive the necessary conditions for equilibrium in the externality market and find the price and level of the externality.

3. (From MWG) A certain lake can be freely accessed by fishermen. When a boat is sent to catch fish, its cost is $r > 0$. The total quantity of fish caught (Q) is a function of the total boats sent (b), that is $Q = f(b)$, with each boat getting $\frac{f(b)}{b}$ fish. We assume concavity of the production function, that is $f'(b) > 0$and $f''(b) < 0$. We also assume a competitive market where the fish are sold at a given price $p > 0$.
   - i) Characterize the equilibrium number of boats that are sent fishing, if fishermen are allowed freely to fish in the lake (hint: First, find the Total Cost function and the Profits function).

ii) Characterize the optimal number of boats that are sent fishing.

4. We have two consumers, A and B with utility functions $U_A = \log x_A + logG$ and $U_B = \log x_A + logG$ respectively. $x_i$represents the amount of a private good that each person consumes, while G represents the total amount of a public good that is offered and $g_A + g_B = G$, with g_A and g_B being the contributions of the two agents for the public good. Also, we assume that both the prices of the public and the private good equal to 1.

   i) Find the best response functions of A and B for the public good. That is, we are looking for a function that, for each person, gives the optimal amount of public good that she demands, with respect to the other person's quantity demanded.

   ii) Solve for the Nash Equilibrium. What will be the optimal contributions to the public good? Show your answer on a diagram with the agents' reaction functions.

   iii) Find the socially optimal (Pareto Optimal) level of the public good and compare it with your previous result.

5. (From MWG) In an economy we have J firms and I individuals. Each firm j generates a level of externality h_j and its profits depend on that externality, that is $\pi_j = \pi_j(h_j)$, while each individual's derived utility function depends in general on the externalities generated by all of the firms, that is $U_i = \Phi_i(h_1, h_2, \dots, h_J) + w_i$. In this case we do not have a homogeneous externality.

   i) Using the proper FOCs derive the Pareto Optimal amounts of externalities for the economy as well as the amounts that will be generated in a competitive equilibrium.

   ii) What tax/subsidy can restore efficiency?

6. (From MWG) Suppose that consumer i's preferences can be represented by the utility function $u_i(x_{1i}, \dots, x_{Li}) = \sum_l log(x_{li})$ (Cobb Douglas preferences).

   i) Derive his demand for good l. What is the wealth effect?

   ii) What happens to the wealth effect as we increase the number of goods? (Calculate the limit as L goes to infinity)

7. (From MWG) Consider an economy with two goods, one consumer and one firm. The initial endowment of the numeraire is $\omega_m > 0$ and the initial endowment of good l is 0. The consumer's quasilinear utility function is $u(x, m) = m + \varphi(x)$, where $\varphi(x) = \alpha + \beta \ln(x)$, with $\alpha, \beta > 0$. The firm's cost function is $c(q) = \sigma q$, with $\sigma > 0$. Also assume that the consumer

Solutions to Problem Set 3

1)

i) When they act independently each will maximize its own profit:

$$max\Pi_0 = p_0q_0 - \frac{q_0^2}{100} + q_h$$

From the FOC we get that $q_0 = 200$

Similarly, $max\Pi_h$ and we get $q_h = 100$

ii) When they merge, they act as one firm:

$$max\Pi = p_0q_0 + p_hq_h - \frac{q_0^2}{100} + q_h - \frac{q_h^2}{100}$$

From the FOCS we have that:

$$\frac{\partial\Pi}{\partial q_o} = 0 \Rightarrow q_0 = 200$$

And $\frac{\partial\Pi}{\partial q_h} = 0 \Rightarrow q_h = 150$

2)

i) Indirect utility function:

$maxu_1$ subject to: $p_1x_1 + m_1 \leq w_1$

$$L = m_1 + 10\ln(1 + x_1) + 5h - h^2 + \lambda(p_1x_1 + m_1 - w_1)$$

From the FOCS we get that:

$$x_1 = \frac{10}{p_1} - 1$$

And $m_1 = w_1 - 10 + p$

So substituting into u_1 we get:

$$v_1 = w_1 - 10 + p + 10\ln\left(\frac{10}{p}\right) + 5h - h^2$$

In a similar manner, $v_2 = w_2 - 5 + p + 5\ln\left(\frac{5}{p}\right) - h^2$

ii) $v_1 = w_1 - 10 + p + 10\ln\left(\frac{10}{p}\right) + 5h - h^2 \Rightarrow \Phi_1(h) = 5h - h^2$

$$v_2 = w_2 - 5 + p + 5\ln\left(\frac{5}{p}\right) - h^2 \Rightarrow \Phi_2(h) = -h^2$$

