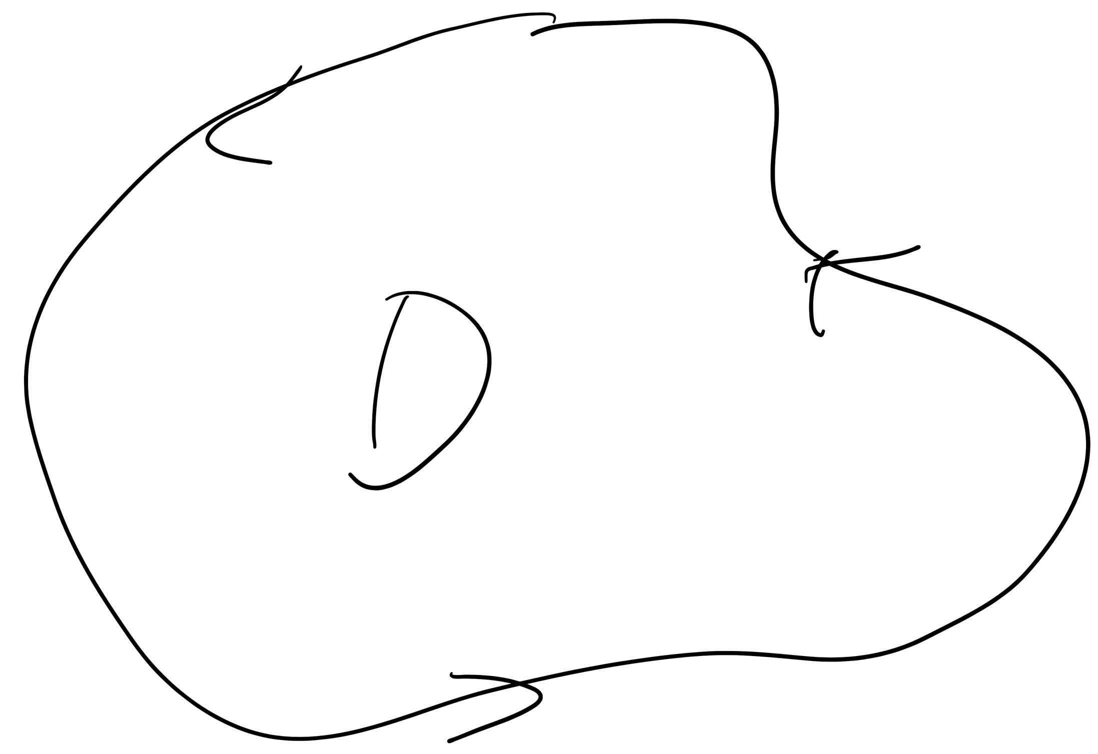
Θεωρημα Green

$$\int_{\partial D} P(x,y)\,dx + Q(x,y)\,dy$$

$$= \iint_D (Q_x - P_y)\,dx\,dy$$



$$\int_{\partial D} (P,Q)\cdot ds$$

$$\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y}$$

$$\int P\,dx + Q\,dy = \iint_D \left(\frac{\partial}{\partial x} Q - \frac{\partial}{\partial y} P\right) dx\,dy$$

18. Let D be a region for which Green's theorem holds. Suppose f is harmonic; that is,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

on D. Prove that

$$\partial_x \qquad \partial_y$$

$$\int_{\partial D} \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = 0.$$

$$= \iint_D \left(-\partial_{xx} f - \partial_{yy} f\right) dx\,dy = 0$$

Θεώρημα D απλό χωρίο.
Το εμβαδόν του ισούται με

$$A(D) = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$$

$$= \int_{\partial D} x\,dy = -\int_{\partial D} y\,dx$$

Απόδ.

$$\int_{\partial D} x\,dy = \int_{\partial D} \overset{\partial_x}{0}\cdot dx + \overset{\partial_y}{x}\,dy$$

$$= \iint_D (\partial_x x - \partial_y 0)\,dx\,dy = \iint_D 1\,dx\,dy$$

$= \text{εμβαδόν}(D)$

$$-\int_{\partial D} y\,dx = -\int_{\partial D} \overset{\partial_x}{y}\,dx + \overset{\partial_y}{0}\,dy =$$

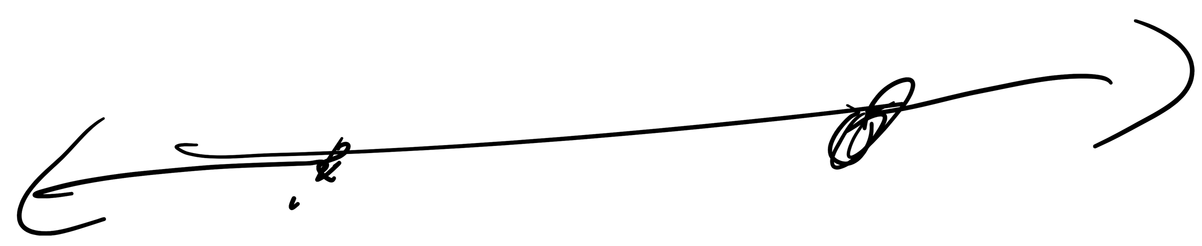
$$= - \iint_D (0 - \partial_y y)\, dx\,dy = \iint_D 1\, dx\,dy$$

