Prerequisites

Almost essential

Game Theory: Strategy and Equilibrium

Games: Mixed Strategies

MICROECONOMICS

Principles and Analysis

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December 2006

Introduction

- Presentation builds on *Game Theory: Strategy and Equilibrium*
- Purpose is to…
  - extend the concept of strategy
  - extend the characterisation of the equilibrium of a game
- Point of taking these steps:
  - tidy up loose ends from elementary discussion of equilibrium
  - lay basis for more sophisticated use of games
  - some important applications in economics

Overview...

An introduction to the issues

- Games: Equilibrium
  - The problem
  - Mixed strategies
  - Applications

Games: a brief review

- Components of a game
  - players (agents) $h = 1,2,\ldots$
  - objectives of players
  - rules of play
  - outcomes
- Strategy
  - s^h: a complete plan for all positions the game may reach
  - S^h: the set of all possible s^h
  - focus on "best response" of each player
- Equilibrium
  - elementary but limited concept – dominant-strategy equilibrium
  - more general – Nash equilibrium
  - NE each player is making the best reply to everyone else

NE: An important result

- In some cases an important result applies
  - where strategy sets are infinite…
  - …for example where agents choose a value from an interval
- THEOREM: If the game is such that, for all agents h, the strategy sets S^h are convex, compact subsets of $\mathbb{R}^n$ and the payoff functions v^h are continuous and quasiconcave, then the game has a Nash equilibrium in pure strategies
- Result is similar to existence result for General Equilibrium

A problem?

- Where strategy sets are finite
  - again we may wish to seek a Nash Equilibrium
  - based on the idea of best reply…
- But some games *apparently* have no NE
  - example – the discoordination game
- Does this mean that we have to abandon the NE concept?
- Can the solution concept be extended?
  - how to generalise…
  - …to encompass this type of problem
- First, a brief review of the example…

Story

Discoordination

This game may seem no more than a frustrating chase round the payoff table. The two players' interests are always opposed (unlike Chicken or the Battle of the Sexes). But it is an elementary representation of class of important economic models. An example is the tax-audit game where Player 1 is the tax authority (“audit”, “no-audit”) and Player 2 is the potentially cheating taxpayer (“cheat”, “no-cheat”). More on this later.

rdination”

Player *a*	[+]	[–]
	3,0	1,2
[–]	0,3	2,1

Player *b*

- *If a plays [–] then b's best response is [+].*
- *If b plays [+] then a's best response is [+].*
- *If a plays [+] then b's best response is [–].*
- *If b plays [–] then a's best response is [–].*
- *Apparently, no Nash equilibrium!*

- *Again there's more to the Nash-equilibrium story here*
- *(to be continued)*

Overview...

Games: Equilibrium

- **The problem**
- Mixed strategies
- Applications

An introduction to the issues

A way forward

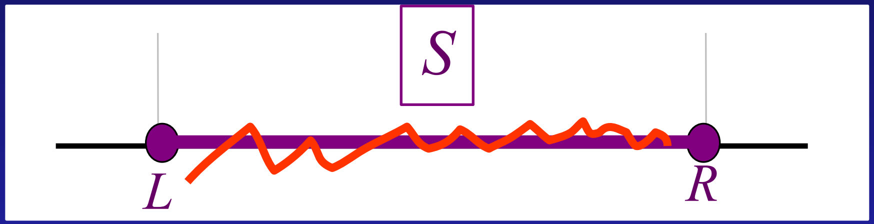
- Extend the concept of strategy
  - New terminology required
- *Pure* strategy
  - the type of strategy that has been discussed so far
  - a deterministic plan for every possible eventuality in the game
- *Mixed* strategy
  - a probabilistic approach to play
  - derived from set of pure strategies
  - pure strategies themselves can be seen as special cases of mixed strategies.

Mixed strategies

- For each player take a set of pure strategies S
- Assign to each member of S a probability π that it will be played
- Enables a “convexification” of the problem
- This means that new candidates for equilibrium can be found…
- …and some nice results can be established
- But we need to interpret this with care…

Strategy space – extended?

