Prerequisites

Almost essential
Firm: Optimisation

Useful, but optional
Firm: Demand and Supply

The Multi-Output Firm

MICROECONOMICS
Principles and Analysis
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Introduction

- This presentation focuses on analysis of firm producing more than one good
  - modelling issues
  - production function
  - profit maximisation
- For the single-output firm, some things are obvious:
  - the direction of production
  - returns to scale
  - marginal products
- But what of multi-product processes?
- Some rethinking required...?
  - nature of inputs and outputs?
  - tradeoffs between outputs?
  - counterpart to cost function?

Overview...

- The Multi-Output Firm
  - **Net outputs**
  - Production possibilities
  - Profit maximisation

A fundamental concept

Multi-product firm: issues

- "Direction" of production
  - Need a more general notation
- Ambiguity of some commodities
  - Is paper an input or an output?
- Aggregation over processes
  - How do we add firm 1's inputs and firm 2's outputs?

Net output

- Net output, written as q_i,
  - if **positive** denotes the amount of good i produced as output
  - if **negative** denotes the amount of good i used up as output
- Key concept
  - treat outputs and inputs symmetrically
  - offers a representation that is consistent
- Provides consistency
  - in aggregation
  - in "direction" of production

We just need some reinterpretation

Approaches to outputs and inputs

NET OUTPUTS
q_1
q_2
...
q_{n-1}
q_n

OUTPUT	INPUTS
	z_1
	z_2
	...
	z_m
q	

- *A standard "accounting" approach*
- *An approach using "net outputs"*
- *How the two are related*
- *A simple sign convention*

$$\begin{bmatrix} q_1 \\ q_2 \\ \cdots \\ q_{n-1} \\ q_n \end{bmatrix} = \begin{bmatrix} -z_1 \\ -z_2 \\ \cdots \\ -z_m \\ +q \end{bmatrix}$$

Outputs:	+	net additions to the stock of a good
Inputs:	−	reductions in the stock of a good

Aggregation

- Consider an industry with two firms
  - Let q_i^f be net output for firm *f* of good *i*, $f = 1,2$
  - Let q_i be net output for whole industry of good *i*
- How is total related to quantities for individual firms?
  - Just add up
  - $q_i = q_i^1 + q_i^2$
- Example 1: both firms produce *i* as output
  - q_i^1 **= 100,** q_i^2 **= 100**
  - q_i **= 200**
- Example 2: both firms use *i* as input
  - q_i^1 **= − 100,** q_i^2 **= − 100**
  - q_i **= − 200**
- Example 3: firm 1 produces *i* that is used by firm 2 as input
  - q_i^1 **= 100,** q_i^2 **= − 100**
  - q_i **= 0**

Net output: summary

- Sign convention is common sense
- If i is an output…
  - addition to overall supply of i
  - so sign is positive
- If i is an inputs
  - net reduction in overall supply of i
  - so sign is negative
- If i is a pure intermediate good
  - no change in overall supply of i
  - so assign it a zero in aggregate

Overview...

- The Multi-Output Firm
  - Net outputs
  - Production possibilities
  - Profit maximisation

A production function with many outputs, many inputs…

Rewriting the production function…

- Reconsider single-output firm example given earlier
  - goods 1,…,m are inputs
  - good m+1 is output
  - $n = m + 1$
- Conventional way of writing feasibility condition:
  - $q \leq \phi(z_1, z_2, \ldots, z_m)$
  - where ϕ is the production function
- Express this in net-output notation and rearrange:
  - $q_n \leq \phi(-q_1, -q_2, \ldots, -q_{n-1})$
  - $q_n - \phi(-q_1, -q_2, \ldots, -q_{n-1}) \leq 0$
- Rewrite this relationship as
  - $\Phi(q_1, q_2, \ldots, q_{n-1}, q_n) \leq 0$
  - where Φ is the implicit production function
- Properties of Φ are implied by those of ϕ…