$$= \text{εμβαδόν}\,(D)$$

$$\int_a^b f'\, dx = f(b) - f(a) \leftarrow \int_{\{a,b\}} f$$

$$\int_D d\omega = \int_{\partial D} \omega$$

Θ. Stokes



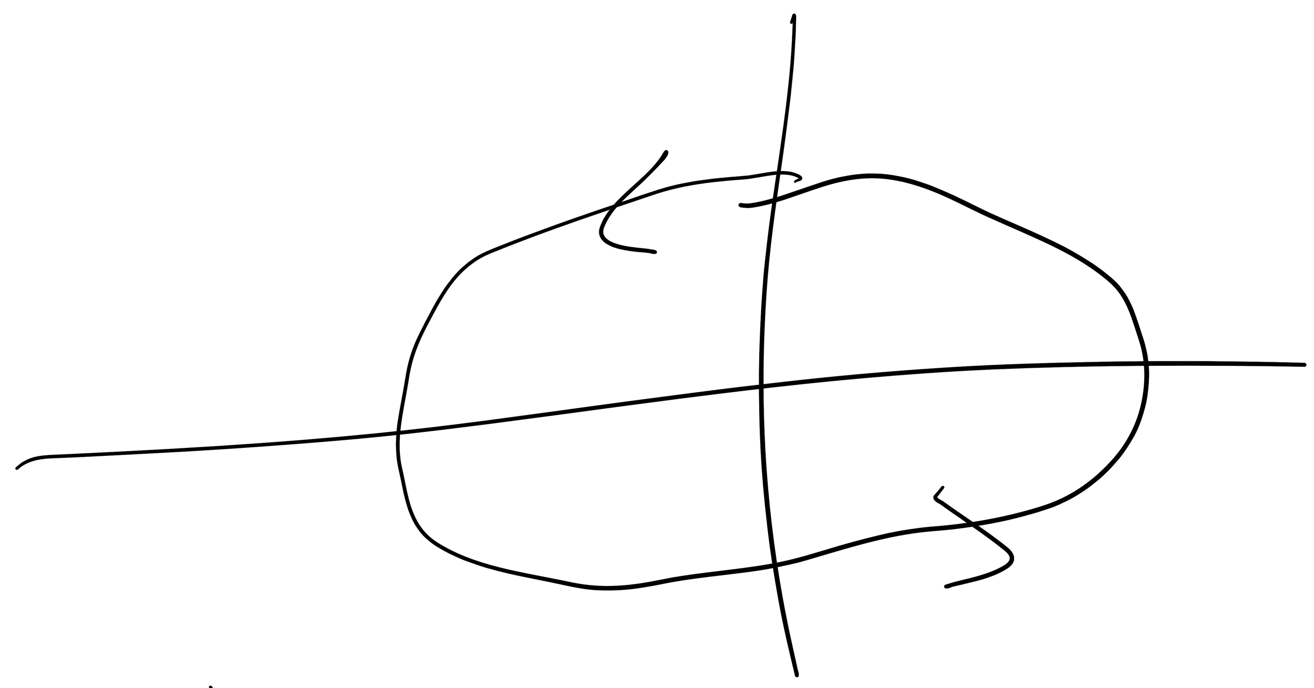
Marsden–Tromba σελ 486 . Θεωρ. 14

Άσκηση Να βρεθεί το εμβαδόν

α) έλλειψης $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$a, b > 0$

Λύση

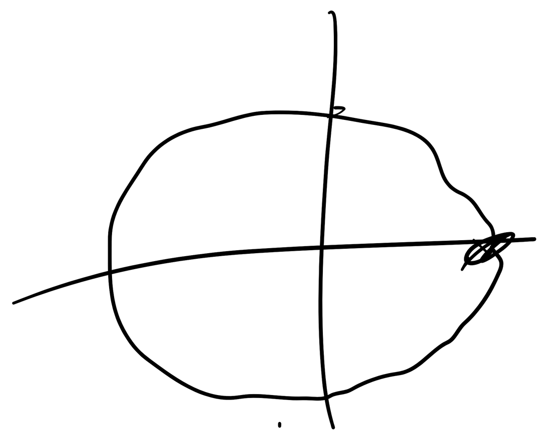


$$D = \left\{ (x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}$$

$$2A(D) = \int_{\partial D} -y\,dx + x\,dy$$

$$= \int_{\partial D} (-y, x) \cdot ds$$

$x^2 + y^2 = 1$



$$\gamma : [0, 2\pi] \to \mathbb{R}^2$$

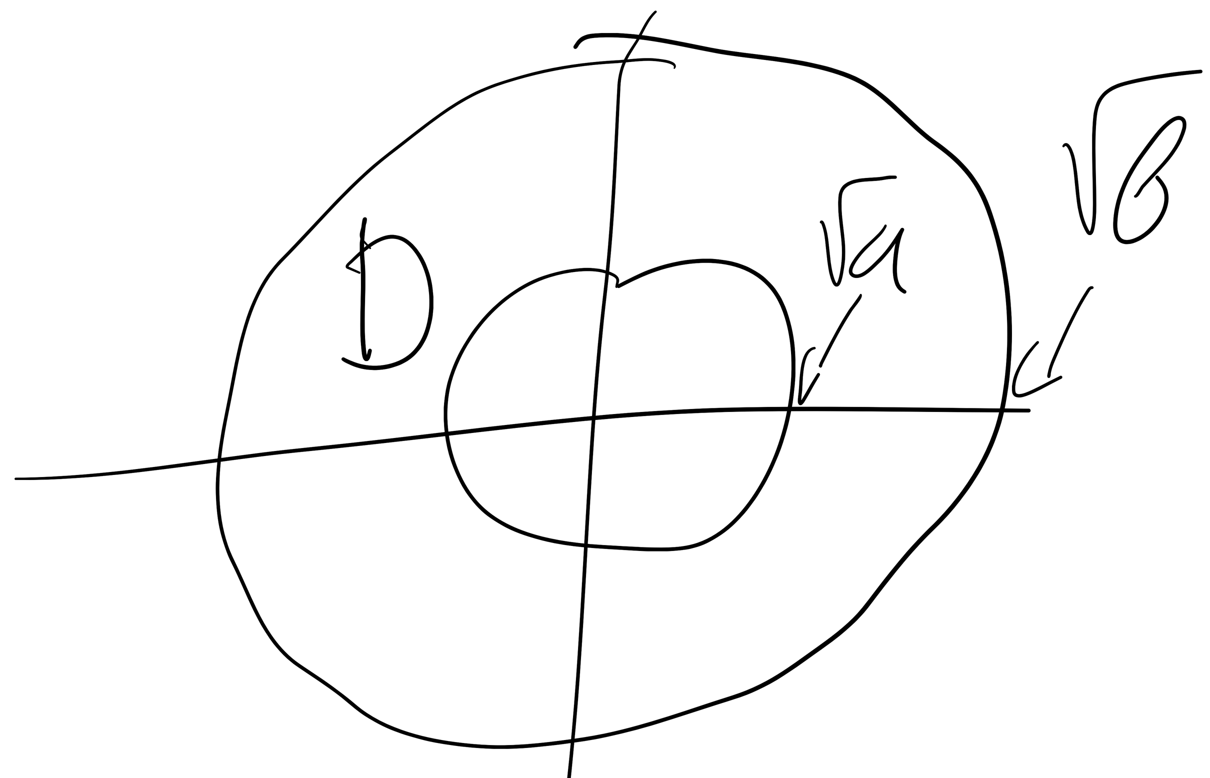
$$\gamma(t) = (a\cos t, b\sin t)$$

$$2A(D) = \int_0^{2\pi} (-b\sin t, a\cos t) \cdot (-a\sin t, b\cos t)\,dt = \int_0^{2\pi} ab\,dt$$

