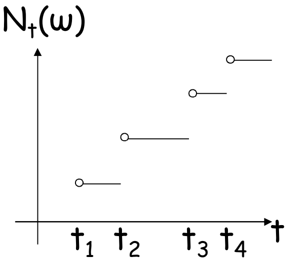
POISSON PROCESSES

COUNTING (ARRIVAL) PROCESS

- $\{N_t(\omega);t>0\}$ defined on some sample space Ω is called a counting process provided that:

  (1) it is non-decreasing
  (2) it increases by jumps only
  (3) it is right continuous
  (4) $N_0(\omega)=0$



POISSON PROCESS

A counting process that satisfies:

1. Each jump is of unit magnitude
2. (independent increments) For any $t,s \geq 0$, $N_{t+s}-N_t$ is independent of $\{N_u(\omega);u\leq t\}$
3. (stationarity) For any $t,s \geq 0$, the distribution of $N_{t+s}-N_t$ is independent of t

Lemma 1 : For every $t \geq 0$, $P\{N_t=0\}=e^{-\lambda t}$ for some $\lambda \geq 0$

Proof : {0 arrivals in [0,t+s]} ⇔
⇔ {0 arrivals in [0,t]} and {0 arrivals in [t,t+s]} , or

$\{N_{t+s}=0\} \Leftrightarrow \{N_t=0\}$ and $\{N_{t+s}-N_t=0\}$
$P\{N_{t+s}=0\}= P\{N_t=0\} P\{N_{t+s}-N_t=0\}$ (indep. increments)
$P\{N_{t+s}=0\}= P\{N_t=0\} P\{N_s=0\}$ (stationary increments) (*)

Let $P\{N_t=0\}=f(t)$, then
(*) $f(t+s)=f(t)f(s)$, $0\leq f(t) \leq 1$, $t,s\geq 0$
the only non-zero f(t) satisfying (*) is $e^{-\lambda t}$, $\lambda\geq 0$

Thus $P\{N_t=0\}= e^{-\lambda t}$, $\lambda\geq 0$

Lemma 2: $\lim_{t\to 0}\frac{1}{t}P\{N_t \geq 2\} = 0$

i.e., Prob{ ≥ 2 arrivals over a small t} $\xrightarrow{t\to 0} 0$ faster that t

Lemma 3: $\lim_{t\to 0}\frac{1}{t}P\{N_t = 1\} = \lambda$ (arrival rate λ)

Proof : $P\{N_t = 1\} = 1 - P\{N_t = 0\} - P\{N_t \geq 2\} \Rightarrow$

$$\lim_{t\to 0}\frac{1}{t}P\{N_t = 1\} = \lim_{t\to 0}\left\{\frac{1}{t} - \frac{e^{-\lambda t}}{t} - \frac{1}{t}P\{N_t \geq 2\}\right\}$$

$$= \lim_{t\to 0}\frac{1-e^{-\lambda t}}{t} = \lambda$$

i.e., Prob{1 arrivals over a small t} $\xrightarrow{t\to 0} \lambda t$

Theorem : If $\{N_t;t\geq 0\}$ is Poisson then

$P\{N_t=k\}=e^{-\lambda t} (\lambda t)^k/k!$, $k=0,1,2,\ldots$ for some $\lambda \geq 0$

Notice on short term behavior : (δ small)

P{0 arrivals in $(t,t+\delta)$} = $1-\lambda\delta+o(\delta^2)$

P{1 arrivals in $(t,t+\delta)$} = $\lambda\delta+o(\delta^2)$

P{>1 arrivals in $(t,t+\delta)$} = $o(\delta^2)$

Moments of Poisson process

- $E\{N_t\} = \sum_{n=0}^{\infty} nP\{N_t = n\} = \sum_{n=0}^{\infty} n \frac{e^{-\lambda t}(\lambda t)^n}{n!} = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda t = \lambda t$
- $E\{N_t^2\} = E\{N_t(N_t - 1) + N_t\} = \sum_{n=0}^{\infty} n(n-1) \frac{e^{-\lambda t}(\lambda t)^n}{n!} + \lambda t$

$$= \sum_{n=0}^{\infty} \underbrace{n(n-1)} \frac{e^{-\lambda t}(\lambda t)^{n-2}(\lambda t)^2}{(n-2)!\underbrace{(n-1)n}} + \lambda t = (\lambda t)^2 + \sum_{n'=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n'}}{n'!} + \lambda t$$

$$= (\lambda t)^2 + \lambda t$$

- $Var\{N_t\} = E\{N_t^2\} - (E\{N_t\})^2 = \lambda t$ (compare with Wiener's)