The production function Φ

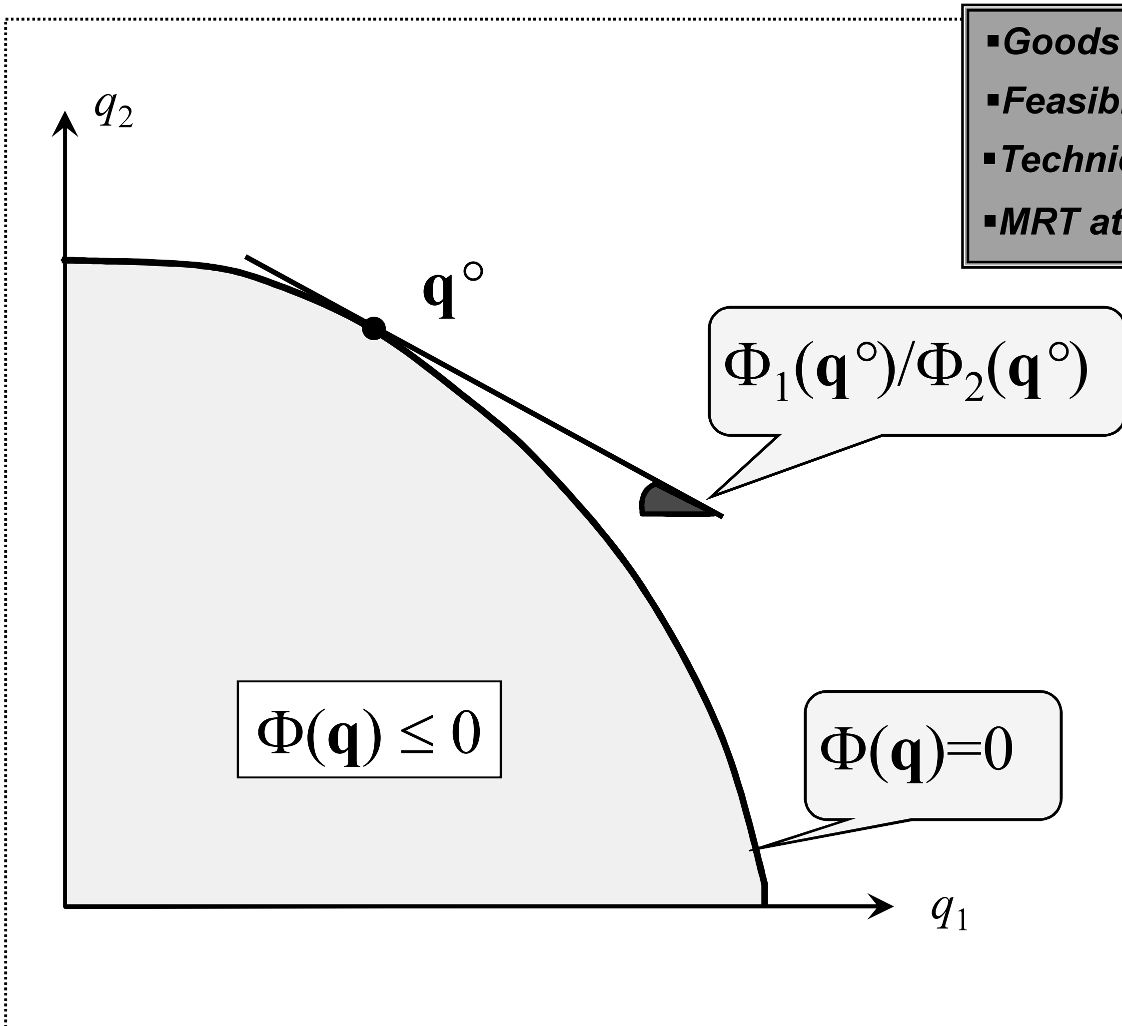
- Recall equivalence for single output firm:
  - $q_n - \phi(-q_1, -q_2, ..., -q_{n-1}) \leq 0$
  - $\Phi(q_1, q_2, ..., q_{n-1}, q_n) \leq 0$
- So, for this case:
  - Φ is increasing in $q_1, q_2, ..., q_n$
  - if ϕ is homogeneous of degree 1, Φ is homogeneous of degree 0
  - if ϕ is differentiable so is Φ
  - for any $i, j = 1,2,\ldots, n-1$ $\text{MRTS}_{ij} = \Phi_j(\mathbf{q})/\Phi_i(\mathbf{q})$
- It makes sense to generalise these…

The production function Φ (more)