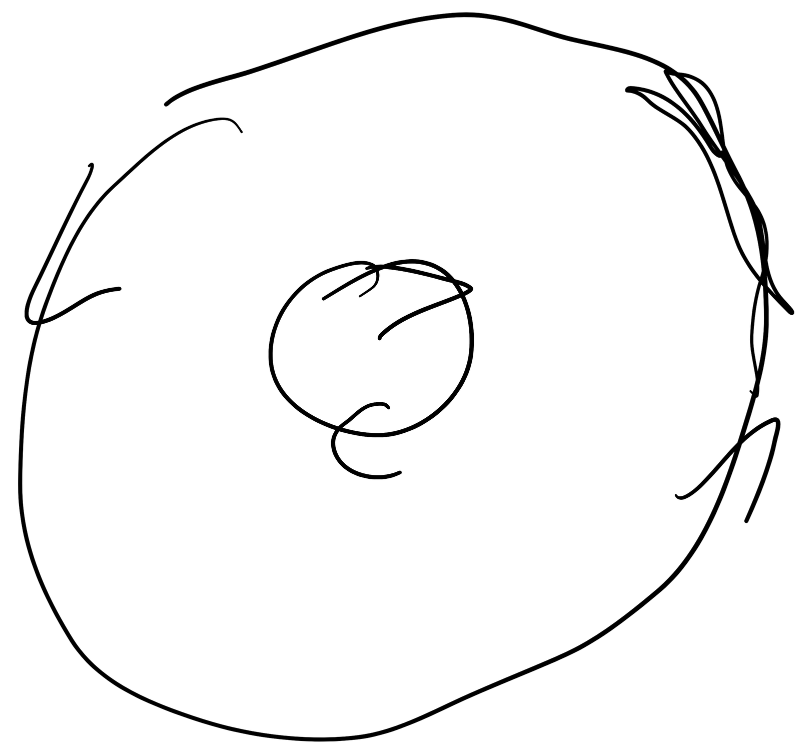
$$= 2\pi ab \Rightarrow A(D) = \pi ab$$

17. Verify Green's theorem for the integrand of Exercise 15 (that is, with $P = 2x^3 - y^3$ and $Q = x^3 + y^3$) and the annular region D described by $a \leq x^2 + y^2 \leq b$, with boundaries oriented as in Figure 8.1.5.

$$\int_{\partial D} P\,dx + Q\,dy \quad G$$



$$= \iint_D (Q_x - P_y)\,dx\,dy$$



$$= \iint_D \left(3x^2 - (-3y^2)\right) dx\,dy$$

$$= 3 \underbrace{\iint_D (x^2+y^2)\,dx\,dy}_{=I}$$

$$A = \{(r,\theta) : r \in [\sqrt{a}, \sqrt{b}], \theta \in [0, 2\pi]\}$$

$$T : A \to D$$

$$T(r,\theta) = (r\cos\theta, r\sin\theta)$$

$$I = \iint_D f(x,y)\,dx\,dy = \iint_{T^{-1}(D)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$= \iint_A r^2 r\,dr\,d\theta = \int_0^{2\pi}\int_{\sqrt{a}}^{\sqrt{b}} r^3\,dr\,d\theta$$

$$= 2\pi \frac{r^4}{4} \Big|_{\sqrt{a}}^{\sqrt{b}} = \frac{\pi}{2}(b^2 - a^2)$$

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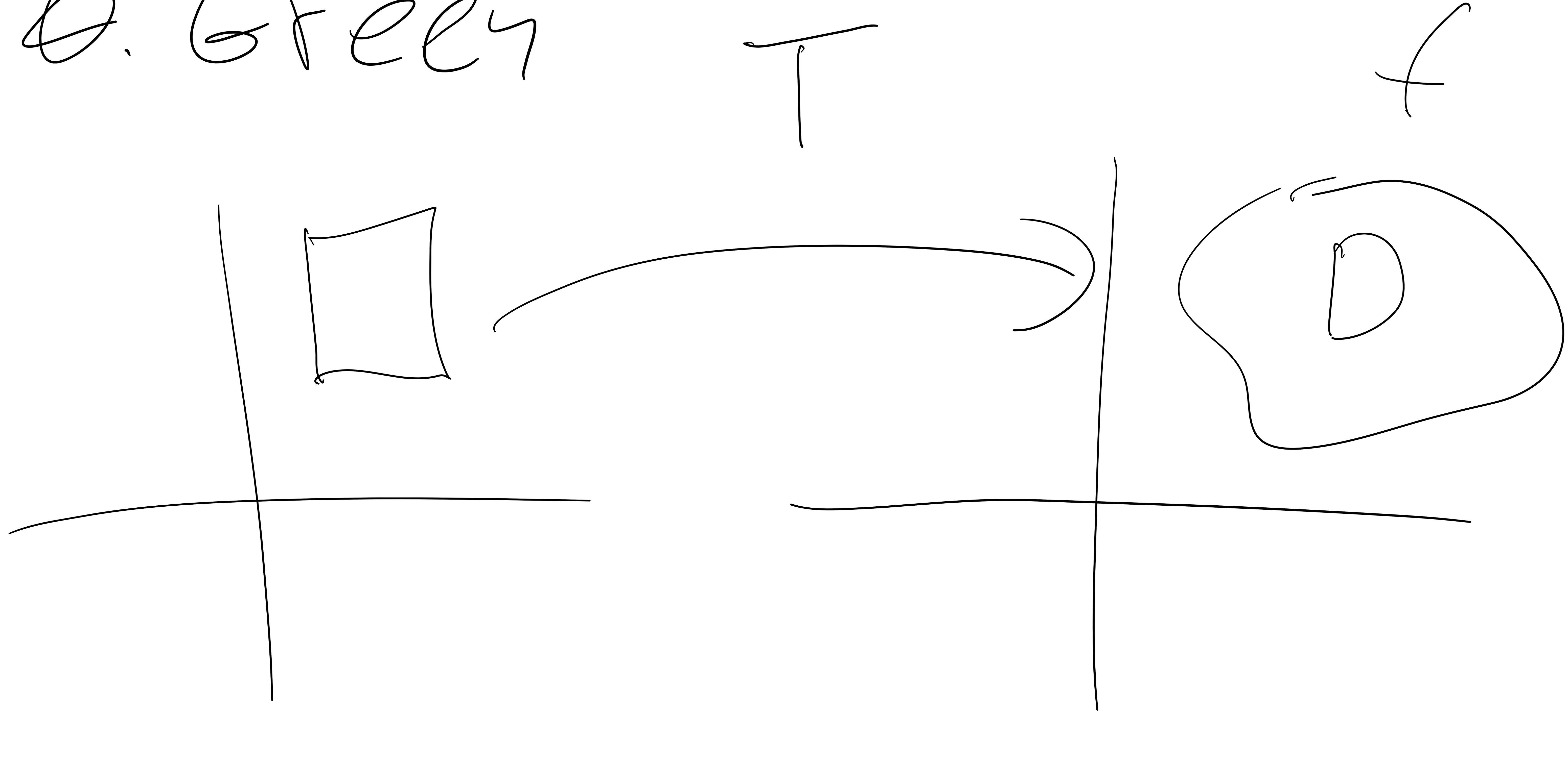
1. Ευθείες, επίπεδα, εξωτερικό γινόμενο
2. Υπολογισμός μερικών παραγώγων (κανόνας αλυσίδας)
3. Ακρότατα
4. Διπλά, τριπλά ολοκληρώματα (Fubini, Αλλαγή μεταβλητής)
5. Επικαμπύλια ολοκληρώματα
6. Θ. Green



