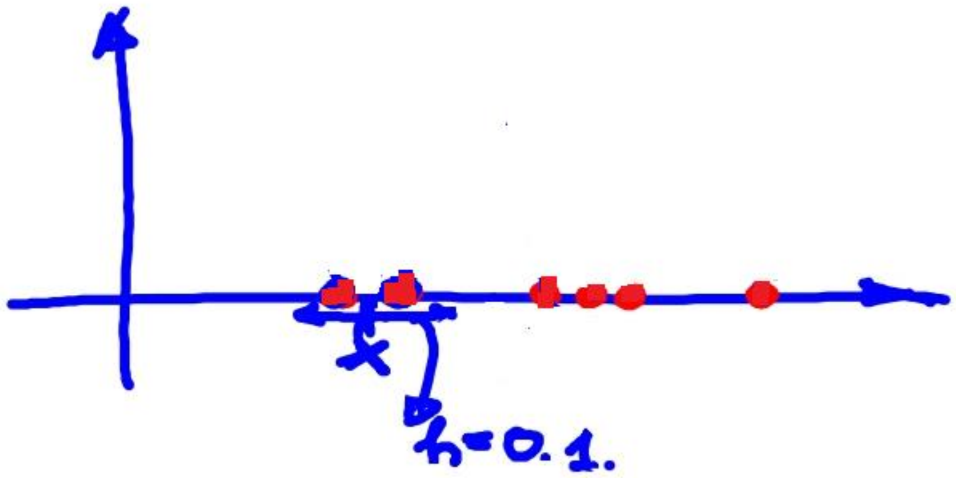
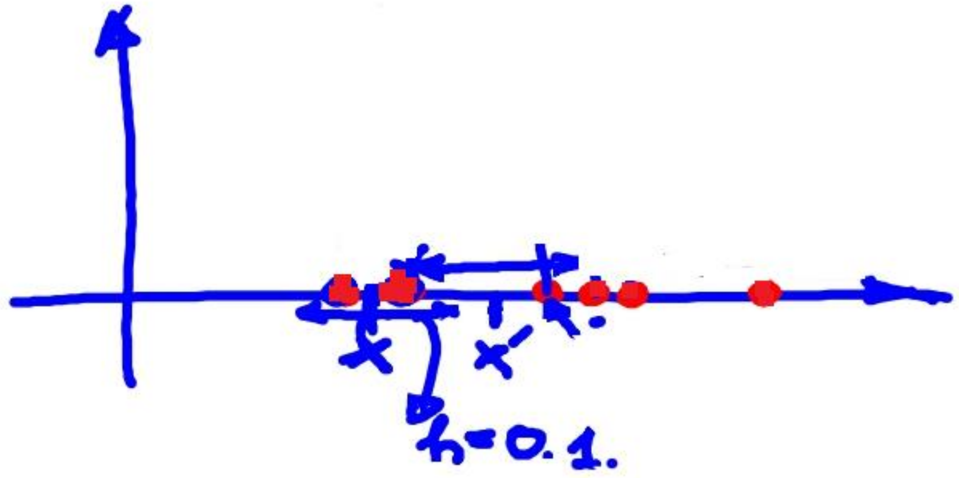
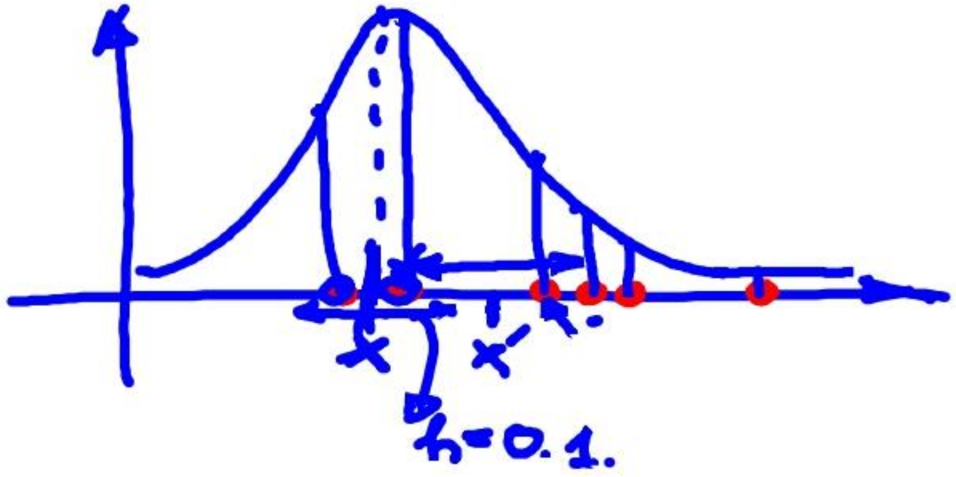
Parzen windows



