

Which is the solutions quality of a LS ?

- for a given neighborhood
- even in exponential time !

Grover '' 92 ''

- TSP with 2 – exchange
- Bipartitioning of a weighted graph with 2 – exchange

$$C_{\text{loc}} \leq C_{AV}$$

local minimum

average cost of all solutions



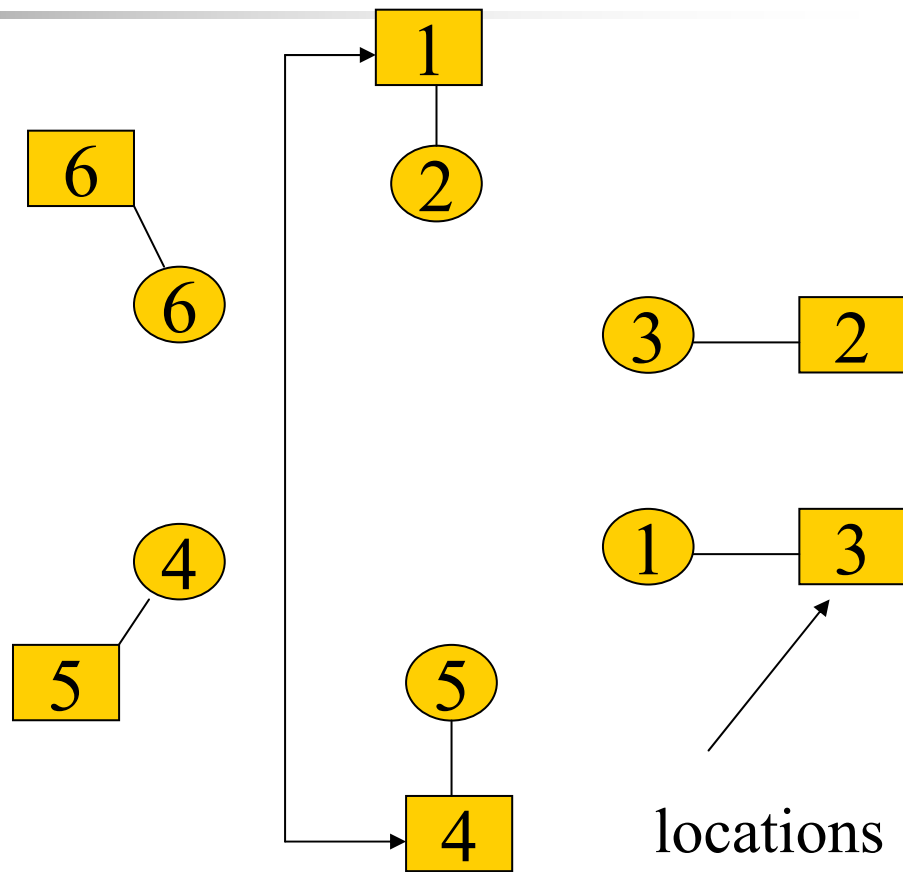
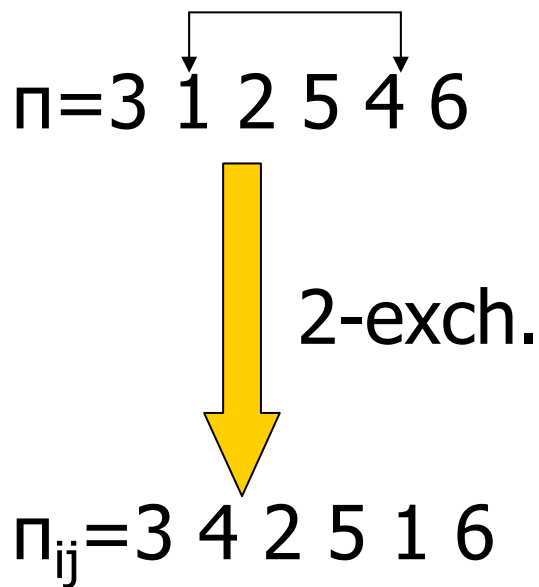
The Quadratic Assignment Problem (QAP)