$$\iint_D f = \iint_{T^{-1}(D)} f(T(u,v))\, J_T \, du\, dv$$

<u>Άσκηση</u> Να βρεθούν οι ∂_{xy}, ∂_x^2, ∂_y^2 των

α) $g(x,y) = x^2 y + \sin(xy)$

β) $s(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$\tan: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

Λύση

α) $\partial_x g = 2xy + \cos(xy) \cdot y$

$\partial_{xx} g = 2y - \sin(xy) \cdot y^2$

$\partial_{xy} g = 2x + \cos(xy)$

$- y \sin(xy)\, x$

β) $\partial_y s = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$

$$\partial_{yy} S = x\left(-\frac{2y}{(x^2+y^2)^2}\right) \qquad \frac{d}{dt}(\tan^{-1}(t)) = \frac{1}{1+t^2}$$

$$= -\frac{2xy}{(x^2+y^2)^2}$$

2. (2 μον.) Έστω $\varphi : \mathbb{R}^2 \to \mathbb{R}$ μία συνάρτηση με συνεχείς μερικές παραγώγους δεύτερης τάξης. Θέτουμε $f(x,y) := \varphi\ (x+y, x-y)$, $(x,y) \in \mathbb{R}^2$.

(α) Δείξτε ότι

$$\left[\frac{\partial f}{\partial x}(x,y)\right] \cdot \left[\frac{\partial f}{\partial y}(x,y)\right] = \left[\frac{\partial \varphi}{\partial x}(x+y, x-y)\right]^2 - \left[\frac{\partial \varphi}{\partial y}(x+y, x-y)\right]^2.$$

(β) Δείξτε ότι

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 \varphi}{\partial x^2}(x+y, x-y) - \frac{\partial^2 \varphi}{\partial y^2}(x+y, x-y).$$

Λύση

Έστω u, v τα ορίσματα της φ. $\varphi(u,v)$

Θέτουμε $u(x,y) = x+y$

$v(x,y) = x-y$

α) $f(x,y) = \varphi(\underbrace{u(x,y), v(x,y)})$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial \varphi}{\partial u}(\ ) \cdot \frac{\partial u}{\partial x}(x,y) + \frac{\partial \varphi}{\partial v}(\ ) \cdot \frac{\partial v}{\partial x}$$

$$= \frac{\partial \varphi}{\partial u}(\ ,\ ) + \frac{\partial \varphi}{\partial v}(\ ,\ )$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial \varphi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial \varphi}{\partial u} - \frac{\partial \varphi}{\partial v}$$

$$\frac{\partial f}{\partial x}(x,y)\cdot\frac{\partial f}{\partial y}(x,y) = \left(\frac{\partial \varphi}{\partial u}(x+y, x-y)\right)^2 - \left(\frac{\partial \varphi}{\partial v}(x+y, x-y)\right)^2$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial \varphi}{\partial u}\left(u(x,y), v(x,y)\right) + \frac{\partial \varphi}{\partial v}\left(u(x,y), v(x,y)\right)$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial^2 \varphi}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 \varphi}{\partial v \partial u}\frac{\partial v}{\partial y}$$

$$+ \frac{\partial^2 \varphi}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 \varphi}{\partial v^2}\frac{\partial v}{\partial y}$$

$$= \frac{\partial^2 \varphi}{\partial u^2} - \frac{\partial^2 \varphi}{\partial v \partial u} + \frac{\partial^2 \varphi}{\partial u \partial v} - \frac{\partial^2 \varphi}{\partial v^2}$$

$$= \frac{\partial^2 \varphi}{\partial u^2}(x+y, x-y) - \frac{\partial^2 \varphi}{\partial v^2}(x+y, x-y)$$