From $max\Phi_1(h)$ we have that $h = 5/2$

iii) $max\Phi_1(h) + \Phi_2(h) = 5h - 2h^2$

From the FOC we get that $h^* = 5/4$

iv) Programs of the two agents:

Agent 1:

$max5h - h^2 - p_hh$ and from the FOC we get $5 - 2h = p_h$

Agent 2:

$\max(-h^2) + p_hh$ and from the FOC: $p_h = 2h$

Solving the system we have that $h = 5/4$ and $p_h = 5/2$

3) MWG chapter 11 exercise 11.D.5 (see solution manual you can find it if you google it)

4)

i) A's maximization problem:

$$\max \log x_A + \log G$$

s.t. $g_A + g_B = G$ and $g_A + x_A = w$

or: $\max \log(w - g_A) + \log(g_A + g_B)$

From the FOC we get $g_A(g_B) = \frac{w - g_B}{2}$

In a similar manner we have that $g_B(g_A) = \frac{w - g_A}{2}$

(we have assumed that they hold the same amount of wealth as it simplifies the expressions)

ii) Because of symmetry it must be that $g_A = g_B$. So by the above equations, $g_A = g_B = w/3$

iii) $\max \log(w - g_A) + \log(g_A + g_B) + \log(w - g_B) + \log(g_A + g_B)$

due to symmetry: $g_A = g_B = g = G/2$

so from the FOC we get: $g = w/2$

In this case $G = w$ while in the previous case, $G = \frac{2w}{3}$ (underprovision)

5) From MWG chapter 11 exercise 11.D.4

6) From MWG chapter 10 exercise 10.C.1

7) From MWG chapter 10 exercise 10.C.2

$g_A = \frac{w - g_A}{2} \Rightarrow g_A = \frac{w}{3}$

$\max \log(w-g) + \log(2g) + \log(w-g) + \log(2g)$

$\max 2\log(w-g) + 2\log(2g)$

FOC $-\frac{2}{w-g} + \frac{2 \cdot 2}{2g} = 0$

$\Rightarrow \frac{2}{w-g} = \frac{2}{g} \Rightarrow w - g = g$

$\Rightarrow w = 2g \Rightarrow g = \frac{w}{2}$

x_i solves $\underset{x_i \in X_i}{\text{Max}}\ u_i(x_i)$ s.t. the old budget constraint

(iii) Market clearing: $\sum_{i=1}^{I} x^*_{li} = \omega_l + \sum_{j=1}^{J} y^*_{lj}$ for each $l = 1, \ldots, L$ - does not depend on prices at all.

10.C.1. (a) The consumer solves

$$\text{Max} \sum_{l=1}^{L} \log x_l \qquad \text{s.t.} \quad \sum_{l=1}^{L} p_l x_l \le w.$$

The first-order condition for the Lagrangean of this program can be written as $x_l = \lambda/p_l$, $l=1,\ldots,L$, where $\lambda > 0$. Substituting in the budget constraint, we find $\lambda = w/L$, therefore the demand function can be written as $x_l(p,w) = \dfrac{w}{Lp_l}$.

The wealth effect is $\partial x_l(p,w)/\partial w = \dfrac{1}{Lp_l}$.

(b) As $L \longrightarrow \infty$, the wealth effect $\partial x_l(p,w)/\partial w \longrightarrow 0$.

10.C.2. (a) The consumer solves

$$\underset{(x,m)}{\text{Max}}\ \alpha + \beta \ln x + m \qquad \text{s.t.} \ p\,x + m \le \omega_m.$$

The first-order condition (assuming interior solution) yields $x(p) = \beta/p$.

The firm solves $\underset{q \ge 0}{\text{Max}}\ pq - \sigma q$.

The firm's first-order condition (assuming interior solution) is $p = \sigma$.

(b) From the two first-order conditions and the consumer's budget constraint, the competitive equilibrium is $p^* = \sigma$, $x^* = \beta/\sigma$, $m^* = \omega_m - \beta$.

10.C.3. (a) Assuming interior solution, the first-order condition is $c'_j(q^*_j) = \lambda > 0$ for all j.

and the FOCs for this program are

$$\frac{\partial S}{\partial q_j} = p(Q) - c_q(q_j,Q) - c_Q(q_j,Q) - \sum_{k \neq j} c_Q(q_k,Q) = 0 ,$$

and we can check that the assumptions ensure that the SOC is satisfied. Let q^o denote the solution to this program (again, independent of j), and let $Q^o = Jq^o$. The optimal Q^o will then be determined by

$$(2) \quad p(Q^o) = c_q(\tfrac{Q^o}{J},Q^o) + c_Q(\tfrac{Q^o}{J},Q^o) + (J-1)c_Q(\tfrac{Q^o}{J},Q^o) .$$