- n locations : distance
- n facilities: flow f_{ij}
- $\pi(i)=k$: facility $i \rightarrow$ location k

minimize the total cost

$$\text{Min}_{\pi \in \Pi} \sum_{i,k=1}^n f_{ik} d_{\pi(i)\pi(k)}$$

The QAP: an example





The QAP

→ PLS-complete (neighb. 2-exchange)
(Schaffer & Yannakakis '91)

2-exchange:

$$\pi = (\pi(1), \dots, \pi(i), \dots, \pi(j), \dots, \pi(n))$$

$$\pi_{ij} = (\pi(1), \dots, \pi(j), \dots, \pi(i), \dots, \pi(n))$$

$$N(\pi) = (n * (n-1)) / 2$$

Notations

- $S(F) = \sum_{i,j} f_{ij}$

- $x = (x_1, x_2, \dots, x_n)$

$$y = (y_1, y_2, \dots, y_n)$$

$$\pi y = (y_{\pi(1)}, y_{\pi(2)}, \dots, y_{\pi(n)})$$

$$\langle x, y \rangle_+ = \max_{\pi} \langle x, \pi y \rangle$$

$$\langle x, y \rangle_- = \min_{\pi} \langle x, \pi y \rangle$$

$$F_k = \sum_i f_{ki} = \sum_i f_{ik}$$

$$\langle F, D \rangle_+ = \langle (F_1, \dots, F_n), (D_1, \dots, D_n) \rangle_+$$

- $C_{AV} = \frac{1}{n!} \sum C(\Pi)$

- $C_{\overline{1 \dots n}}$

Solution quality for the QAP ?

E. Angel & V.Z. '96 (DAM 98)

Th₁ :

$$C_{\text{loc}} \leq \frac{\langle F, D \rangle_+}{S(F)S(D)} n C_{\text{AV}} = \frac{\langle F, D \rangle_+}{2(n-1)} \leq \frac{n}{2} C_{\text{AV}}$$

Th₂ :

$$\leq \frac{n}{2} C_{\text{AV}} \text{ in } O\left(n \log \frac{S(F) \times S(D)}{2}\right) \text{ iterat.}$$



Idea (deepest local search)

$$\delta_{ij}(\pi) = C(\pi_{ij}) - C(\pi)$$

$$\nabla^2 C(\pi) = \frac{1}{n(n-1)} \sum_{i,j} \delta_{ij}(\pi) \leq -\frac{4}{n}(C(\pi) - C_{AV}) + a$$

$$\forall \pi \exists i, j \text{ t. q.}$$

$$\delta_{ij}(\pi) \leq \nabla^2 C(\pi)$$

$$C(\pi) \rightarrow C(\pi') \leq C(\pi) - \frac{4}{n}(C(\pi) - C_{AV}) + a$$

$$1 = 1 - \frac{4}{n}(1 - C_{AV}) + a$$

$$C_{AV} = \frac{S(F) \times S(D)}{2n(n-1)}$$

$$1) \frac{C_{\max}}{\frac{n}{2} C_{AV}} \rightarrow \infty$$

ex.

$$F = D = \begin{pmatrix} 0 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & 0 & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n^2 \\ 1 & \cdot & \cdot & \cdot & 1 & n^2 & 0 \end{pmatrix}$$

$$\frac{n}{2} C_{AV} = \frac{(n-1)(3n+2)^2}{4}$$

$$C_{\max} = \frac{n(n-1)}{2} - 1 + n^4$$

2) Traveling Salesman Problem (2-exchange)