Άσκηση A) Να υπολογιστεί το

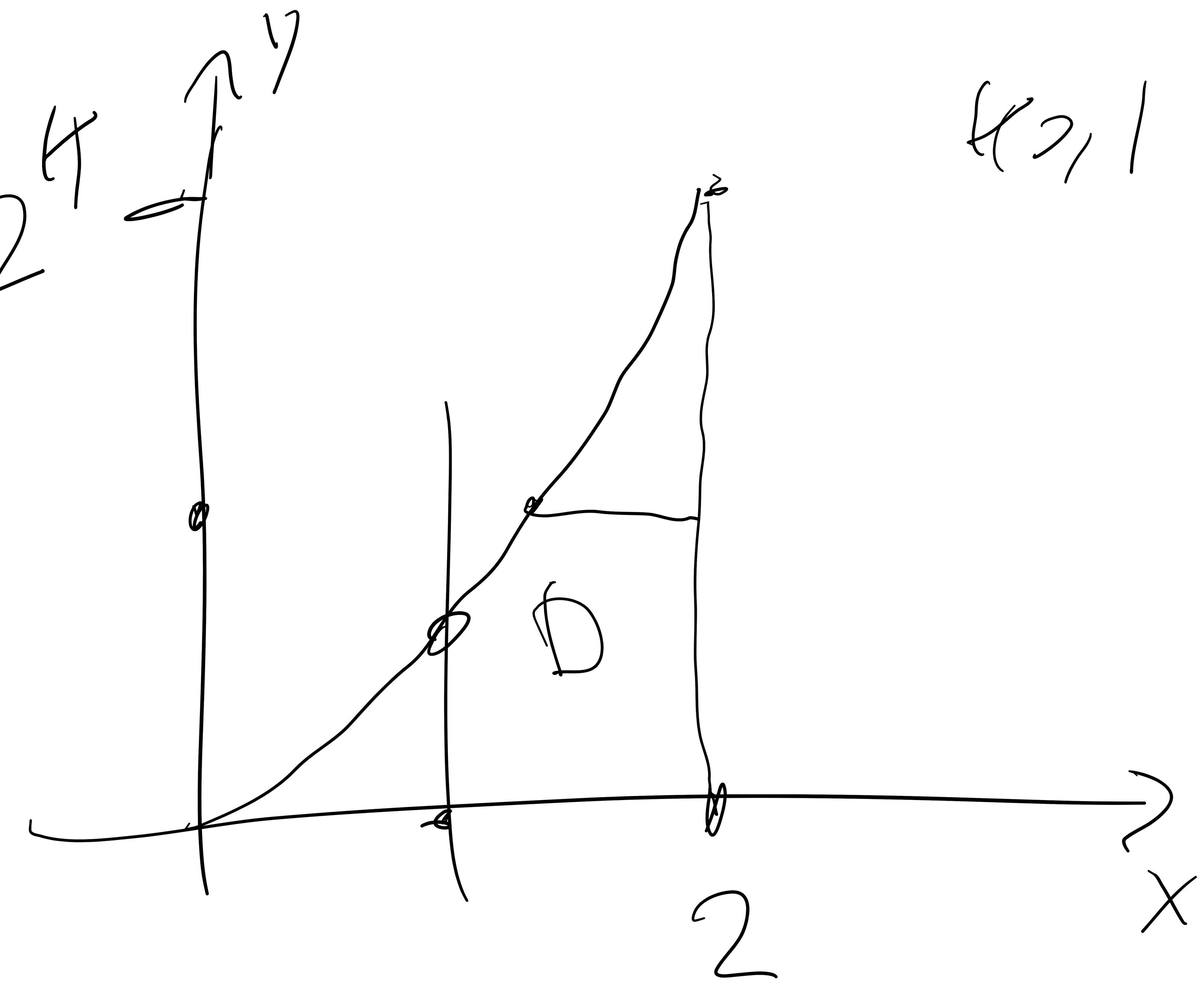
$$I=\int_0^{2^{\kappa}}\int_{\sqrt[\kappa]{y}}^{2} e^{x^{\kappa+1}}\,dx\,dy \quad , \kappa\in\mathbb{R}$$

B) Ποιος ο όγκος του χωρίου κάτω από το γράφημα της $z=\kappa(1-\sqrt{x^2+y^2})$ και μεταξύ των επιπέδων $z=0$, $z=\kappa$.

Λύση

$\kappa \geq 1$

$$y^{\frac{1}{\kappa}} \leq x < 2$$



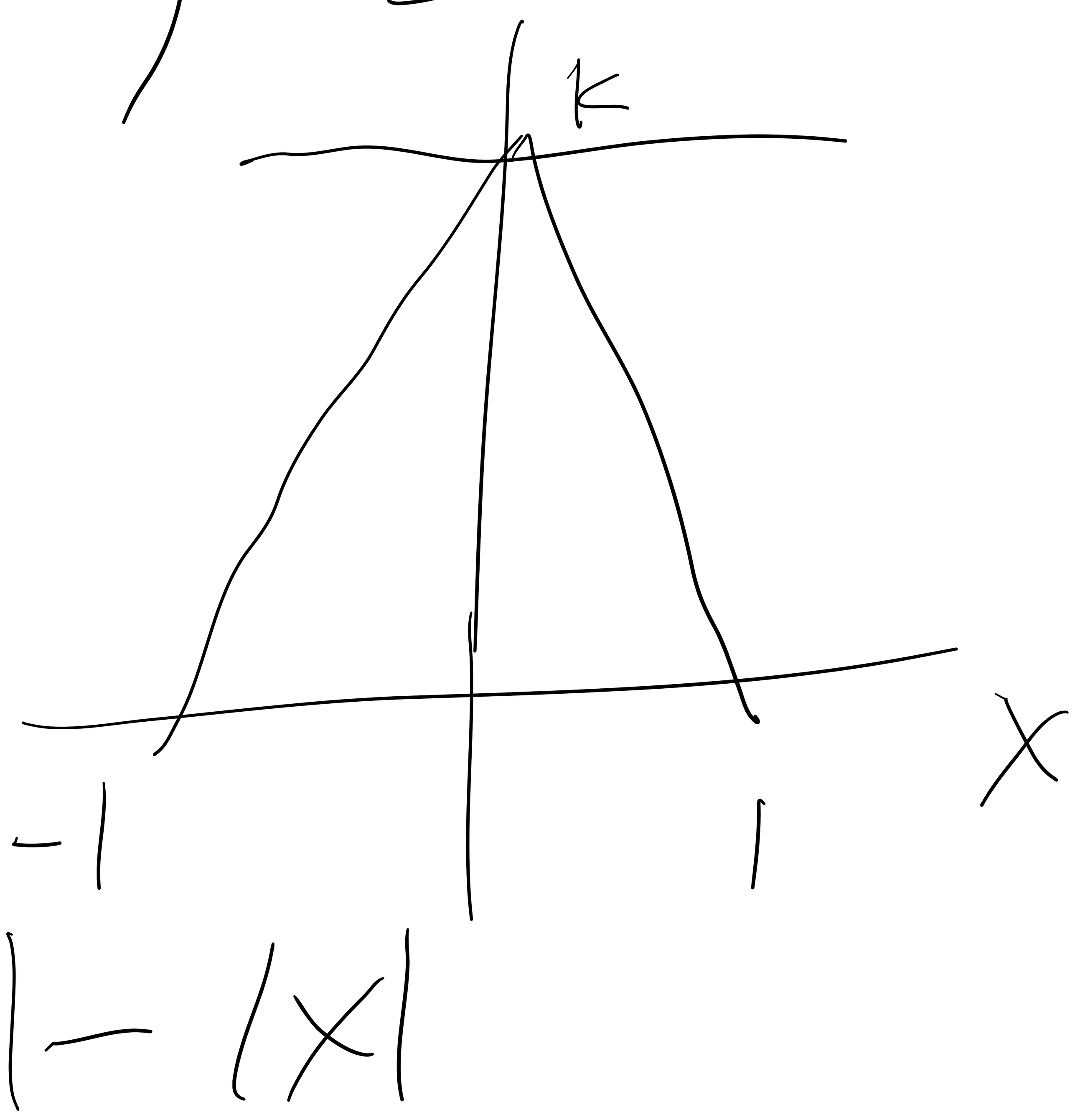
$$D=\{(x,y):\ 0\leq x\leq 2,\ 0\leq y\leq x^{\kappa}\}$$

