## MICROECONOMICS
*Principles and Analysis*

# THE MULTI-OUTPUT FIRM

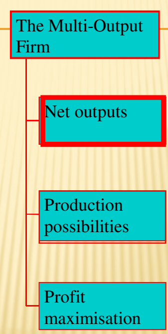


## INTRODUCTION

- This presentation focuses on analysis of firm producing more than one good
  - modelling issues
  - production function
  - profit maximisation
- For the single-output firm, some things are obvious:
  - the direction of production
  - returns to scale
  - marginal products
- But what of multi-product processes?
- Some rethinking required...?
  - nature of inputs and outputs?
  - tradeoffs between outputs?
  - counterpart to cost function?



## OVERVIEW...




*A fundamental concept*



## MULTI-PRODUCT FIRM: ISSUES

- “Direction” of production
  - Need a more general notation
- Ambiguity of some commodities
  - Is paper an input or an output?
- Aggregation over processes
  - How do we add firm 1’s inputs and firm 2’s outputs?

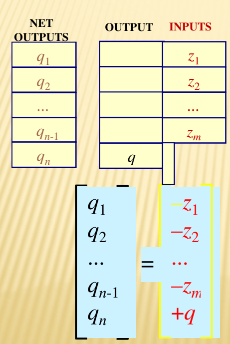


## NET OUTPUT

- Net output, written as $q_i$,
  - if **positive** denotes the amount of good *i* produced as output
  - if **negative** denotes the amount of good *i* used up as output
- Key concept
  - treat outputs and inputs symmetrically
  - offers a representation that is consistent
- Provides consistency
  - in aggregation
  - in “direction” of production



## APPROACHES TO OUTPUTS AND INPUTS




- *A standard “accounting” approach*
- *An approach using “net outputs”*
- *How the two are related*
- *A simple sign convention*

| | | |
|---|---|---|
| Outputs: | + | net additions to the stock of a good |
| Inputs: | – | reductions in the stock of a good |

## AGGREGATION

- Consider an industry with two firms
  - Let $q_i^f$ be net output for firm *f* of good *i*, $f = 1,2$
  - Let $q_i$ be net output for whole industry of good *i*
- How is total related to quantities for individual firms?
  - Just add up
  - $q_i = q_i^1 + q_i^2$
- Example 1: both firms produce *i* as output
  - $q_i^1 = 100, q_i^2 = 100$
  - $q_i = 200$
- Example 2: both firms use *i* as input
  - $q_i^1 = -100, q_i^2 = -100$
  - $q_i = -200$
- Example 3: firm 1 produces *i* that is used by firm 2 as input
  - $q_i^1 = 100, q_i^2 = -100$
  - $q_i = 0$

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## NET OUTPUT: SUMMARY

- Sign convention is common sense
- If *i* is an output…
  - addition to overall supply of *i*
  - so sign is positive
- If *i* is an inputs
  - net reduction in overall supply of *i*
  - so sign is negative
- If *i* is a pure intermediate good
  - no change in overall supply of *i*
  - so assign it a zero in aggregate

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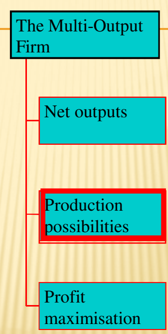
## OVERVIEW…




*A production function with many outputs, many inputs…*

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## REWRITING THE PRODUCTION FUNCTION…

- Reconsider single-output firm example given earlier
  - goods 1,…,*m* are inputs
  - good *m*+1 is output
  - $n = m + 1$
- Conventional way of writing feasibility condition:
  - $q \leq \phi(z_1, z_2, \ldots, z_m)$
  - where $\phi$ is the production function
- Express this in net-output notation and rearrange:
  - $q_n \leq \phi(-q_1, -q_2, \ldots, -q_{n-1})$
  - $q_n - \phi(-q_1, -q_2, \ldots, -q_{n-1}) \leq 0$
- Rewrite this relationship as
  - $\Phi(q_1, q_2, \ldots, q_{n-1}, q_n) \leq 0$
  - where $\Phi$ is the implicit production function
- Properties of $\Phi$ are implied by those of $\phi$…

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## THE PRODUCTION FUNCTION $\Phi$