- For a vector **q** of net outputs
  - **q** is feasible if $\Phi(\mathbf{q}) \leq 0$
  - **q** is technically efficient if $\Phi(\mathbf{q}) = 0$
  - **q** is infeasible if $\Phi(\mathbf{q}) > 0$
- For all feasible **q**:
  - $\Phi(\mathbf{q})$ is increasing in $q_1, q_2, \ldots, q_n$
  - if there is CRTS then Φ is homogeneous of degree 0
  - if ϕ is differentiable so is Φ
  - for any two inputs i, j, $\mathrm{MRTS}_{ij} = \Phi_j(\mathbf{q})/\Phi_i(\mathbf{q})$
  - for any two outputs i, j, the marginal rate of transformation of i into j is $\mathrm{MRT}_{ij} = \Phi_j(\mathbf{q})/\Phi_i(\mathbf{q})$
- Illustrate the last concept using the *transformation curve*…

Firm's transformation curve

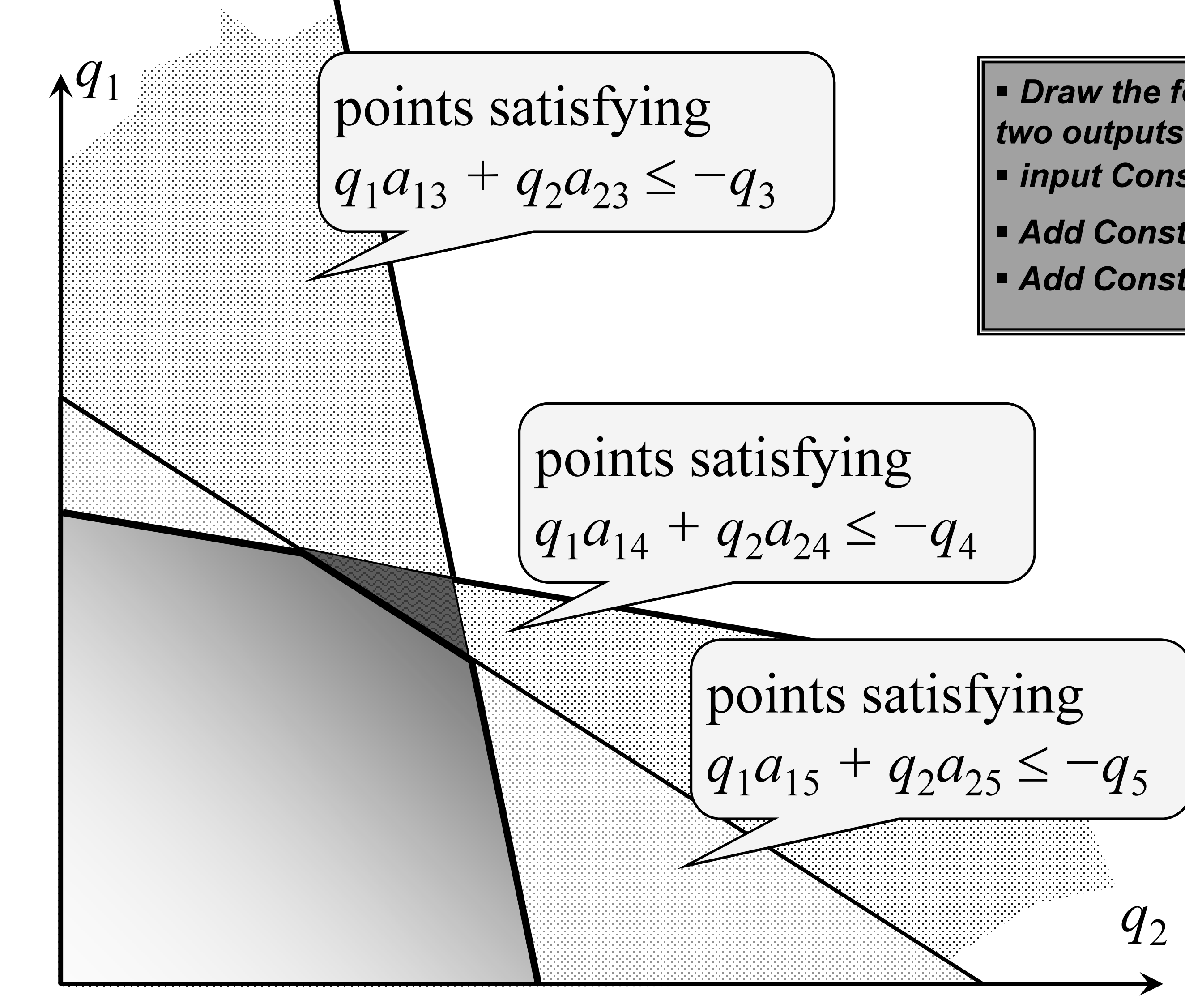
- *Goods 1 and 2 are outputs*
- *Feasible outputs*
- *Technically efficient outputs*
- *MRT at q°*