- Use the example of strategy space in *Game Theory: Basics*
- In the simplest case S is just two blobs "*L*eft" and "*R*ight"

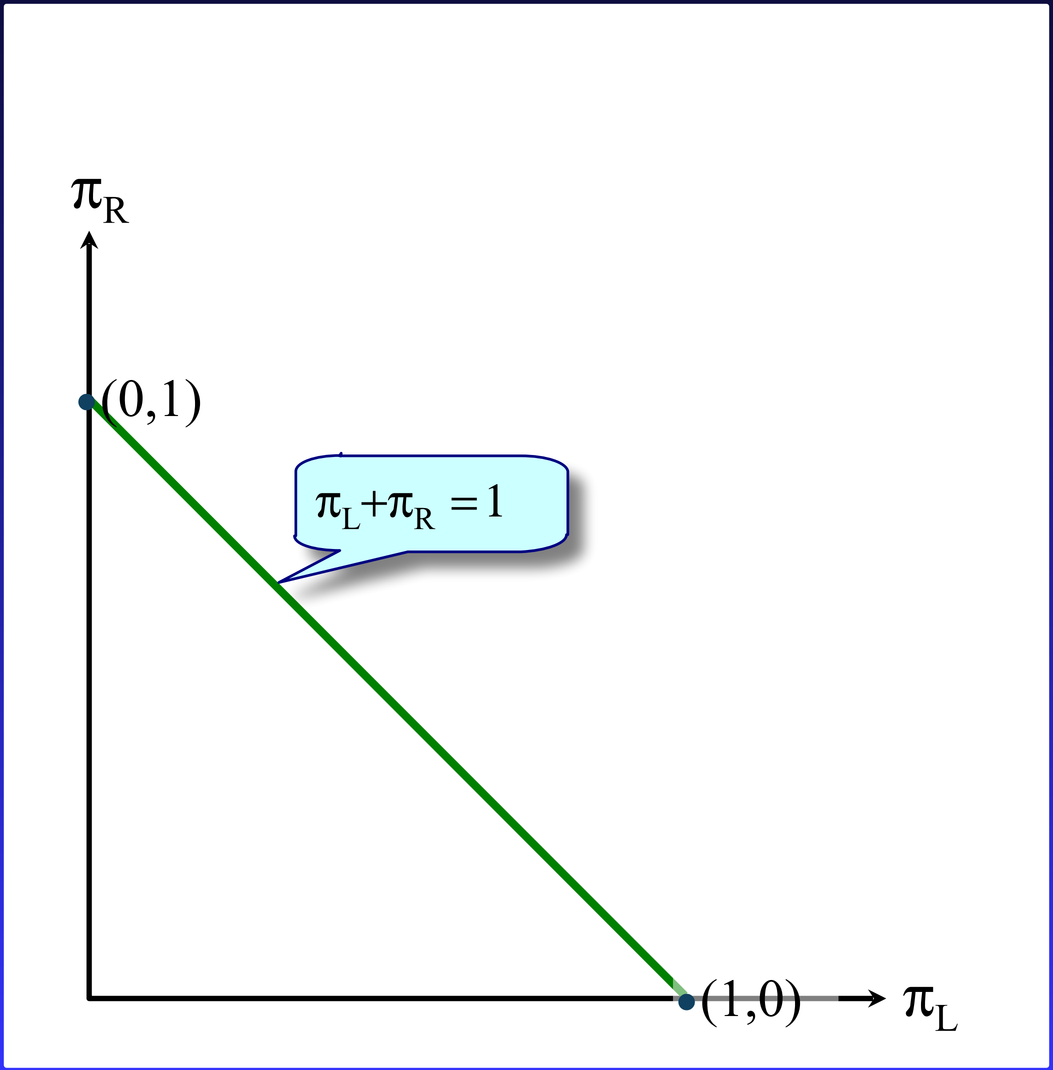


- Suppose we introduce the probability π.
- Could it effectively change the strategy space like this?
- This is misleading
- There is no "half-left" or "three-quarters-right" strategy.
- Try a different graphical representation

Strategy – a representation

- Draw a diagram in the space of the *probabilities*.
- Start by enumerating each strategy in the set *S*.
  - If there are *n* of these we'll need an *n*-dimensional diagram.
  - Dimension *i* corresponds to the probability that strategy *i* is played.
- Then plot the points (1,0,0,…), (0,1,0,…), (0,0,1,…),…
- Each point represents the case where the corresponding pure strategy is played.
- Treat these points like "radio buttons":
  - You can only push one down at a time
  - Likewise the points (1,0,0,…), (0,1,0,…), (0,0,1,…),… are mutually exclusive
- Look at this in the case $n = 2$…