$\kappa \neq -1$

$$I=\int_0^2\int_0^{x^{\kappa}} e^{x^{\kappa+1}}\,dy\,dx$$

$$=\int_0^2 e^{x^{\kappa+1}}x^{\kappa}\,dx=\frac{1}{\kappa+1}\int_0^2\left(e^{x^{\kappa+1}}\right)'dx$$

$$= \frac{1}{k+1}\left(e^{2^{k+1}} - e^{0}\right)$$

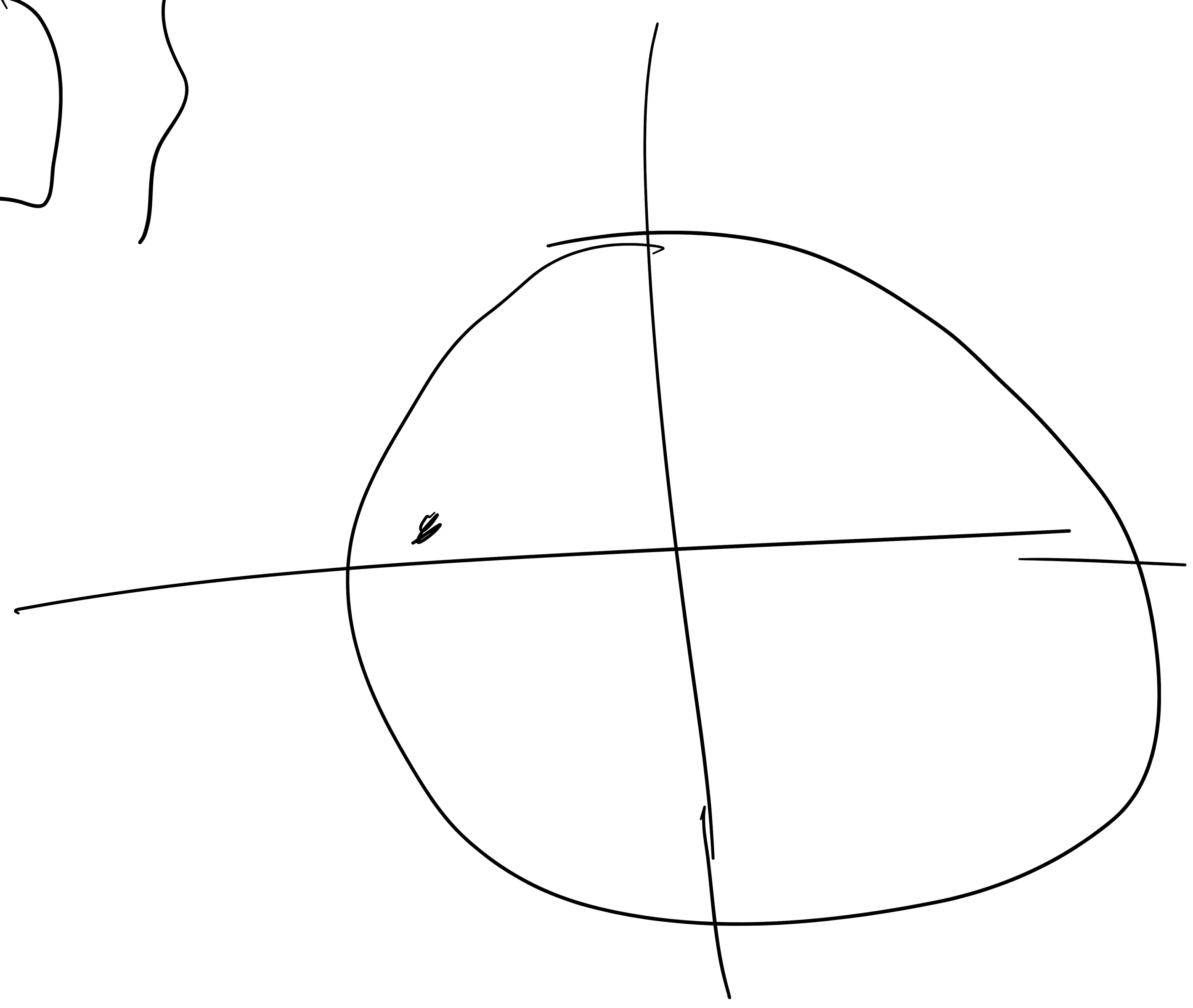


B) Το χωρίο είναι το

$$D = \{ (x,y,z) : x^2+y^2 \le 1,\ 0 \le z \le k\left(1-\sqrt{x^2+y^2}\right) \}$$

$$\text{όγκος}(D) = \iiint_D 1 \, dx\,dy\,dz$$

$$G = \{ (r,\theta,z) : r \in [0,1],\ \theta \in [0,2\pi],\ z \in [0, k(1-r)] \}$$



όγκος (D)

$$= \iiint_D 1 \, dx\,dy\,dy =$$

$$\iiint_G r \, dr\,d\theta\,dz = \int_0^1 \int_0^{2\pi} \int_0^{k(1-r)} r \, dz\,d\theta\,dr$$