An example with five goods

- Goods 1 and 2 are outputs
- Goods 3, 4, 5 are inputs
- A linear technology
  - fixed proportions of each input needed for the production of each output:
  - $q_1 a_{1i} + q_2 a_{2i} \leq -q_i$
  - where a_{ji} is a constant $i = 3,4,5$, $j = 1,2$
  - given the sign convention $-q_i > 0$
- Take the case where inputs are fixed at some arbitrary values…

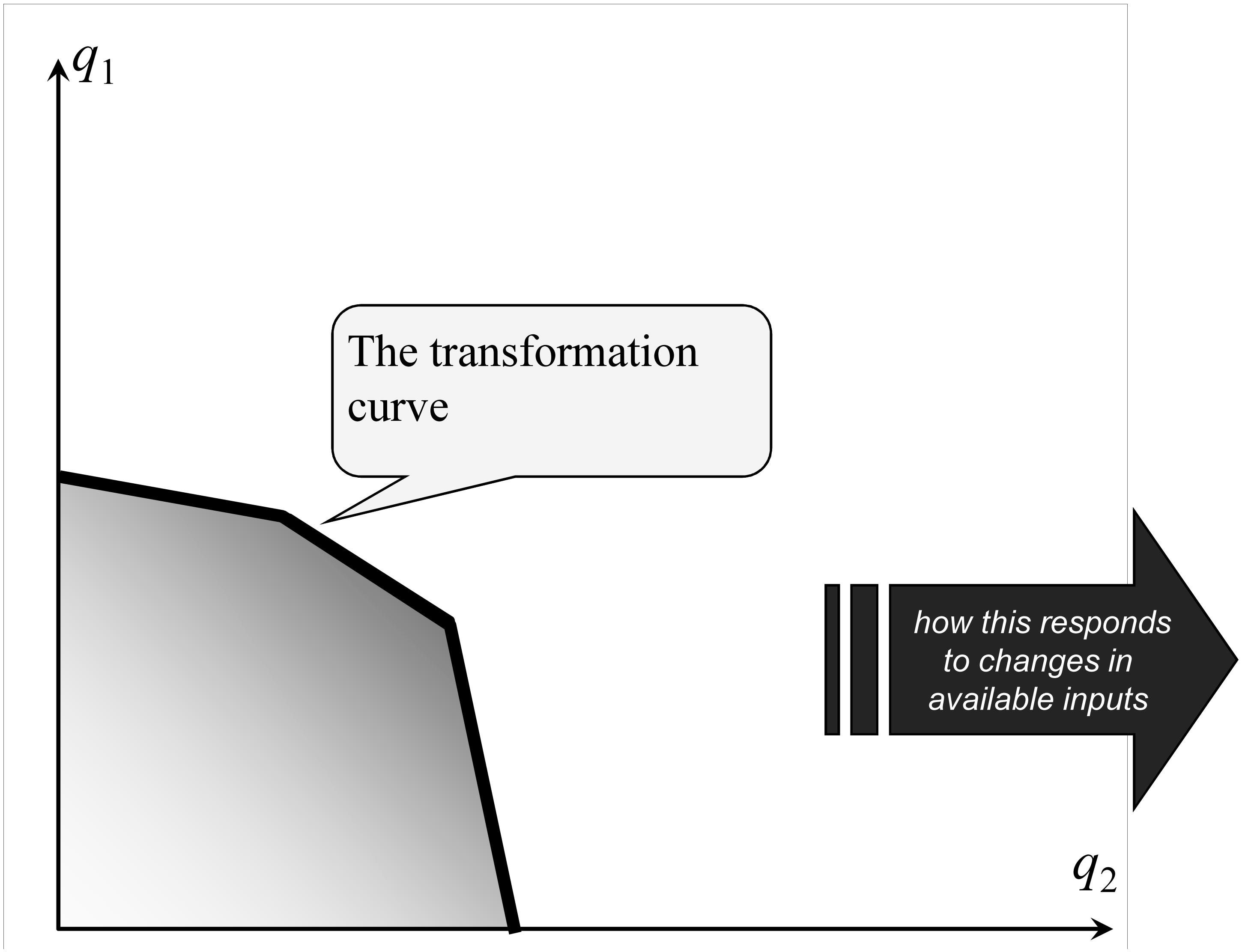
The three input constraints



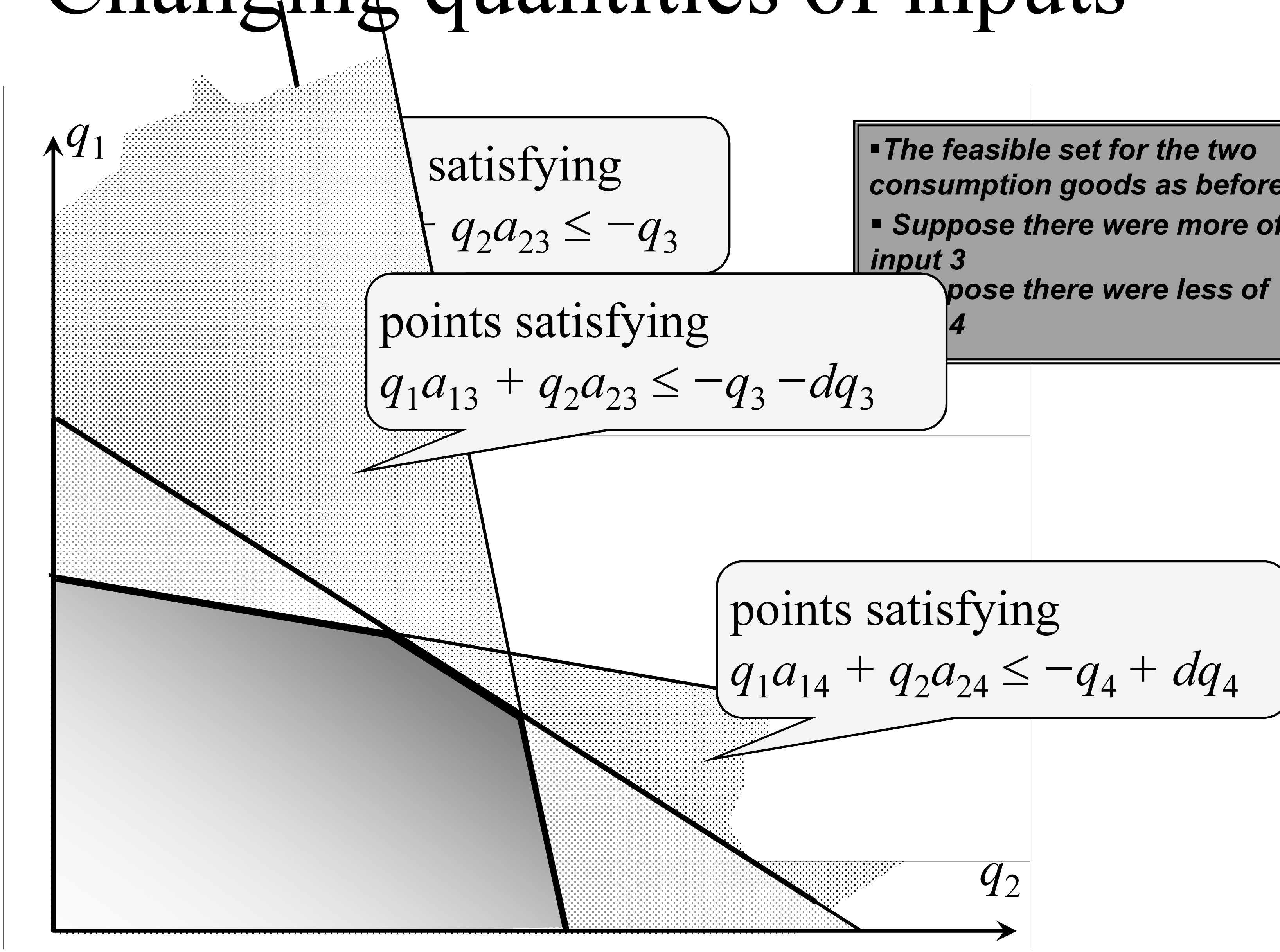
- ***Draw the feasible set for the two outputs:***
- ***input Constraint 3***
- ***Add Constraint 4***
- ***Add Constraint 5***