(Again, by $c_q > 0$, $c_Q < 0$, and $c_q + Jc_Q > 0$, a solution will exist.) Since $c_Q < 0$, (2) implies that $p(Q^o) < c_q(\frac{Q^o}{J},Q^o) + c_Q(\frac{Q^o}{J},Q^o)$. Also, since $p'(Q)\leq 0$, and $\frac{d}{dQ}\left[c_q(\frac{Q}{J},Q) + c_Q(\frac{Q}{J},Q)\right] = \frac{1}{J}\left(c_{qq}(\frac{Q}{J},Q) + (J+1)c_{qQ}(\frac{Q}{J},Q) + Jc_{QQ}(\frac{Q}{J},Q)\right) > 0$ (by assumption), then we must have that $Q^o > Q^*$. This is intuitive since firms ignore the positive externality that they create, and we have an under-production competitive equilibrium. To restore efficiency the government can subsidize production with a subsidy of $s = -(J-1)c_Q(\frac{Q^o}{J},Q^o)$. Firm j's FOC will then be $p - (J-1)c_Q(\frac{Q^o}{J},Q^o) = c_q(q_j,Q) + c_Q(q_j,Q)$, and it is easy to see that $Q = Q^o$ and $p = p(Q^o)$ will cause $q_j = \frac{Q^o}{J}$ to solve this FOC.

11.D.4 For the Pareto optimal outcome we solve

$$\underset{\{h_i\}}{\text{Max}} \; \sum_{i=1}^{I} \phi_i(h_1,\ldots,h_J) + \sum_{j=1}^{J} \pi_j(h_j) ,$$

which yields the FOCs $\sum_{i=1}^{I}\left(\frac{\partial \phi_i(h_1^o,\ldots,h_J^o)}{\partial h_j}\right) \leq \pi_j'(h_j^o)$ with equality if $h_j^o > 0$ for all $j=1,\ldots J$. On the other hand, in a competitive equilibrium each firm maximizes profits individually, and we get the FOC shown in condition (11.D.1) in the textbook. To restore the Pareto optimal outcome in a competitive equilibrium, we must set an individual tax for each j of $t_j = -\sum_{i=1}^{I}\left(\frac{\partial \phi_i(h_1^o,\ldots,h_J^o)}{\partial h_j}\right)$. Each firm will face the same tax rate if and only if we have $\sum_{i=1}^{I}\left(\frac{\partial\phi_i(h_1^o,\ldots,h_J^o)}{\partial h_j}\right) = \sum_{i=1}^{I}\left(\frac{\partial\phi_i(h_1^o,\ldots,h_J^o)}{\partial h_k}\right)$ for all j,k.

11.D.5 [First Printing Errata: the assumption that $f(0)=0$ should be added.]

(a) This is a model of free entry so fishermen will send out boats as long as there are positive profits from doing so. Therefore, the equilibrium number of boats, b^*, will be reached when $p\cdot\frac{f(b^*)}{b^*} - r = 0$, or, $\frac{f(b^*)}{b^*} = \frac{r}{p}$. This condition is that average revenue equals average cost. (We ignore integer problems, but if we are to give the integer equilibrium number then it is b^* such that $p\cdot\frac{f(b^*)}{b^*} - r \geq 0$ and $p\cdot\frac{f(b^*+1)}{b^*+1} - r < 0$.)

(b) To characterize the optimal number of boats we must solve for maximum total surplus, i.e., $\text{Max}_b\ p\cdot f(b) - r\cdot b$, the FOC is $p\cdot f'(b^o) - r \leq 0$, which is necessary and sufficient since the SOC, $p\cdot f''(b) < 0$, is satisfied. Therefore, the condition for the optimal number of boats is $f'(b^o) = \frac{r}{p}$, i.e., that marginal revenue equals marginal (and in this case average) cost. Assuming that $f(0) = 0$ ensures that $b^o \leq b^*$ (equality only at 0).

(c) To restore efficiency we need the equilibrium condition satisfied at b^o, i.e., we need the tax level to satisfy $\frac{f(b^*)}{b^*} = \frac{r+t}{p}$, or $t = p\cdot\frac{f(b^*)}{b^*} - r$.

(d) Clearly, if owned by a single individual, the problem to be solved is exactly that solved in part **(b)** above, which results in b^o.

11.D.6 **(a)** First, if the firm decides to go off and generate any level of the externality, absent of an agreement, it solves $\text{Max}_h\ p(h) = \alpha + \beta h - \mu h^2$, the (necessary and sufficient) FOC is $\beta - 2\mu h^* = 0$, or $h^* = \frac{\beta}{2\mu}$. This yields the firm profits of $\pi(h^*) = \alpha + \frac{\beta^2}{4\mu}$, which is the firm's reservation profits.

A coalition of θI consumers making a take-it-or-leave-it offer to the firm