Two pure strategies in *S*



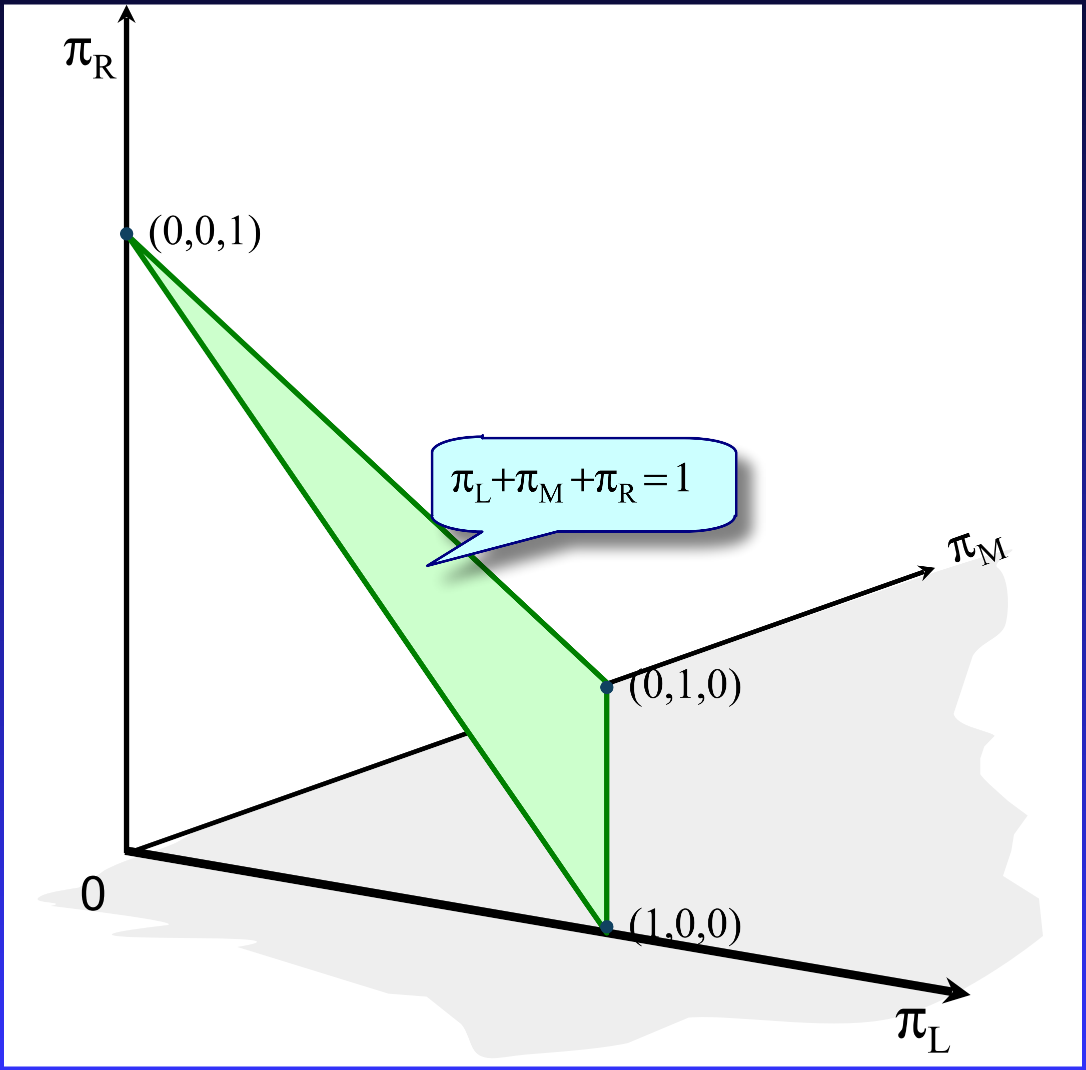
- *Probability of playing* L
- *Probability of playing* R
- *Playing L with certainty*
- *Playing R with certainty*
- *Cases where* $0 < \pi < 1$