- *Intersection is the feasible set for the two outputs*

The resulting feasible set



Changing quantities of inputs



- *The feasible set for the two consumption goods as before:*
- *Suppose there were more of input 3*
- *...pose there were less of ...4*

Overview...

Integrated approach to optimisation

- The Multi-Output Firm
  - Net outputs
  - Production possibilities
  - **Profit maximisation**

Profits

- The basic concept is (of course) the same
  - Revenue – Costs
- But we use the concept of net output
  - this simplifies the expression
  - exploits symmetry of inputs and outputs
- Consider an "accounting" presentation…

Accounting with net outputs

- Suppose goods 1,...,m are inputs and goods m+1 to n are outputs

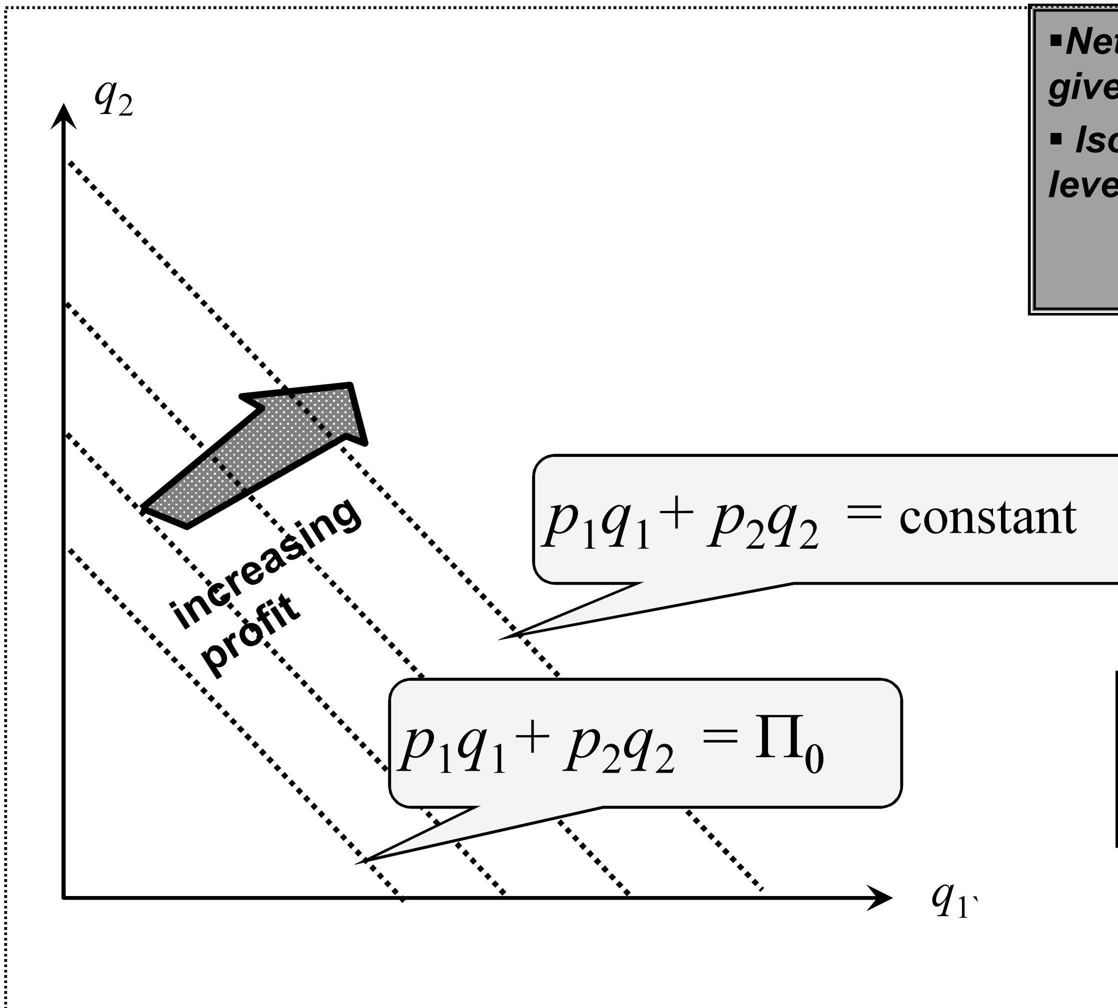
$$\sum_{i=m+1}^{n} p_i q_i$$ **Revenue**

