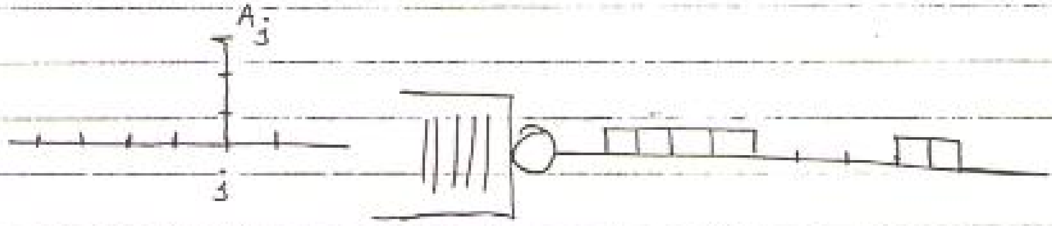
Example 3 (Buffer occupancy process)



$\{A_j\}$ $\{Q_j\}$

Consider the discrete-time, single-server queueing system modeling the transmission of information over a fixed rate link.

A_j = # of message arrival at time instant j

$P\{A_j = k / A_{j-1}, A_{j-2}, \ldots\} = P\{A_j = k\}$, i.e. $\{A_j\}$ is a independent process (assumed to be independent from anything else).

S_e = service-time requirement (positive integer) of e^{th} message. Again, $\{S_e\}$ is an indep. process

Q_j = # of messages in the queue at time j

Case 1 : Under the above assumptions $\{Q_j\}_{j \geq 0}$ is not a M.C. Q_{j+1} is determined by Q_j plus the number of new arrivals minus the number of departures. The former is determined without any past info. The latter requires info regarding the amount of remaining service for the message under service. This info is not contained in Q_j, thus $\{Q_j\}$ is not a M.C.

Case 2: Assume that $P\{S_e = 1\} = 1$. Then as long as a message is present (indicated by Q_j) one message will always depart in the next slot. Thus $\{Q_j\}$ is a M.C.

Case 3 Assume that S_e is geometrically distributed,

$$\text{i.e.} \quad P\{S_e = k\} = q^{k-1} p \qquad k = 1, 2, 3, \ldots$$

This is the probability that the first success is achieved in the k^{th} trial in a sequence of independent trials with success probability p and fail prob. $q = 1-p$.

Notice that $P\{S_e > k\} = q^k$ and

$$P\{S_e > k+n / S_e > n\} = \frac{q^{k+n}}{q^n} = q^k = P\{S_e > k\}$$

That is, the geometric RV also has the memoryless property (as the exponential).

In this case $\{Q_j\}$ is a M.C. since the remaining service time is not dependent on the amount of service completed and it is equal to $P\{S_e\} = q^{k-1}p$.

Case 4 : Consider case 1 but observe the occupancy process at time slots n at which a departure occurs. Then $\{Q_n\}$ is a M.C.

Case 5 : Assume that $P\{S_e = 1\} = 1$ and that

$P\{A_j = k / A_{j-1}, \dots\} = P\{A_j = k / A_{j-1} = m\} = P(m, k)$,

i.e. $\{A_j\}$ is a M.C. Then $\{Q_j\}$ is not a M.C. since new arrivals (A_j) cannot be determined without info about the previous ones (A_{j-1}).

But $\{A_j, Q_j\}$ is a M.C.

Example 4 (Queueing Analysis).

Consider the queueing system described in Example 3 with $P\{S_e = 1\} = 1$, $P\{A_j = k / A_{j-1}, \ldots\} = P\{A_j = k\}$ and assuming infinite capacity. The evolution of the occupancy process is described by

$$Q_j = (Q_{j-1} - 1)^+ + A_j = \begin{cases} Q_{j-1} - 1 + A_j & \text{if } Q_{j-1} > 0 \\ Q_{j-1} + A_j & \text{if } Q_{j-1} = 0 \end{cases}$$

$\{Q_j\}$ is a M.C.

$$P\{Q_j = i\} = P\{Q_{j-1} = 0, A_j = i\} + P\{Q_{j-1} = 1, A_j = i\} + \sum_{k=2}^{\infty} P\{Q_{j-1} = k, A_j = i - k + 1\}$$

or

$$P_i^j = \sum_{k=0}^{1} P_k^{j-1} a_i + \sum_{k=2}^{\infty} P_k^{j-1} a_{i-k+1} \qquad (*)$$

where $P_i^j = P\{Q_j = i\}$, $a_i = P\{A_j = i\}$

Under stability conditions $\lim_{j \to \infty} P_i^j = \pi_i$, where π_i, $i = 0, 1, 2, \ldots$, can be obtained from (from $(*)$)

$$\pi_i = \sum_{k=0}^{1} \pi_k a_i + \sum_{k=2}^{\infty} \pi_k a_{i-k+1}, \qquad i \geq 0 \qquad (**)$$

$\sum \pi_i = 1$