- *Pure strategy means being at one of the two points (1,0) or (0,1)*
- *But what of these points...?*

Mixed strategy – a representation

- Just as the endpoints (1,0) and (0,1) represent the playing of the "pure" strategies *L* and *R*...
- ...so any point on the line joining them represents a probabilistic mixture of *L* and *R*:
  - The middle of the line is the case where the person spins a fair coin before choosing *L* or *R*.
  - $\pi_L = \pi_R = ½$.
- Consider the extension to the case of 3 pure strategies:
  - Strategies consist of the actions "*L*eft", "*M*iddle", "*R*ight"
  - We now have three "buttons" (1,0,0), (0,1,0), (0,0,1).
- Again consider the diagram:

Three pure strategies in *S*



- ***Third axis corresponds to probability of playing "Middle"***
- ***Three "buttons" for the three pure strategies***
- ***Introduce possibility of having $0 < \pi < 1$***

Strategy space again

- Allowing for the possibility of "mixing"...
- ...a player's strategy space consists of a pair:
  - a collection of pure strategies (as before)
  - a collection of probabilities
- Of course this applies to each of the players in the game
- How does this fit into the structure of the game?
- Two main issues:
  - modelling of payoffs
  - modelling and interpretation of probabilities

The payoffs

- We need to take more care here
  - a question of the nature of "utility"
- If pure strategies only are relevant
  - payoffs can usually be modelled simply
  - usually can be represented in terms of ordinal utility
- If players are acting probabilistically
  - consider how to model *prospective* payoffs
  - take into account preferences under uncertainty
  - use expected utility?
- Cardinal versus ordinal utility
  - if we take expectations over many cells of the payoff table…
  - …we need a cardinal utility concept
  - can transform payoffs υ only by scale and origin: $a + b\upsilon$
  - otherwise expectations operator is meaningless

Probability and payoffs

- Expected utility approach induces a simple structure
- We can express resulting payoff as
  - sum of …
  - (utility associated with each button x
  - probability each button is pressed)
- So we have a neat linear relationship
  - payoff is linear in utility associated with each button
  - payoff is linear in probabilities
  - so payoff is linear in strategic variables
- Implications of this structure?

Reaction correspondence

- A simple tool
  - build on the idea of the *reaction function* used in oligopoly…
  - …given competitor’s quantity, choose your own quantity
- But, because of linearity need a more general concept
  - reaction *correspondence*
  - multivalued at some points
  - allows for a “bang-bang” solution
- Good analogies with simple price-taking optimisation
  - think of demand-response with straight-line indifference curves…
  - …or straight-line isoquants
- But computation of equilibrium need not be difficult

Mixed strategies: computation

- To find optimal mixed-strategy:
  - take beliefs about probabilities used by other players
  - calculate expected payoff as function of these and one's own probabilities
  - find response of expected payoff to one's own probability
  - compute reaction correspondence
- To compute mixed-strategy equilibrium
  - take each agent's reaction correspondence
  - find equilibrium from intersection of reaction correspondences
- Points to note
  - beliefs about others' probabilities are crucial
  - stage 4 above usually leads to $\pi = 0$ or $\pi = 1$ except at some special point…
  - …acts like a kind of tipping mechanism

Mixed strategies: result

- The linearity of the problem permits us to close a gap
- We have another existence result for Nash Equilibrium

- THEOREM Every game with a finite number of pure strategies has an equilibrium in mixed strategies.

The random variable

- Key to the equilibrium concept: probability
- But what is the nature of this entity?
  - an explicit generating model?
  - subjective idiosyncratic probability?
  - will others observe and believe the probability?
- How is one agent's probability related to another?
  - do each choose independent probabilities?
  - or is it worth considering a correlated random variable?
- Examine these issues using two illustrations

Overview...