  Note : $E\{N_t\} = Var\{N_t\} = \lambda t$ (high for large t)

- $R_N(t_1, t_2) = E\{N_{t_1} N_{t_2}\} = E\{N_{t_1}^2 + (N_{t_2} - N_{t_1})N_{t_1}\} =$

$$= \lambda t_1 + (\lambda t_1)^2 + \lambda(t_2 - t_1)\lambda t_1 = \lambda t_1 + \lambda^2 t_1 t_2 \quad (t_1 < t_2)$$

$$= \lambda \min\{t_1, t_2\} + \lambda^2 t_1 t_2$$ (Compare with Wiener's)

- $k_N(t_1, t_2) = R_N(t_1, t_2) - E\{N_{t_1}\}E\{N_{t_2}\} = \lambda \min\{t_1, t_2\}$

  (Compare with Wiener's)

Note : Above moments similar to those of the Wiener process as a result of the independent increment property, common to both.

Corrolary : if $\{N_t; t \geq 0\}$ is Poisson, then

$$P\{N_{t+s} - N_t = k \mid N_u; u \leq t\} \underset{\text{ind. incr.}}{=} P\{N_{t+s} - N_t = k\}$$

$$\underset{\text{stationary incr.}}{=} P\{N_s = k\} = \frac{e^{-\lambda s}(\lambda s)^k}{k!}$$

Alternative definition of Poisson Process - A

(by checking in samples the validity of the independent and stationary increment property)

$\{N_t;t\geq 0\}$ is Poisson with rate λ <u>iff</u>

(a) $N_t(\omega)$ has unit magnitude jumps for almost all ω
(b) $\forall t,s\geq 0$, $E\{N_{t+s}-N_t|N_u;u\leq t\}=\lambda s$

Alternative definition of Poisson Process - B

(by checking in samples the validity of the Poisson distribution)

$\{N_t;t\geq 0\}$ is Poisson with rate λ <u>iff</u>

$P\{N_B=k\}=e^{-\lambda b}(\lambda b)^k/k!$, $k=0,1,2,\ldots$

For any subset B of $\mathbb{R}_+$ that is the union of a finite number of disjoint intervals whose length sums up to b.

Uniformity of distribution of time of given arrival(s) or Conditional distribution of arrival(s)

- Assume we are told that one event occurred at some point within $(0, t)$. The question is: when is it more likely for that event to have occurred?

$$\begin{aligned}\Pr(Y_1 < s | N(t) = 1) &= \frac{\Pr(Y_1 < s, N(t) = 1)}{\Pr(N(t) = 1)} \\ &= \frac{\Pr(1 \text{ event in } (0, s)) \Pr(0 \text{ events in } (s, t))}{\Pr(1 \text{ event in } (0, t)} \\ &= \frac{\frac{(\lambda s)^1}{1!} e^{-\lambda s} \times \frac{(\lambda(t-s))^0}{0!} e^{-\lambda(t-s)}}{\frac{(\lambda t)^1}{1!} e^{-\lambda t}} = \frac{s}{t} \quad (19)\end{aligned}$$

- Thus, Y is uniformly distributed on $(0, t)$: $Y \sim U(0, t)$
- This result was expected given the memoryless nature of the Poisson process.

Proposition: (Uniformity of the distribution of the time of Poisson arrival occurrences over an interval)

Let $A_1 \cup A_2 \cup ... \cup A_n = B$, $\{A_i\}_{i=1}^n$ disjoint, $|A_i| = a_i$ (length)

$k_1 + k_2 + ... + k_n = k$, all in N, $|B| = b$. Then (multinomial distr)

$$P\{\underbrace{N_{A_1} = k_1, N_{A_2} = k_2, ..., N_{A_n} = k_n}_{c} \mid \underbrace{N_B = k}_{d}\} = \frac{k!}{k_1!...k_n!}\left(\frac{a_1}{b}\right)^{k_1} ... \left(\frac{a_n}{b}\right)^{k_n}$$

$$\text{Proof}: P\{c \mid d\} = \frac{P\{c, d\}}{P\{d\}} = \frac{P\{c\}}{P\{d\}} = \frac{P\{N_{A_1} = k_1\}...P\{N_{A_n} = k_n\}}{P\{N_B = k\}}$$

$\{N_{A_i}\}_i$ independent since $\{A_i\}_i$ disjoint.