$$\hat{p}(x) = \frac{2}{6 \cdot 0.1} = \frac{20}{6} = 3.33$$





Maximum likelihood example 2.

$$p(x) = \frac{1}{(2\pi)^{1/2}\sigma} \cdot \exp\left[-\frac{(x-\theta)^2}{2\sigma^2}\right].$$

$x_1, x_2, \ldots, x_N$

Log likelihood:

$$\Pi = \ln\left[\prod_{i=1}^{N}\left[\frac{1}{(2\pi)^{1/2}\sigma}\exp\left[-\frac{(x_i-\theta)^2}{2\sigma^2}\right]\right]\right]$$

$$= -N\ln\left[(2\pi)^{1/2}\sigma\right] - \sum_{i=1}^{N}\frac{(x_i-\theta)^2}{2\sigma^2}. = C - N\ln\sigma - \sum_{i=1}^{N}\frac{(x_i-\theta)^2}{2\sigma^2}$$

$$\frac{\partial \Pi}{\partial \sigma} = 0 \Rightarrow -\frac{N}{\sigma} + \sum_{i=1}^{N} \frac{(x_i - \theta)^2}{\sigma^3} = 0 \Rightarrow$$

$$\boxed{\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \theta)^2}$$

$$\frac{\partial^2 \Pi}{\partial \sigma^2} = +\frac{N}{\sigma^2} - 3\left[\frac{\sum_i (x_i - \theta)^2}{\sigma^4}\right]$$

$$= +\frac{N}{\sigma^2}\left[1 - 3\frac{\cancel{\hat{\sigma}^2}}{\cancel{\sigma^2}}\right] = -\frac{2N}{\sigma^2} < 0$$

Maximum likelihood, example 3.

Gamma distribution.

$$p(x \mid a, b) = \frac{x^{a-1}}{\Gamma(a)\, b^{a}} \exp\left[-\frac{x}{b}\right].$$

$$\Gamma(z) = \int_0^\infty e^{t}\, t^{z-1}\, dt$$

$$x_1, x_2, \ldots, x_N$$

Likelihood:

$$\prod_{i=1}^{N} \frac{x_i^{a-1}}{\Gamma(a)\, b^{a}} \exp\left(-\frac{x_i}{b}\right)$$

$$\boxed{\begin{aligned} &\ln(cd) = \ln c + \ln d. \\ &\ln(c^{d}) = d \ln c. \end{aligned}}$$

Log likelihood:

$$\mathcal{L} = (a-1)\sum_i \ln x_i - N \ln \Gamma(a) - Na \ln b - \frac{1}{b}\sum_i x_i$$

$$\frac{\partial \mathcal{L}}{\partial b} = 0 \Rightarrow -\frac{Na}{b} + \frac{1}{b^2}\sum_i x_i = 0$$

$$\Rightarrow \quad a = \frac{\sum_i x_i}{N b} \Rightarrow b = \frac{\overline{X}}{a}.$$

$$\Pi = N(\alpha-1)\overline{\ln x} - N\ln\Gamma(\alpha) - \alpha N - \alpha N \ln\bar{x} + \alpha N \ln\alpha$$

$$\boxed{\frac{\partial \Pi}{\partial \alpha} = 0} \Longrightarrow \text{find } \alpha.$$

For example: $X = \{1,2,3,4,5,6,7,8\}$

$\bar{x} = 4.5$ $\qquad\qquad$ $\alpha \simeq 3$

$\overline{\ln x} = 1.3256.$ $\qquad\qquad$ $b \simeq 1.5.$

MAP example.