- Recall equivalence for single output firm:
  - $q_n - \phi(-q_1, -q_2, \ldots, -q_{n-1}) \leq 0$
  - $\Phi(q_1, q_2, \ldots, q_{n-1}, q_n) \leq 0$
- So, for this case:
  - $\Phi$ is increasing in $q_1, q_2, \ldots, q_n$
  - if $\phi$ is homogeneous of degree 1, $\Phi$ is homogeneous of degree 0
  - if $\phi$ is differentiable so is $\Phi$
  - for any $i, j = 1,2,\ldots, n-1$ $\mathrm{MRTS}_{ij} = \Phi_j(\mathbf{q})/\Phi_i(\mathbf{q})$
- It makes sense to generalise these…

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## THE PRODUCTION FUNCTION $\Phi$ (MORE)

- For a vector **q** of net outputs
  - **q** is feasible if $\Phi(\mathbf{q}) \leq 0$
  - **q** is technically efficient if $\Phi(\mathbf{q}) = 0$
  - **q** is infeasible if $\Phi(\mathbf{q}) > 0$
- For all feasible **q**:
  - $\Phi(\mathbf{q})$ is increasing in $q_1, q_2, \ldots, q_n$
  - if there is CRTS then $\Phi$ is homogeneous of degree 0
  - if $\phi$ is differentiable so is $\Phi$
  - for any two inputs *i*, *j*, $\mathrm{MRTS}_{ij} = \Phi_j(\mathbf{q})/\Phi_i(\mathbf{q})$
  - for any two outputs *i*, *j*, the marginal rate of transformation of *i* into *j* is $\mathrm{MRT}_{ij} = \Phi_j(\mathbf{q})/\Phi_i(\mathbf{q})$
- Illustrate the last concept using the *transformation curve*…

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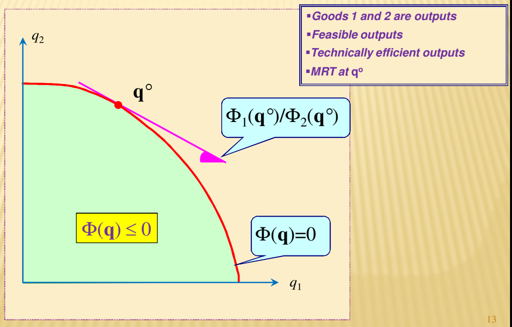
## FIRM'S TRANSFORMATION CURVE

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## AN EXAMPLE WITH FIVE GOODS

- Goods 1 and 2 are outputs
- Goods 3, 4, 5 are inputs
- A linear technology
  - fixed proportions of each input needed for the production of each output:
  - $q_1 a_{1i} + q_2 a_{2i} \le -q_i$
  - where $a_{ji}$ is a constant $i$ = 3,4,5, $j$ = 1,2
  - given the sign convention $-q_i > 0$
- Take the case where inputs are fixed at some arbitrary values…

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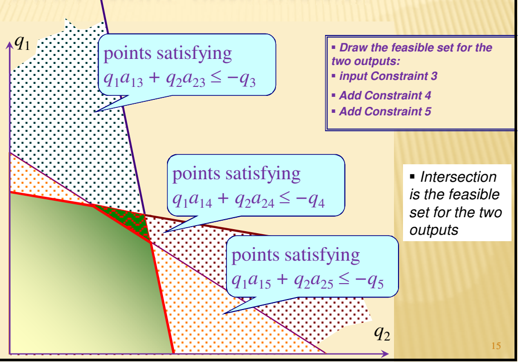
## THE THREE INPUT CONSTRAINTS

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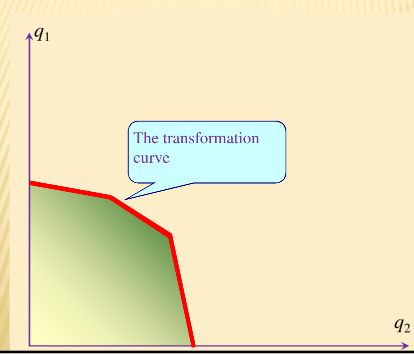
## THE RESULTING FEASIBLE SET

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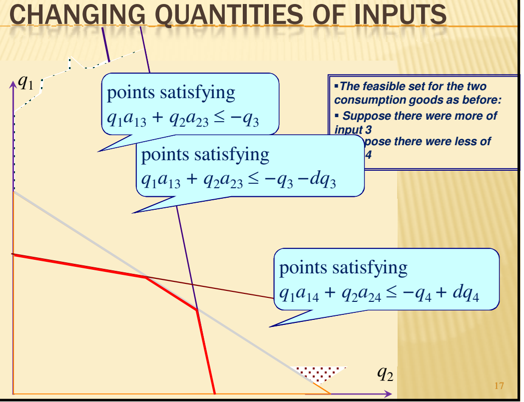
## CHANGING QUANTITIES OF INPUTS