(write each prob as a Poisson over each sub-interval and manipulate)

On the estimation of λ :

Strong Law of Large Numbers (SLLN) justifies : $\lambda = \lim_{t\to\infty} \frac{N_t(\omega)}{t}$ a.e.

Proof : Use discrete unit time axis, n

$$(*)\ N_n = \underbrace{N_1 + (N_2 - N_1) + (N_3 - N_2) + \ldots + (N_n - N_{n-1})}_{\text{are all i.i.d. with mean } \lambda \Rightarrow \text{SLLN holds}}$$

On the limiting behavior of N_t :

From (*) $\Rightarrow N_t$ is the sum of i.i.d. RV's $\Rightarrow$

Central Limit Theorem implies

$$\lim_{t\to\infty} P\left\{\frac{N_t - \lambda t}{\sqrt{\lambda t}} \le x\right\} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \quad \text{(PDF for N(0,1))}$$

(good approximation for $\lambda t \ge 10$ - λt is the variance)

Proposition : (distribution of interarrival times)

$$P\{T_{n+1} - T_n \le s \mid T_0, T_1, \ldots, T_n\} = 1 - e^{-\lambda t} \quad s \ge 0$$

i.e., it is exponentially distributed and independent of past arrival times.

Proof :

$$P\{T_{n+1} - T_n > s \mid T_0, T_1, \ldots, T_n\} = P\{N_{T_{n+s}} - N_{T_n} = 0 \mid T_0, T_1, \ldots, T_n\}$$

$$= P\{N_{T_{n+s}} - N_{T_n} = 0 \mid N_u ; u \le T_n\} = \text{(indep. incr. of Poisson)}$$

$$= P\{N_{T_{n+s}} - N_{T_n} = 0\} = e^{-\lambda s}$$

Memorylessness of interarrival times

Since exponential, the distribution of $T_{n+1}-T_n$ is memoryless:

$$P\{T_{n+1}-T_n>t+s \mid T_{n+1}-T_n>t\}=P\{T_{n+1}-T_n>s\}$$

i.e., knowing that t time units have passed since the last arrival does not affect the time when the next arrival will occur, which remains exponential with the same parameter λ.

Stated differently: No matter which time instant t I observe the system, the evolution of future arrival times is independent of t and past arrival times.

Thus, **there is no need to maintain any record regarding past arrivals to determine future ones (great simplification in system modeling).**

Minimum and comparison of exponential variables

- Let $X_1, \ldots, X_n$ be a set of n independent and exponentially distributed random variables with rates $\lambda_1, \ldots, \lambda_n$ respectively.

$$\Pr(\min\{X_1, X_2, \ldots, X_n\} > t) = e^{-\sum_i \lambda_i t} \quad (8)$$

- Let $X_1 \sim exp(\lambda_1)$ and $X_2 \sim exp(\lambda_2)$

$$\Pr(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2} \quad (9)$$

Example: Reliability

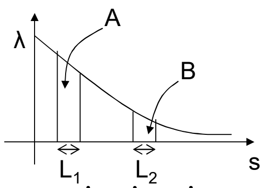
- Let us consider a car with three components subject of failure: The engine with rate 1 failure every 10 years, the electricity system with rate 2 failures every 5 years and the air conditioning with rate 1 failure every 2 years.
  - Assume we start at time $t = 0$. Find the probability to have at least one failure on the first year:
    We use random variable $X_{failure}$ to model the time until the first failure. This is exponentially distributed with rate
    $\lambda_t = \frac{1}{10} + \frac{2}{5} + \frac{1}{2} = 1$ failure per year
    Therefore: $P(X_{failure} \leq 1 year) = 1 - e^{-1}$
  - Find the probability that the engine fails before any other component:
    $P(X_1 \leq min\{X_2, X_3\}) = \frac{\lambda_1}{\lambda_1+(\lambda_2+\lambda_3)} = \frac{1/10}{1/10+2/5+1/2} = \frac{1}{10}$
  - Find the probability that the engine fails twice in a row before any other failure:
    $\frac{1}{10}\frac{1}{10}$