Games: Equilibrium

The problem

Mixed strategies

Applications

- **The audit game**
- Chicken

An example where only a mixed strategy can work…

Illustration: the audit game

- Builds on the idea of a discoordination game
- A taxpayer chooses whether or not to report income y
  - pays tax ty if reports
  - pays 0 if does not report and concealment is not discovered
  - pays tax plus fine F if does not report and concealment is discovered
- Tax authority (TA) chooses whether or not to audit taxpayer
  - incurs resource cost c if it audits
  - receives due tax ty plus fine F if concealment is discovered
- Examine equilibrium
  - first demonstrate no equilibrium in pure strategies
  - then the mixed-strategy equilibrium
- First examine best responses of each player to the other…

Audit game: normal form

- Each chooses one of two actions
- (taxpayer, TA) payoffs
- If taxpayer conceals then TA will audit
- If TA audits then taxpayer will report
- If taxpayer reports then TA won't audit
- If TA doesn't audit then taxpayer will conceal

Taxpayer \ Tax Authority	[Audit]	[Not audit]
[conceal]	$[1-t]y-F,\ ty+F-c$	$y,\ 0$
[report]	$[1-t]y,\ ty-c$	$[1-t]y,\ ty$

- $ty + F - c > 0$
- $[1-t]y > [1-t]y - F$
- $ty - c > ty$
- $y > [1-t]\,y$

- No equilibrium in pure strategies

mixed strategies

Audit game: mixed strategy approach

- Now suppose each player behaves probabilistically
  - taxpayer conceals with probability π^a
  - TA audits with probability π^b
- Each player maximises expected payoff
  - chooses own probability…
  - …taking as given the other's probability
- Follow through this process
  - first calculate expected payoffs
  - then compute optimal π given the other's π
  - then find equilibrium as a pair of probabilities

Audit game: taxpayer's problem

- Payoff to taxpayer, given TA's value of π^b:
  - if conceals: $\upsilon^a = \pi^b [y - ty - F] + [1 - \pi^b]\, y \quad = y - \pi^b ty - \pi^b F$
  - if reports: $\upsilon^a = y - ty$
- If taxpayer selects a value of π^a, calculate expected payoff
  - $E\upsilon^a = \pi^a [y - \pi^b ty - \pi^b F] + [1 - \pi^a]\, [y - ty]$

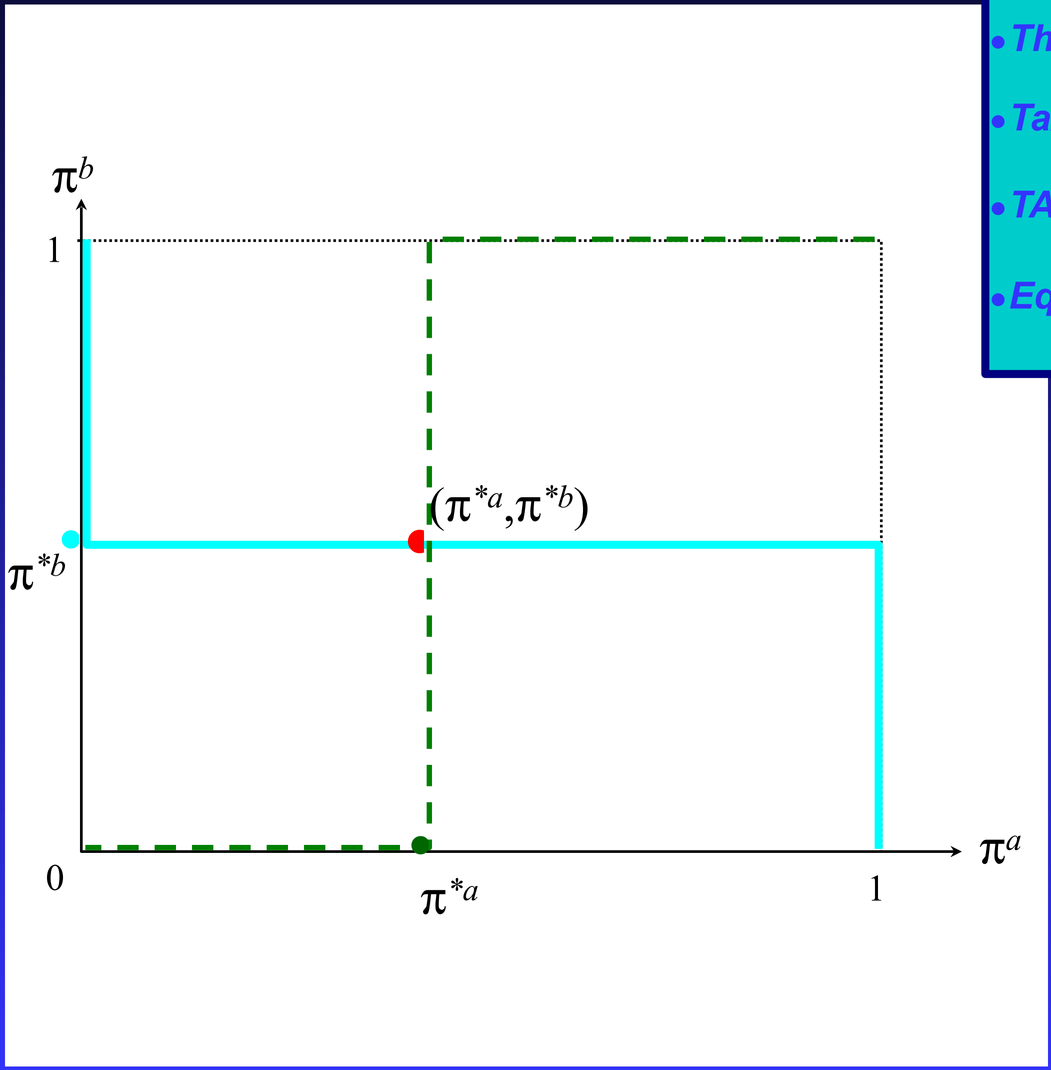
    $= [1 - t]\, y + \pi^a\, [1 - \pi^b]\, ty - \pi^a \pi^b F$
- Taxpayer's problem: choose π^a to max $E\upsilon^a$
- Compute effect on $E\upsilon^a$ of changing π^a :
  - $\partial E\upsilon^a / \partial \pi^a = [1 - \pi^b] ty - \pi^b F$
  - define $\pi^{*b} = ty / [ty + F]$
  - then $E\upsilon^a / \partial \pi^a$ is positive if $\pi^b < \pi^{*b}$, negative if ">"
- So optimal strategy is
  - set π^a to its max value 1 if π^b is low (below π^{*b})
  - set π^a to its min value 0 if π^b is high