- $D = (d_{ij})$ adj. matrix

- $F = (f_{ij}) = \begin{pmatrix} 0 & 1 & 0 & . & . & . & . & . & 0 & 1 \\ 1 & 0 & 1 & 0 & . & . & . & . & . & 0 \\ 0 & 1 & 0 & 1 & 0 & . & . & . & . & . \\ 0 & 0 & 1 & 0 & 1 & 0 & . & . & . & . \\ . & . & 0 & 1 & 0 & 1 & 0 & . & . & . \\ . & . & . & 0 & 1 & 0 & 1 & 0 & . & . \\ . & . & . & . & 0 & 1 & 0 & 1 & 0 & 0 \\ . & . & . & . & . & 0 & 1 & 0 & 1 & 0 \\ 0 & . & . & . & . & . & 0 & 1 & 0 & 1 \\ 1 & 0 & . & . & . & . & . & 0 & 1 & 0 \end{pmatrix}$

$$S(F) = 2n$$

$$\langle F, D \rangle_+ = 2S(D)$$



Particular cases (TSP)

Th₁ ⇒ conclusion :

$$C_{\overline{\text{loc}}} \leq C_{AV}$$

[Grover]

$$C_{\overline{\text{loc}}} \leq \frac{\langle F, D \rangle_+}{S(F)S(D)} n C_{AV}$$



3) Bipartitioning

- weighted graph $G(V,E)$
- 2-exchange

$$|A| = |B| = \frac{|V|}{2} (\pm 1)$$

$$- D = (d_{ij}) \text{ adj. matrix}$$

Particular cases (Bipartitioning)

$$F = (f_{ij}) = \begin{pmatrix} O & U \\ U & O \end{pmatrix} \begin{matrix} \updownarrow n/2 \\ \updownarrow n/2 \end{matrix}$$

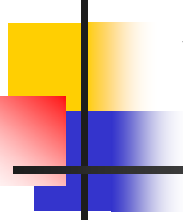
$$U = (u_{ij}) = \begin{cases} 1, & i, j = 1, \dots, \frac{n}{2} \\ 0 & \text{o t h e r w i s e} \end{cases}$$

$$C_{loc} \leq C_{AV}$$

$\frac{n}{2} S(D)$

$$C_{loc} \leq \frac{\langle F, D \rangle + n C_{AV}}{S(F)S(D)} \quad \text{[Grover]}$$

$\frac{n}{2}$



m-densest (m-lightest)

◦ Graph $G(V, E)$, $|V| = n$

consider $m (\leq n)$, $m = 1, \dots, n$

" Find a sub - graph $G'_G(V', E')$ with $|V'| = m$
and $|E'|$ maximum (minimum)

• NP - complete

- maximum clique

- maximum Independent set

Results (2-exchange)

$$F = \begin{pmatrix} \boxed{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \updownarrow m \\ \updownarrow n-m \end{matrix} \quad D = (d_{ij})$$

$d_1 \geq d_2 \geq \dots \geq d_n$ degrees of G

$$C_{loc}^+ \geq \frac{(m-1)}{2(n-1)} (d_n + d_{n-1} + \dots + d_{n-m+1})$$

$$C_{loc}^- \leq \frac{(m-1)}{2(n-1)} (d_1 + d_2 + \dots + d_m)$$

$$\underline{\underline{\text{Th}_1}} \left\{ \begin{array}{l} S(F) = m(m-1) \\ S(D) = 2|E| \\ C_{AV} = \frac{m(m-1)}{n(n-1)} |E| \\ \langle F, D \rangle_+ = (m-1)(d_1 + \dots + d_m) \end{array} \right.$$



5) Stable Maximum

(Meyers "72")

" If $d_1 \geq d_2 \geq \dots \geq d_n$ and $d_1 + d_2 + \dots + d_k \leq n - k$,
for $2 \leq k \leq n$ then stable maximal $\geq k$ "

(Angel & V.Z. "96")

" If $d_1 + d_2 + \dots + d_k \leq \left\lfloor \frac{2a(n-1)}{k-1} \right\rfloor_*$ for a integer,

$2 \leq k \leq n$

$$\lfloor x \rfloor_* = \begin{cases} x - 1 & \text{if } x \text{ integer} \\ \lfloor x \rfloor & \text{otherwise} \end{cases}$$

stable maximal $\geq k - a + 1$ "