$$-\sum_{i=1}^{m} p_i [-q_i]$$ **– Costs**

$$\sum_{i=1}^{n} p_i q_i$$ **= Profits**

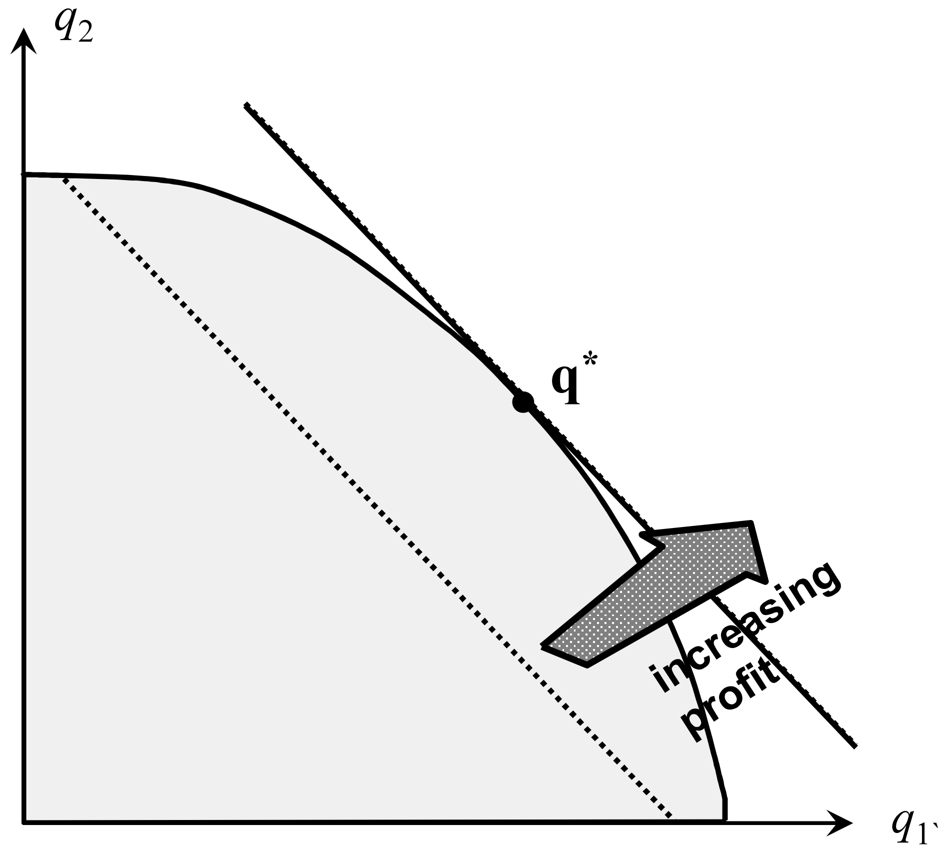
- ***Cost of inputs (goods 1,...,m)***
- ***Revenue from outputs (goods m+1,...,n)***
- ***Subtract cost from revenue to get profits***

Iso-profit lines...