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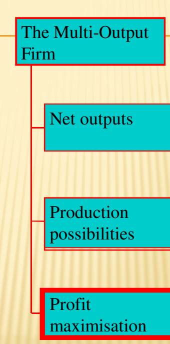
## OVERVIEW…

*Integrated approach to optimisation*

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## PROFITS

- The basic concept is (of course) the same
  - Revenue – Costs
- But we use the concept of net output
  - this simplifies the expression
  - exploits symmetry of inputs and outputs
- Consider an "accounting" presentation…

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## ACCOUNTING WITH NET OUTPUTS

- Suppose goods 1,…,$m$ are inputs and goods $m$+1 to $n$ are outputs

$$\sum_{i=m+1}^{n} p_i q_i \qquad \textbf{Revenue}$$

$$-\sum_{i=1}^{m} p_i [-q_i] \qquad \textbf{Costs}$$

$$\sum_{i=1}^{n} p_i q_i \qquad = \textbf{Profits}$$

- Cost of inputs (goods 1,…,m)
- Revenue from outputs (goods m+1,…,n)
- Subtract cost from revenue to get profits

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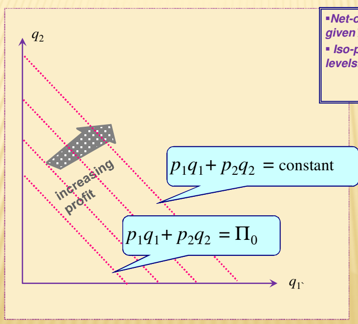
## ISO-PROFIT LINES…




- Net-output vectors yielding a given $\Pi_0$.
- Iso-profit lines for higher profit levels.

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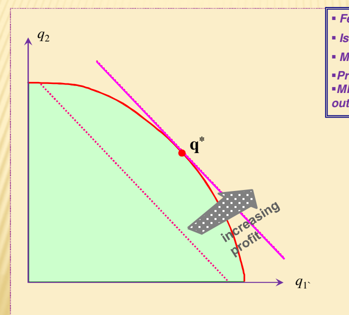
## PROFIT MAXIMISATION: MULTI-PRODUCT FIRM (1)




- Feasible outputs
- Isoprofit line
- Maximise profits
- Profit-maximising output
- MRTS at profit-maximising output

- Here $q_1^*>0$ and $q_2^*>0$
- **q*** is technically efficient
- Slope at **q*** equals price ratio

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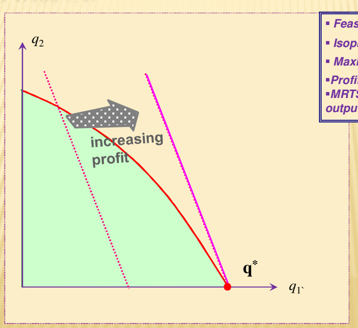
## PROFIT MAXIMISATION: MULTI-PRODUCT FIRM (2)




- Feasible outputs
- Isoprofit line
- Maximise profits
- Profit-maximising output
- MRTS at profit-maximising output

- Here $q_1^*>0$ but $q_2^* = 0$
- **q*** is technically efficient
- Slope at **q*** ≤ price ratio

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## MAXIMISING PROFITS

- Problem is to choose **q** so as to maximise

$$\sum_{i=1}^{n} p_i q_i \quad \text{subject to} \quad \Phi(\mathbf{q}) \le 0$$

- Lagrangean is

$$\sum_{i=1}^{n} p_i q_i - \lambda\,\Phi(\mathbf{q})$$

- FOC for an interior maximum is
  - $p_i - \lambda\,\Phi_i(\mathbf{q}) = 0$

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## MAXIMISED PROFITS

- Introduce the *profit function*
  - the solution function for the profit maximisation problem

$$\Pi(\mathbf{p}) = \max_{\{\Phi(\mathbf{q})\leq 0\}} \sum_{i=1}^{n} p_i q_i = \sum_{i=1}^{n} p_i q_i^*$$

- Works like other solution functions:
  - non-decreasing
  - homogeneous of degree 1
  - continuous
  - convex
- Take derivative with respect to $p_i$ :
  - $\Pi_i(\mathbf{p}) = q_i^*$
  - write $q_i^*$ as net supply function
  - $q_i^* = q_i(\mathbf{p})$

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## SUMMARY

- Three key concepts
- Net output
  - simplifies analysis
  - key to modelling multi-output firm
  - easy to rewrite production function in terms of net outputs
- Transformation curve
  - summarises tradeoffs between outputs
- Profit function
  - counterpart of cost function

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