Gaussian.

$$p(\theta|X) = p(X|\theta)\, p(\theta) / p(X)$$

Likelihood.

$$p(X|\theta)\, p(\theta) = \frac{1}{(2\pi)^{\frac{N+1}{2}} \sigma_n^N \sigma_0} \exp\left[-\sum_{i=1}^{N} \frac{(x_i-\theta)^2}{2\sigma_n^2}\right] \exp\left[-\frac{(\theta-\theta_0)^2}{2\sigma_0^2}\right]$$

$$= \frac{1}{(2\pi)^{\frac{N+1}{2}} \sigma_n^N \sigma_0} \exp\underbrace{\left[-\sum_{i=1}^{N} \frac{(x_i-\theta)^2}{2\sigma_n^2} - \frac{(\theta-\theta_0)^2}{2\sigma_0^2}\right]}_{A}$$

$$A = \frac{-\theta^2 + 2\theta\theta_0 - \theta_0^2}{2\sigma_0^2} - \sum_i \frac{x_i^2 - 2\theta x_i + \theta^2}{2\sigma_n^2}$$

$$= \frac{-\theta^2\sigma_n^2 + 2\theta\theta_0\sigma_n^2 - \theta_0^2\sigma_n^2 - \sum_i \left[x_i^2\sigma_0^2 - 2\theta x_i\sigma_0^2 + \theta^2\sigma_0^2\right]}{2\sigma_0^2\sigma_n^2}.$$

Numerator of A:

$$-\theta^2\left(\sigma_n^2 + N\sigma_0^2\right) + 2\theta\left(\theta_0\sigma_n^2 + \sigma_0^2\sum_i x_i\right) + C_0$$

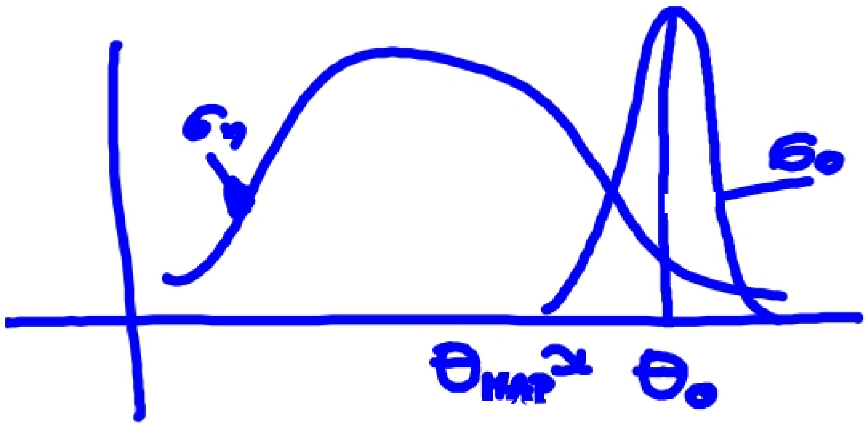
$$A = \frac{-\theta^2 + \dfrac{2\theta\left(\theta_0 \sigma_n^2 + \sigma_0^2 \sum_i x_i\right)}{\sigma_n^2 + N\sigma_0^2} + C}{\dfrac{2\sigma_n^2 \sigma_0^2}{\sigma_n^2 + N\sigma_0^2}}$$

$$-\frac{(\theta - k)^2}{2\Lambda^2}$$

Completing the square.