Burstiness of Poisson arrivals

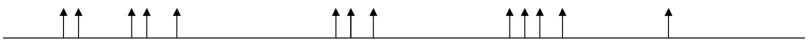
Let τ be the generic RV for $\tau_n = T_{n+1} - T_n$

$P\{\tau \leq s\} = 1 - e^{-\lambda s} \Leftrightarrow f_\tau(s) = \lambda e^{-\lambda s}$

Since $f_\tau(s)$ decreases with s ⇒
$P\{\tau \approx L_1\} \approx A > B \approx P\{\tau \approx L_2\}$ (for $L_1 < L_2$)



Thus, **short interarrival times occur more frequently than long ones.**



Thus **arrivals appear in bursts (clusters) (Poisson is a fairly bursty arrival process)**
Make the distinction between "burstiness of arrivals" and "uniformity of the times of arrivals over an interval"

Alternative definition of Poisson Process - C

A counting (arrival) process is Poisson iff the associated interarrival times are independent and identically distributed exponential RVs.

Moments of interarrival times:

$$E\{T_{n+1} - T_n\} = \int_0^{\infty} e^{-\lambda t} dt = \frac{1}{\lambda}$$

$$Var\{T_{n+1} - T_n\} = \frac{1}{\lambda^2}$$

On the arrival (not interarrival) times Poisson Process

Moments of arrival times, T_n

$$T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \dots + (T_n - T_{n-1})$$

$$E\{T_n\} = \frac{n}{\lambda^2} \quad , \quad Var\{T_n\} = \frac{n}{\lambda^2} \quad (\text{since } T_n - T_{n-1} \text{ indep.})$$

Distribution of arrival times, T_n (Erlang-n)

$$P\{T_n \le t\} = P\{N_t \ge n\} = 1 - P\{N_t < n\} = 1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

$$f(t) = \frac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!} \quad , \quad t \ge 0$$

Note:

Erlang-n is the distribution of the interarrival times of groups of n Poisson arrivals

Erlang-n is the distribution of the sum of n exponential and identical RV's

Example :

$\{U_k\}_{k\geq 1}$ are car interarrival times (assumed indep.)

U_k is erlang - 2 distributed, i.e.,

$$P\{U_k \leq t\} = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t} \quad , \quad t \geq 0$$

U_k may be viewed as the sum of two interarrival times in a Poisson process $\{N_t\}$ with rate λ. Let $\{T_k\}_{k\geq 1}$ be the arrival times of that Poisson.

Then

$$U_1 = T_2 \quad , \quad U_2 = T_4 - T_2 \quad , \quad U_3 = T_6 - T_4 \quad , \ldots$$

If M_t = # cars that passed by before or at t

$$P\{M_t = k\} = P\{2k \leq N_t < 2(k+1)\} = P\{N_t = 2k\} + P\{N_t = 2k+1\}$$

$$= \frac{e^{-\lambda t}(\lambda t)^{2k}}{2k!} + \frac{e^{-\lambda t}(\lambda t)^{2k+1}}{(2k+1)!} \quad , \quad k = 0,1,2,\ldots$$

Example : (hardware lifetime and replacement cost)

Assume exponential lifetime of a piece of hardware which is replaced by an identical upon failure. Replacement cost is β and the discount rate of money is α (i.e., 1\$ spent at time t has present value $e^{-\alpha t}$, α is the interest rate). Find the expected cost.

N_t = # of failures in [0, t]

$\{N_t\}$ is Poisson due to the indep. and expon. lifetimes

$T_n(\omega)$ = nth failure for realization ω

$\beta e^{-\alpha T_n(\omega)}$ = present value of the cost of nth replacement

$C(\omega)$ = total cost for the (particular) realization ω

$$C(\omega) = \sum_{n=1}^{\infty} \beta e^{-\alpha T_n(\omega)} \quad , \quad \omega \in \Omega$$

$$E\{C\} = E\{C(\omega)\} = \beta \sum_{n=1}^{\infty} E\{e^{-\alpha T_n(\omega)}\} = \beta \sum_{n=1}^{\infty} E\{e^{-\alpha T_n}\}$$

to use the i.i.d. property of inter - failures write :