- ***Net-output vectors yielding a given Π_0.***
- ***Iso-profit lines for higher profit levels.***

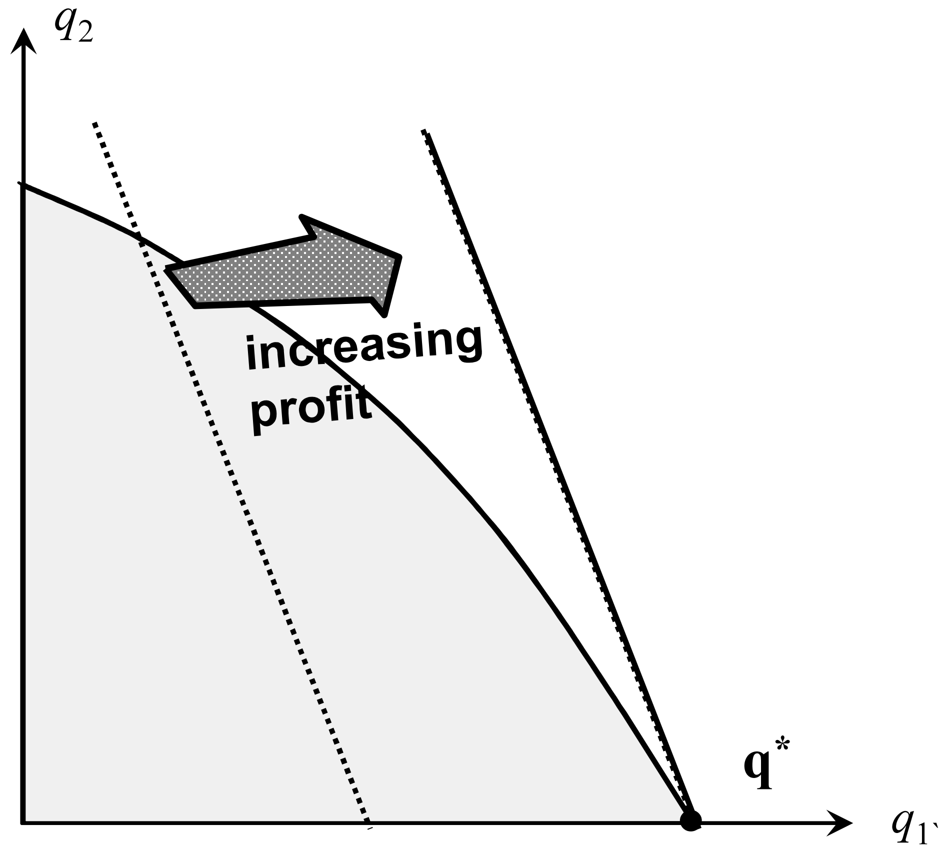
Profit maximisation: multi-product firm (1)



- *Feasible outputs*
- *Isoprofit line*
- *Maximise profits*
- *Profit-maximising output*
- *MRTS at profit-maximising output*

- *Here* $q_1^*>0$ *and* $q_2^*>0$
- **q*** *is technically efficient*
- *Slope at* **q*** *equals price ratio*

Profit maximisation: multi-product firm (2)



- *Feasible outputs*
- *Isoprofit line*
- *Maximise profits*
- *Profit-maximising output*
- *MRTS at profit-maximising output*

- *Here* $q_1^*>0$ *but* $q_2^* = 0$
- $\mathbf{q}^*$ *is technically efficient*
- *Slope at* $\mathbf{q}^*$ $\leq$ *price ratio*

Maximising profits

- Problem is to choose **q** so as to maximise

$$\sum_{i=1}^{n} p_i q_i \quad \text{subject to} \quad \Phi(\mathbf{q}) \leq 0$$

- Lagrangean is

$$\sum_{i=1}^{n} p_i q_i - \lambda \Phi(\mathbf{q})$$

- FOC for an interior maximum is
  - $p_i - \lambda \Phi_i(\mathbf{q}) = 0$

Maximised profits

- Introduce the *profit function*
  - the solution function for the profit maximisation problem

$$\Pi(\mathbf{p}) = \max_{\{\Phi(\mathbf{q}) \leq 0\}} \sum_{i=1}^{n} p_i q_i = \sum_{i=1}^{n} p_i q_i^*$$

- Works like other solution functions:
  - non-decreasing
  - homogeneous of degree 1
  - continuous
  - convex
- Take derivative with respect to p_i :
  - $\Pi_i(\mathbf{p}) = q_i^*$
  - write q_i^* as net supply function
  - $q_i^* = q_i(\mathbf{p})$

Summary

- Three key concepts
- Net output
  - simplifies analysis
  - key to modelling multi-output firm
  - easy to rewrite production function in terms of net outputs
- Transformation curve
  - summarises tradeoffs between outputs
- Profit function
  - counterpart of cost function