Audit game: TA's problem

- Payoff to TA, given taxpayer's value of π^a:
  - if audits: $\upsilon^b = \pi^a [ty + F - c] + [1 - \pi^a][ty - c] = ty - c + \pi^a F$
  - if does not audit: $\upsilon^b = \pi^a \cdot 0 + [1 - \pi^a]\, ty = [1 - \pi^a]\, ty$
- If TA selects a value of π^b, calculate expected payoff
  - $\mathrm{E}\upsilon^b = \pi^b [ty - c + \pi^a F] + [1 - \pi^b]\,[1 - \pi^a]\, ty$

    $= [1 - \pi^a]\, ty + \pi^a \pi^b [ty + F] - \pi^b c$
- TA's problem: choose π^b to max $\mathrm{E}\upsilon^b$
- Compute effect on $\mathrm{E}\upsilon^b$ of changing π^b :
  - $\partial \mathrm{E}\upsilon^b / \partial \pi^b = \pi^a [ty + F] - c$
  - define $\pi^{*a} = c / [ty + F]$
  - then $\mathrm{E}\upsilon^b / \partial \pi^b$ is positive if $\pi^a < \pi^{*a}$, negative if ">"
- So optimal strategy is
  - set π^b to its min value 0 if π^a is low (below π^{*a})
  - set π^b to its max value 1 if π^a is high

Audit game: equilibrium



- *The space of mixed strategies*
- *Taxpayer's reaction correspondence*
- *TA's reaction correspondence*
- *Equilibrium at intersection*

- $\pi^a = 1$ if $\pi^b < \pi^{*b}$

  $\pi^a = 0$ if $\pi^b > \pi^{*b}$

- $\pi^b = 0$ if $\pi^a < \pi^{*a}$

  $\pi^b = 1$ if $\pi^a > \pi^{*a}$

Overview...

Mixed strategy or correlated strategy…?