$$T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \ldots + (T_n - T_{n-1})$$

$$E\{C\} = \beta \sum_{n=1}^{\infty} E\left\{ e^{-\alpha T_1} e^{-\alpha(T_2 - T_1)} \ldots e^{-\alpha(T_n - T_{n-1})} \right\}$$

$$= \beta \sum_{n=1}^{\infty} E\{e^{-\alpha T_1}\} E\{e^{-\alpha(T_2 - T_1)}\} \ldots E\{e^{-\alpha(T_n - T_{n-1})}\} = \beta \sum_{n=1}^{\infty} [E\{e^{-\alpha T_1}\}]^n$$

$$E\{e^{-\alpha T_1}\} = \int_0^{\infty} e^{-\alpha T_1} \lambda e^{-\lambda t} dt = \frac{\lambda}{\lambda + \alpha} \quad \text{, thus}$$

$$E\{C\} = \beta \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda + \alpha} \right)^n = \beta \frac{\frac{\lambda}{\lambda + \alpha}}{1 - \frac{\lambda}{\lambda + \alpha}} = \frac{\beta \lambda}{\alpha}$$

The result may be derived by setting $f(t) = \beta e^{-\alpha t}$, $f(T_n) = \beta e^{-\alpha T_n}$ and using the following (see next page)

In particular, if the mean lifetime is 5,000 hours, replacement cost 800 dollars, and the interest rate 24 percent per year, then $\beta = 800$, $\lambda = 1/5{,}000$, $\alpha = 0.24/(365 \times 24) = 0.01/365$, and hence $E[C] = 800 \times 36{,}500/5{,}000 = 5840$ dollars. □

Proposition : for a non - negative function $f(\cdot)$ on R_+

$$E\left\{ \sum_{n=1}^{\infty} f(T_n) \right\} = \lambda \int_0^{\infty} f(t) dt$$

where T_n is the occurrence time (arrival) of the nth event in a Poisson process with rate λ.

Proof :

$$E\{f(T_n)\} = \int_0^{\infty} f(t) \frac{\lambda e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!} dt \quad , \quad \text{thus}$$

$$E\left\{ \sum_{n=1}^{\infty} f(T_n) \right\} = \sum_{n=1}^{\infty} \int_0^{\infty} f(t) \frac{\lambda e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!} dt$$

$$= \int_0^{\infty} \lambda f(t) \sum_{n=1}^{\infty} \frac{e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!} dt = \lambda \int_0^{\infty} f(t) dt$$

Stopping time of an arrival process

A RV T is a stopping time of an arrival process $\{N_t\}$ if the occurrence of the event $\{T \le t\}$ is determined by $\{N_u; u \le t\}$, i.e., by knowing the history of the arrival process up to t.

Example 1: T_5=the time of the 5^{th} arrival is stopping time since $\{N_u; u \le t\}$ determines if $\{T_5 \le t\}$.

Example 2: T=first time interarrival time exceeds some value C. It is a stopping time.

Poisson arrivals over [T,T+s] , for T a RV

If T=t (a fixed random point in time) we know that
$N_{T+s}-N_T$ is independent from $\{N_u; u \le s\}$ and Poisson with rate λs.

If T is not fixed but a RV then above holds <u>if T is stopping time</u> and then
$P\{N_{T+s}-N_T=k \mid N_u; u \le T\}=e^{-\lambda s}(\lambda s)^k/k!$, k=0,1,...

e.g., $T=T_n$ is a stopping time

T= time of occurrence of the largest interarrival is not a stopping time since $\{T \le t\}$ cannot be determined by $\{N_u; u \le t\}$ since the future evolution of the arrival process is needed as well.

Example: Buses arrive as Poisson with λ=0.2 per minute. Inspector arrives at time of the 5^{th} bus arrival after time t_0 and will stay for 60 minutes. Find the distribution of buses to arrive within these 60 minutes.