$$= \int_0^1 \int_0^{2\pi} r\, k(1-r)\, d\theta\, dr =$$

$$= 2\pi k \int_0^1 r(1-r)\, dr = 2\pi k\left(\frac{1}{2} - \frac{1}{3}\right)$$

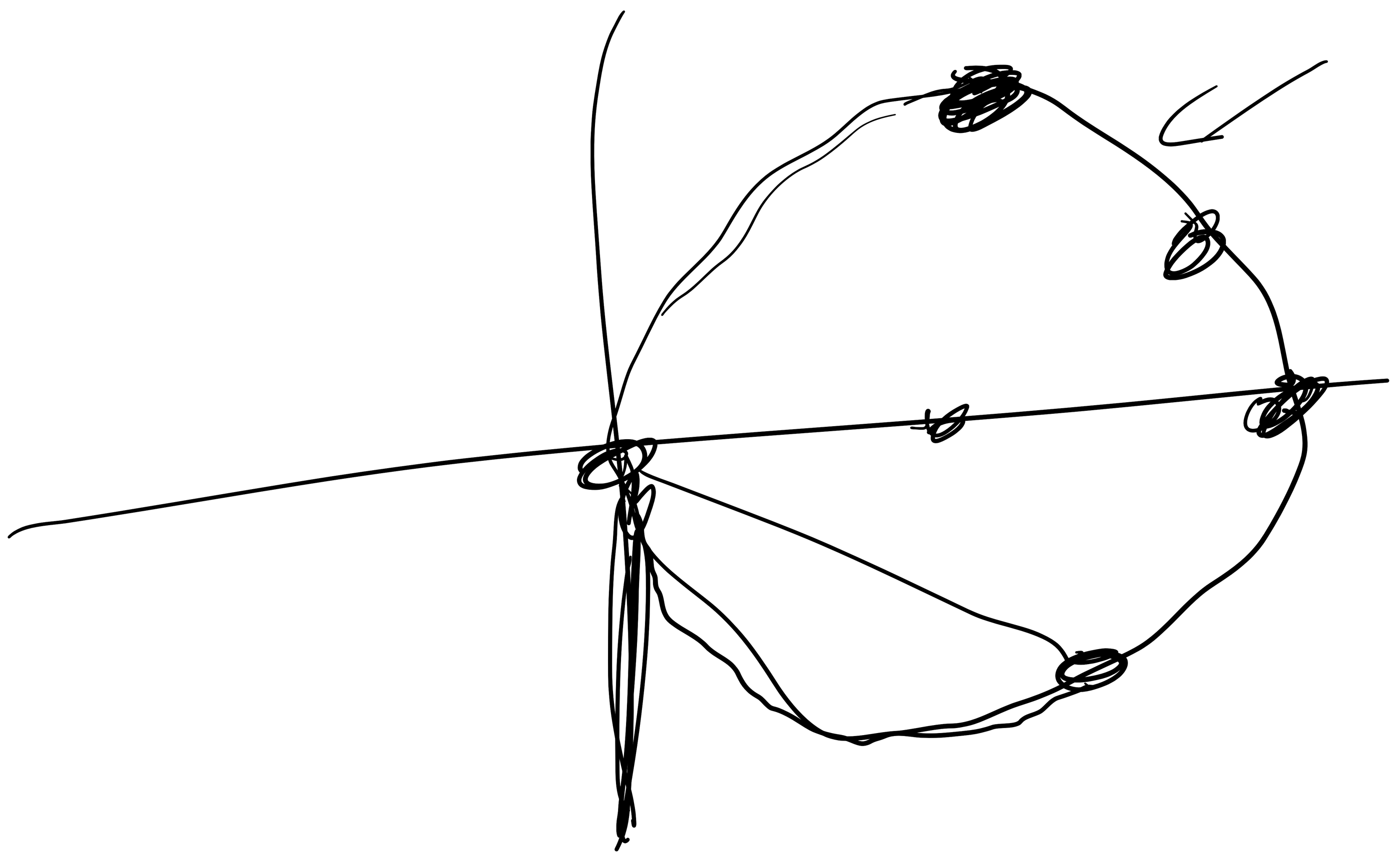
$$= \frac{\pi k}{3}$$

3. (25 Βαθμοί) Έστω $a > 0$. (α) Ζωγραφίστε στο επίπεδο τον κύκλο με εξίσωση $(x-a)^2 + y^2 = a^2$. Να βρεθεί η εξίσωσή του σε πολικές συντεταγμένες. [Δηλαδή γραφεί η εξίσωση του κύκλου με μεταβλητές τα r, θ που ορίζονται από τις $x = r\cos\theta, y = r\sin\theta$.]

(β) Να υπολογιστεί με χρήση πολικών συντεταγμένων το ολοκλήρωμα

$$\iint_D \sqrt{x^2+y^2}\, dxdy,$$

όπου $D := \{(x, y) \in \mathbb{R}^2 : (x-a)^2 + y^2 \le a^2\}$.



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2\cos^2\theta + \cancel{a^2} - 2ar\cos\theta + r^2\sin^2\theta = \cancel{a^2}$$