Games: Equilibrium

- The problem
- Mixed strategies
- Applications
  - The audit game
  - **Chicken**

Chicken game again

- A number of possible background stories
  - think of this as individuals' contribution to a public project
  - there's the danger that one may contribute, while the other "free rides"…
  - ...and the danger that nobody contributes at all
  - but this isn't quite the classic "public good problem" (later)
- Two players with binary choices
  - call them "contribute" and "not contribute"
  - denote as [+] and [−]
- Payoff structure
  - if you contribute and the other doesn't, then you get 1 the other gets 3
  - if both of you contribute, then you both get 2
  - if neither of you contribute, then you both get 0
- First, let's remind ourselves of pure strategy NE…

Chicken game: normal form

Player *a* (rows), Player *b* (columns)

	[+]	[–]
[+]	2,2	1,3
[–]	3,1	0,0

- *If a plays [–] then b's best response is [+]*
- *If b plays [+] then a's best response is [–]*
- *Resulting NE*
- *By symmetry, another NE*

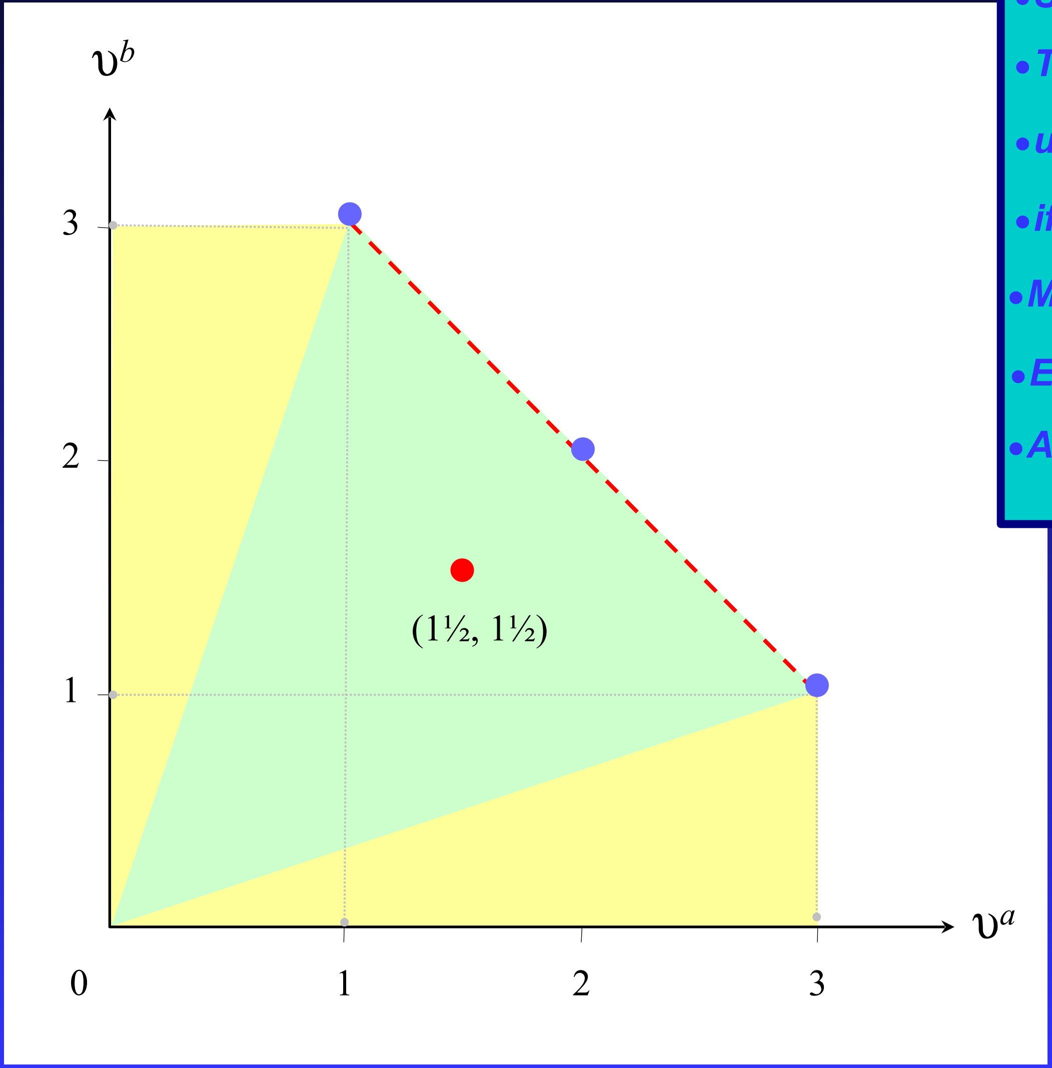
- *Two NE's in pure strategies*
- *Up to this point utility can be taken as purely ordinal*