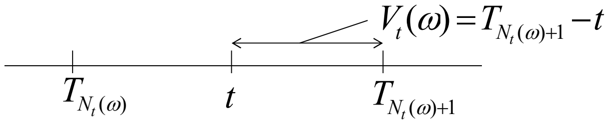
Answer:

Time of arrival of inspector $T = T_{N_{t_0}(\omega)+5} \quad \omega \in \Omega$

T is a stopping time. Thus,

$$P\{N_{T+60} - N_T = k\} = \frac{e^{-12} 12^k}{k!} \quad , \quad k = 0,1,\ldots$$

Forward Recurrence Times



V_t = (remaining) time between current time t and the next arrival

Theorem: $P\{V_t \le u \mid N_s; s \le t\} = 1 - e^{-\lambda u} \quad , \quad u \ge 0$

(i.e., the distribution of V_t is the same as that of an interarrival)

Proof: $\{V_t \le u\} = \{T_{N_t+1} - t \le u\} = \{T_{N_t+1} - t > u\}^c$

$$= \{T_{N_t+1} > t + u\}^c = \{N_{t+u} - N_t = 0\}^c$$

Thus, $P\{V_t \le u \mid N_s; s \le t\} = P\{\{N_{t+u} - N_t = 0\}^c \mid N_s; s \le t\}$

$$= 1 - P\{N_{t+u} - N_t = 0 \mid N_s; s \le t\} = 1 - P\{N_{t+u} - N_t = 0\}$$

$$= 1 - P\{N_u = 0\} = 1 - e^{-\lambda u}$$

Note: $\mathrm{E}\{V_t\} = \frac{1}{\lambda}$ and $E\{T_{n+1} - T_n\} = \frac{1}{\lambda}$

$$E\{T_{N_t+1} - T_{N_t}\} = E\{T_{N_t+1} - t + t - T_{N_t}\} = E\{V_t\} + E\{t - T_{N_t}\}$$
$$= \frac{1}{\lambda} + E\{t - T_{N_t}\}$$

That is, the interarrival interval that we happen to observe (that includes t) is larger, on average, than an ordinary such interval between two arrivals!!!

Example :

The time between the two consecutive arrivals containing our time of arrival to the bus stop is almost twice as large, on the average, as the typical bus interarrival time, assuming Poisson bus arrivals.

(Reason why the bus is always more late than usual
- or claimed by the company -
when we arrive at the bus stop)

Uniqueness of Poisson superposition property: If L & M are renewal processes and their superposition N is renewal, then all 3 are Poisson (renewal process: i.i.d. but not necessarily exponential interarrival times).

Decomposition of a Poisson process: $N=\{N_t;t\geq 0\}$ Poisson with rate λ, $\{X_n;n=1,2,\ldots\}$ Bernoulli with param. p

$\{S_n;\ n=1,2,\ldots\}$=# of successes in n trials

$N_t(\omega)$ trials (i.e., arrivals that are split based on p) are carried out in [0,t]

$M_t(\omega)=S_{N_t(\omega)}(\omega)$ is the number of successes over [0,t]

$L_t(\omega)= N_t(\omega)-M_t(\omega)$ is the number of failures over [0,t]

Theorem: $M=\{M_t;t\geq 0\}$ & $L=\{L_t;t\geq 0\}$ are Poisson with rate λp and $\lambda(1-p)$, respectively and M & L are independent.

Proof : Suffices to show that

$$P\{M_{t+s}-M_t=m, L_{t+s}-L_t=k \mid M_u, L_u; u\leq t\}=$$
$$=\frac{e^{-\lambda ps}(\lambda ps)^m}{m!}\cdot\frac{e^{-\lambda(1-p)s}(\lambda(1-p)s)^k}{k!}\quad,\quad k,m=0,1,2\ldots$$

$\forall\quad t,s\geq 0.$

$\{M_{t+s}-M_t=m, L_{t+s}-L_t=k\}\Leftrightarrow$ (bring in N_t)

$\{N_{t+s}-N_t=m+k, M_{t+s}-M_t=m\}\Leftrightarrow$ (bring in S_{N_t})

$\{N_{t+s}-N_t=m+k, S_{N_{t+s}}-S_{N_t}=m\}=\mathbf{A}$

Now, $\{M_u, L_u; u \le t\} \Leftrightarrow \{N_u; u \le t, X_1, X_2, ..., X_{N_t}\} = \mathbf{B}$