$$r^2 - 2ar\cos\theta = 0$$

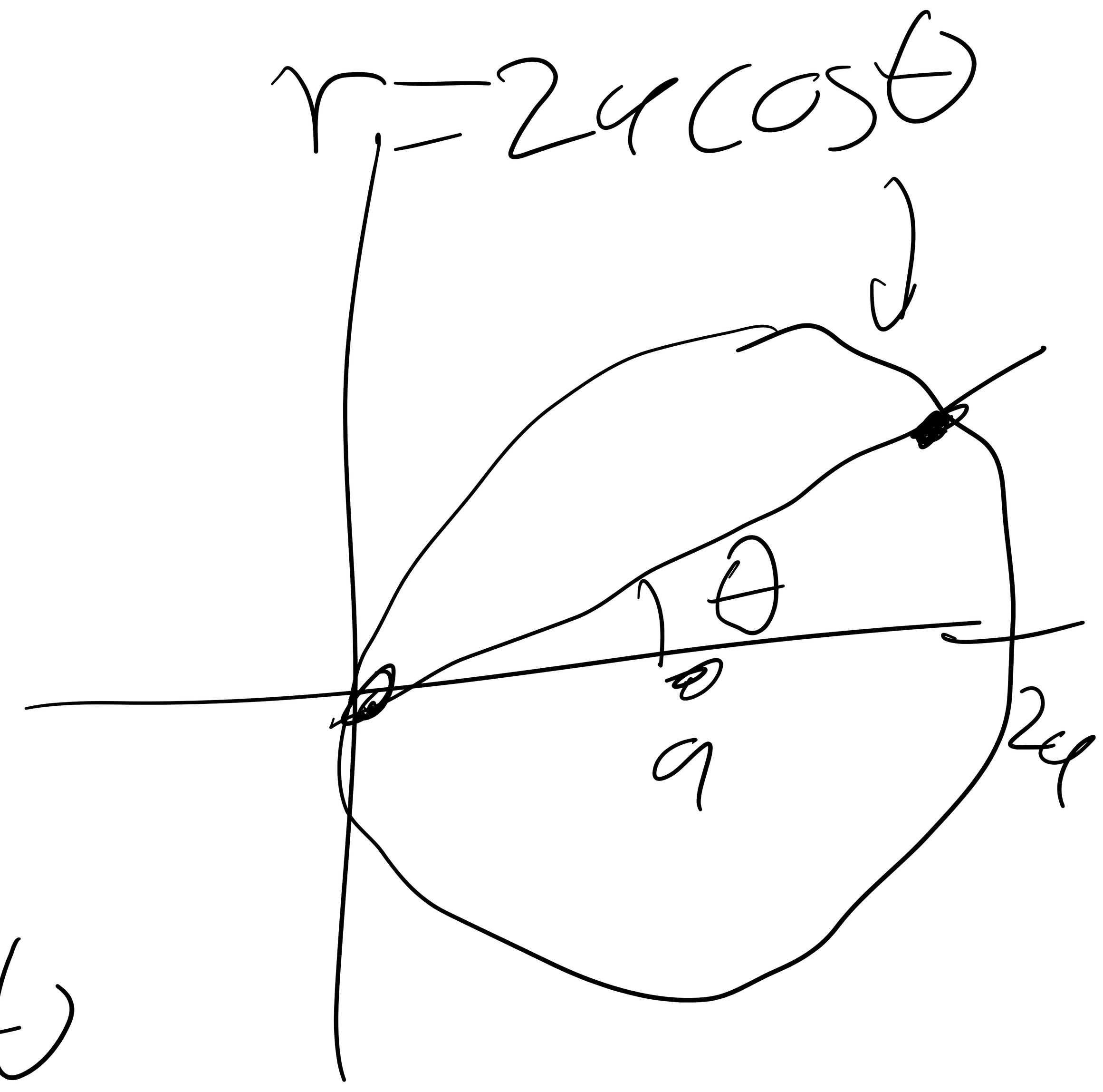
$$r = 0$$

ή

$$r = 2a\cos\theta$$

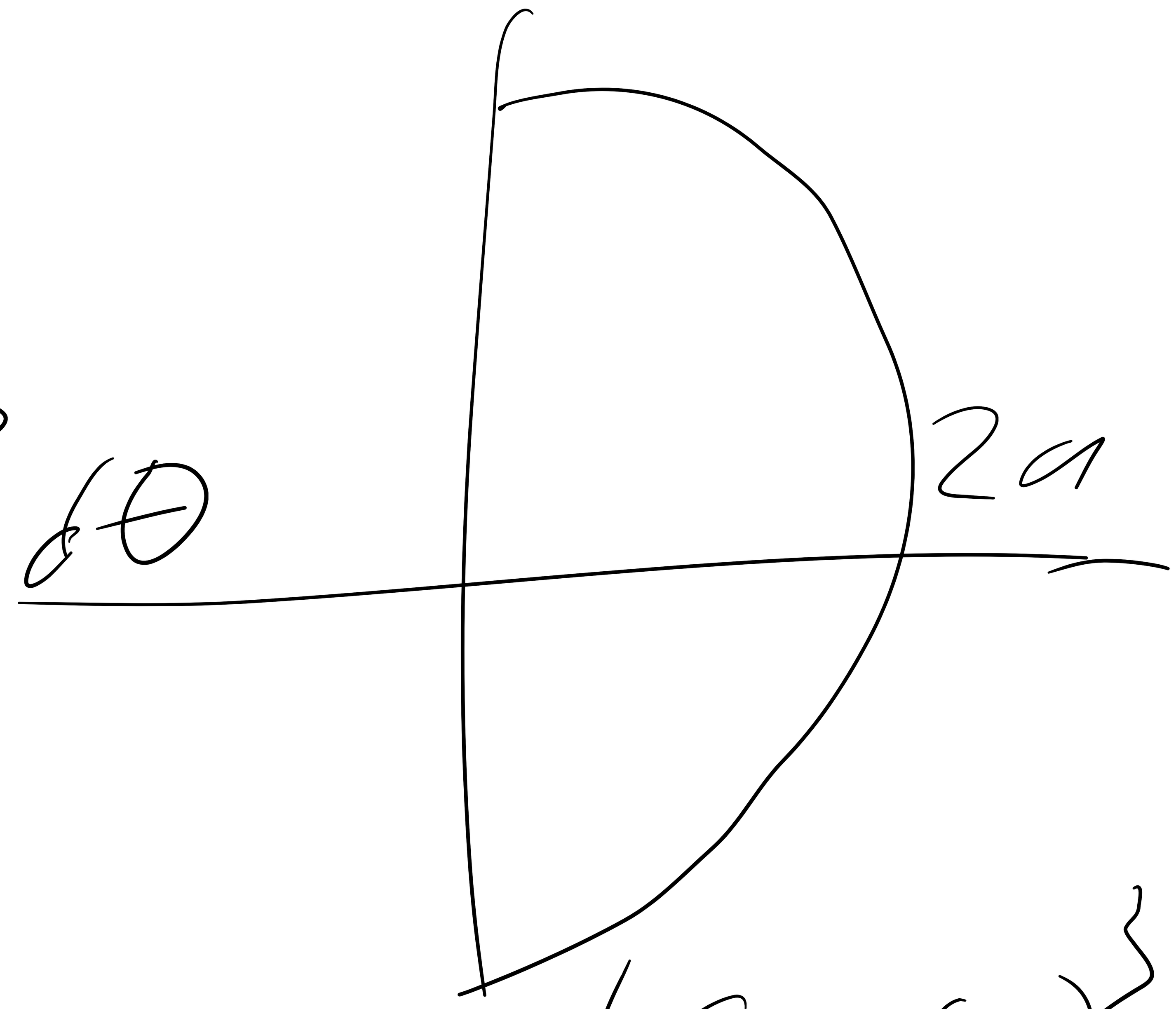
$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$r = 2a\cos\theta, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



e) $$\iint_D \sqrt{x^2+y^2}\,dx\,dy$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r \cdot r\,dr\,d\theta$$



$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2a\cos\theta)^3}{3}\,d\theta$$

$(\cos\theta)^3$

$u = \sin\theta$

$$= \frac{8a^3}{3}\, 2 \int_0^{\frac{\pi}{2}} (1-\sin^2\theta)\,(\sin\theta)'\,d\theta$$

$$= \frac{16a^3}{3} \int_0^1 (1-u^2)\,du =$$

$$= \frac{16a^3}{3}\left(1 - \frac{1}{3}\right) = \frac{32a^3}{9}$$