mixed strategies

Chicken: mixed strategy approach

- Each player behaves probabilistically:
  - *a* plays [+] with probability π^a
  - *b* plays [+] with probability π^b
- Expected payoff to *a* is
  - $E\upsilon^a = \pi^a\,[2\cdot\pi^b + 1\cdot[1-\pi^b]] + [1-\pi^a][3\cdot\pi^b + 0\cdot[1-\pi^b]] = \pi^a + 3\pi^b - 2\pi^a\pi^b$
- Differentiating:
  - $dE\upsilon^a / d\pi^a = 1 - 2\pi^b$
  - which is positive (resp. negative) if $\pi^b < ½$ (resp. $\pi^b > ½$)
- So *a*'s optimal strategy is $\pi^a = 1$ if $\pi^b < ½$, $\pi^a = 0$ if $\pi^b > ½$
- Similar reasoning for *b*
- Therefore mixed-strategy equilibrium is
  - $(\pi^a, \pi^b) = (½, ½)$
  - where payoffs are $(\upsilon^a, \upsilon^b) = (1½, 1½)$

Chicken: payoffs



- Space of utilities
- Two NEs in pure strategies
- utilities achievable by randomisation
- if utility is thrown away…
- Mixed-strategy NE
- Efficient outcomes
- An equitable solution?

- *Utility here must have <u>cardinal</u> significance*
- *Obtained by taking ½ each of the two pure-strategy NEs*
- *How can we get this?*

Chicken game: summary

- If the agents move sequentially then get a pure-strategy NE
  - outcome will be either (3,1) or (1,3)
  - depends on who moves first
- If move simultaneously: a coordination problem?
- Randomisation by the two agents?
  - independent action does not help much
  - produces payoffs (1½, 1½)
- But if they use *the same* randomisation device:
  - play [+] with *the same* probability π
  - expected payoff for each is $\upsilon^a = \pi + 3\pi - 2\pi^2 = 2\pi[1 - \pi]$
  - maximised where $\pi = ½$
- Appropriate randomisation seems to solve the coordination problem

Another application?

- Do mixed strategies this help solve Prisoner's Dilemma?
- A reexamination
  - again model as individuals' contribution to a public project
  - two players with binary choices: contribute [+], not-contribute [–]
  - close to standard public-good problem
- But payoff structure crucially different from "chicken"
  - if you contribute and the other doesn't, you get 0 the other gets 3
  - if both of you contribute, then you both get 2
  - if neither of you contribute, then you both get 1
- We know the outcome in pure strategies:
  - there's a NE ([–], [–])
  - but payoffs in NE are strictly dominated by those for ([+], [+])
- Now consider mixed strategy…

PD: mixed-strategy approach

- Again each player behaves probabilistically:
  - a plays [+] with probability π^a
  - b plays [+] with probability π^b
- Expected payoff to a is
  - $E\upsilon^a = \pi^a [2 \cdot \pi^b + 0 \cdot [1 - \pi^b]] + [1 - \pi^a][3 \cdot \pi^b + 1 \cdot [1 - \pi^b]] = 1 + 2\pi^b - \pi^a$
  - clearly $E\upsilon^a$ is decreasing in π^a
- Optimal strategies
  - from the above, a will set π^a to its minimum value, 0
  - by symmetry, b will also set π^b to 0
- So we are back to the non-cooperative solution :
  - $(\pi^a, \pi^b) = (0,0)$
  - both play [-] with certainty
- Mixed-strategy approach does not resolve the dilemma

Assessment

- Mixed strategy: a key development of game theory
  - closes a hole in the NE approach
  - but is it a theoretical artifice?
- Is mixed-strategy equilibrium an appropriate device?
  - depends on the context of the microeconomic model
  - degree to which it's plausible that agents observe and understand the use of randomisation
- Not the last word on equilibrium concepts
  - as extra depth added to the nature of game…
  - …new refinements of definition
- Example of further developments
  - introduction of time, in dynamic games
  - introduction of asymmetric information