Notice that $N_{t+s} - N_t$ & $S_{N_{t+s}} - S_{N_t}$ (and thus $\mathbf{A}$) are indep. of $\mathbf{B}$

$$P(\mathbf{A}) = \sum_{n=0}^{\infty} P\{N_t = n, N_{t+s} - N_t = m+k, S_{N_{t+s}} - S_{N_t} = m\}$$

$$= \sum_{n=0}^{\infty} P\{N_t = n, N_{t+s} = m+k+n, S_{m+k+n} - S_n = m\}$$

$$= \sum_{n=0}^{\infty} P\{N_t = n, N_{t+s} = n+m+k\} P\{S_{m+k+n} - S_n = m\}$$

$$= \sum_{n=0}^{\infty} P\{N_t = n, N_{t+s} - N_t = m+k\} P\{S_{m+k} = m\}$$

$$= P\{N_{t+s} - N_t = m+k\} P\{S_{m+k} = m\}$$

$$= \frac{e^{-\lambda s}(\lambda s)^{m+k}}{(m+k)!} \cdot \frac{(m+k)!}{m!k!} p^m (1-p)^k$$

Example: N is the Poisson arrival process of cars (rate λ). N^1, N^2,…, N^5 are the arrival processes of cars with 1,2,…,5 passengers (0.3,0.3,0.2,0.1,0.1 are the passenger occupancy probabilities for 1,2,…,5).

N^1, N^2,…, N^5 are Poisson with rates 0.3λ,0.3λ,0.2λ,0.1λ,0.1λ.

Expected # of passengers per unit time:

$E\{N^1+2N^2+3N^3+4N^4+5N^5\}=$
0.3λ+2*0.3λ+3*0.2λ+4*0.1λ+5*0.1λ

Addition and sampling of Poisson processes

- Two more (very important) properties of the Poisson process:

Addition: Let $N_1(t)$ and $N_2(t)$ be two Poisson processes with rates λ_1 and λ_2 respectively. Then, the process $N_{tot} = N_1(t) + N_2(t)$ is also a Poisson process with rate $\lambda_{tot} = \lambda_1 + \lambda_2$

Sampling: Let $N(t)$ be a Poisson process with rate λ, and let $0 \leq p \leq 1$ be some sampling probability. Then, the process obtained by sampling the events of $N(t)$ with probability p is also a Poisson process with rate λp. The events left apart is also a Poisson process with rate $\lambda(1-p)$.

Addition and sampling of Poisson processes. Example

- Consider an Ethernet switch with 24 input ports and 2 output ports. Each input port injects packets to the switch as a Poisson process with rate $\lambda = 10$ packets/sec. The routing tables state that 70% of the packets must go to output port number 1, while the other 30% must go to the second output port.

- The counting process at the switch is also Poisson because it is the combination of 24 Poisson processes. The total rate is then $\lambda_{tot} = 24 \times 10 = 240$ packets/sec.
- The counting process of outgoing packets on the two ports are also Poisson because the are random samples of a Poisson process. The first output port has got a rate of $\lambda_{o1} = 0.7 \times 240 = 168$ packets/sec, and the second $\lambda_{o2} = 0.3 \times 240 = 72$ packets/sec.

Compound Poisson process

(allowing jumps of <u>any</u> size in a Poisson process)

Definition A: $Z=\{Z_t;t\geq 0\}$ is a compound Poisson provided that:

(a) $Z_t(\omega)$ has only finitely many jumps in any finite interval (a.e.)

(b) for all $t,s\geq 0$, $Z_{t+s}-Z_t$ is indep. of $\{Z_n;u\leq t\}$

(c) for all $t,s\geq 0$, the distribution of $Z_{t+s}-Z_t$ depends on s (indep. of t)

Note:

- if $N=\{N_t;t\geq 0\}$ is the process that counts the number of jumps in $(0,t]$, then (b) & (c) ==> N is Poisson.
- Z & N differ in the fact that jumps in Z are not all equal to one (1) but are RV's $\{X_1,X_2,\ldots\}$.
  (b) & (c) ==> $\{X_1,X_2,\ldots\}$ are i.i.d. and, thus, indep. of $\{T_1,T_2,\ldots\}$.
- If $\{T_1,T_2,\ldots\}$ are Poisson arrivals times & $\{X_1,X_2,\ldots\}$ are i.i.d. RV's indep. of $\{T_1,T_2,\ldots\}$ then the sum of all X_i such that $T_i\leq t$, Z_t, forms a compound Poisson process.

Definition: Z is a compound Poisson <u>iff</u> its jump times form a Poisson process & the magnitudes of its jumps are i.i.d RV's independent of the jump times.