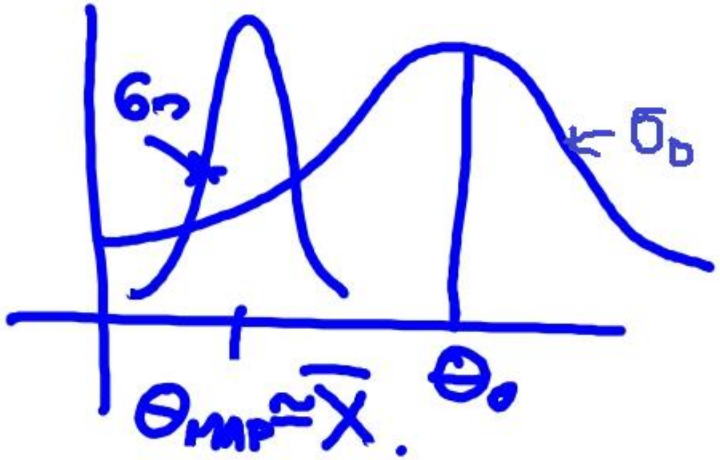
$$\frac{dA}{d\theta} = 0 \quad \Rightarrow -2\theta + \frac{2\left(\theta_0 \sigma_n^2 + \sigma_0^2 \sum_i x_i\right)}{\sigma_n^2 + N\sigma_0^2} = 0$$

$$\theta_{MAP} = \frac{N\sigma_0^2 \overline{X} + \sigma_n^2 \theta_0}{N\sigma_0^2 + \sigma_n^2}$$



$$N \to \infty \implies \theta_{MAP} \to \frac{N\sigma_0^2 \overline{X}}{N\sigma_0^2} = \overline{X}.$$

$$N = 0 \implies \theta_{MAP} = \frac{\sigma_n^2 \theta_0}{\sigma_n^2} = \theta_0.$$



$$\sigma_n \gg \sigma_0 \implies \theta_{MAP} \simeq \theta_0.$$

$$\sigma_n \ll \sigma_0 \implies \theta_{MAP} \simeq \overline{X}$$

Full Bayesian inference:

I add and subtract $\dfrac{\left[\theta_0 \sigma_n^2 + \sigma_0^2 \sum_i x_i\right]^2}{(\sigma_n^2 + N\sigma_0^2)^2}$ to the numerator of A in order to complete the square.

$$A \sim \frac{-\left(\theta - \frac{\theta_0 \sigma_n^2 + \sigma_0^2 \sum_i x_i}{\sigma_n^2 + N\sigma_0^2}\right)^2}{2\frac{\sigma_n^2 \sigma_0^2}{\sigma_n^2 + N\sigma_0^2}} + C$$

variance mean.

Estimate of θ:

$$\theta_N = \frac{N\sigma_0^2 \bar{x} + \sigma_n^2 \theta_0}{N\sigma_0^2 + \sigma_n^2} \rightarrow$$

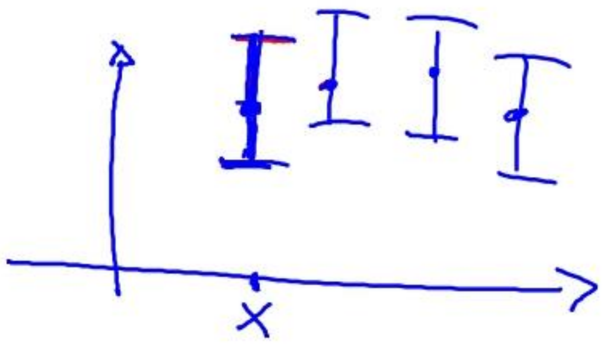
Identical to MAP.

"uncertainty"

variance

$$\sigma_N^2 = \left[\frac{\sigma_n^2 \sigma_0^2}{\sigma_n^2 + N\sigma_0^2} \right].$$

$p(x|X)$



$$p(x|\mathcal{X}) = \int p(x|\theta)\, p(\theta|\mathcal{X})\, d\theta$$

$$= C \int e^{-\frac{(x-\theta)^2}{2\sigma_n^2}}\, e^{-\frac{(\theta-\theta_N)^2}{2\sigma_N^2}}\, d\theta$$