Example

$n=10$

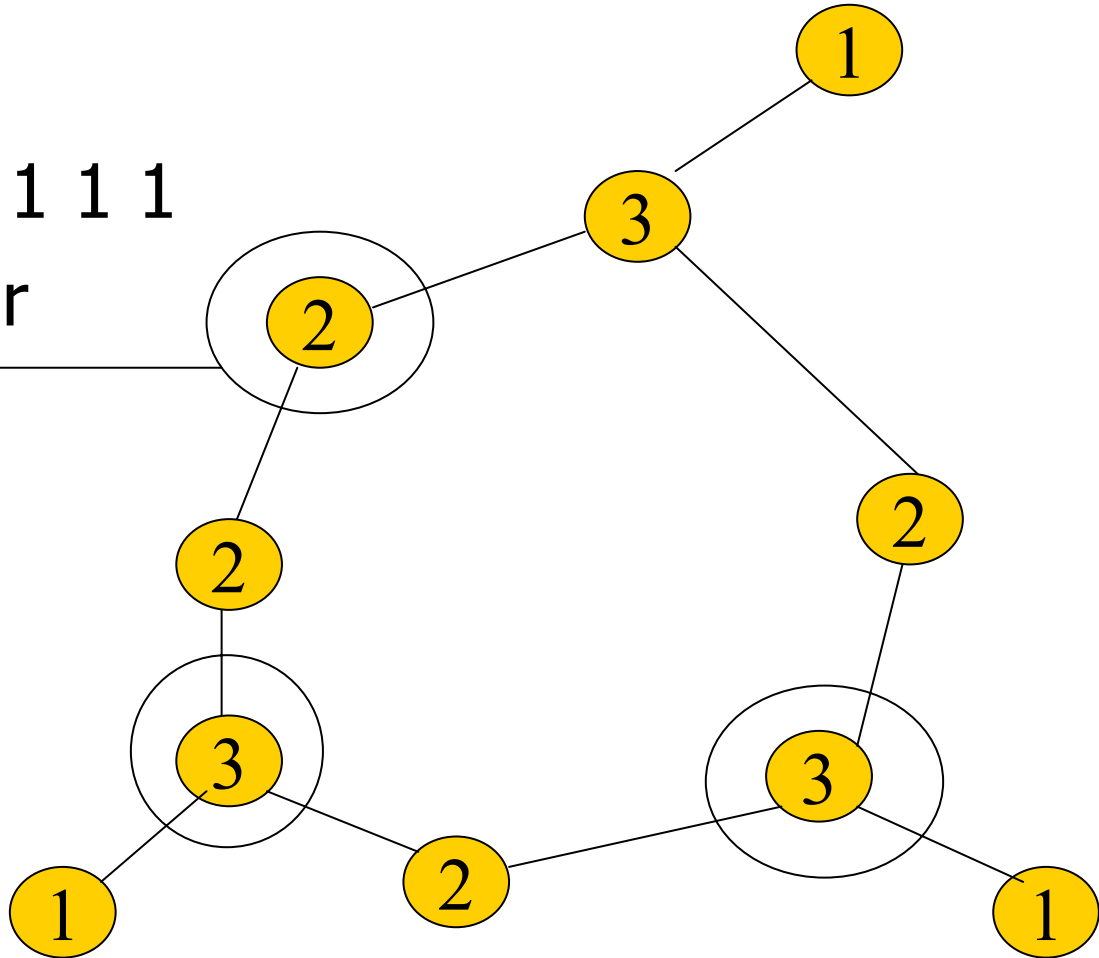
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$6 \leq 10 - 2$ Meyer

$a=3$

$K=5$

$k-a+1=3$





Comparison

- Meyer's result is deductive
- This result is constructive
- This condition is more flexible
- Result interesting in practice

" C_{loc} includes at most $a - 1$ edges "

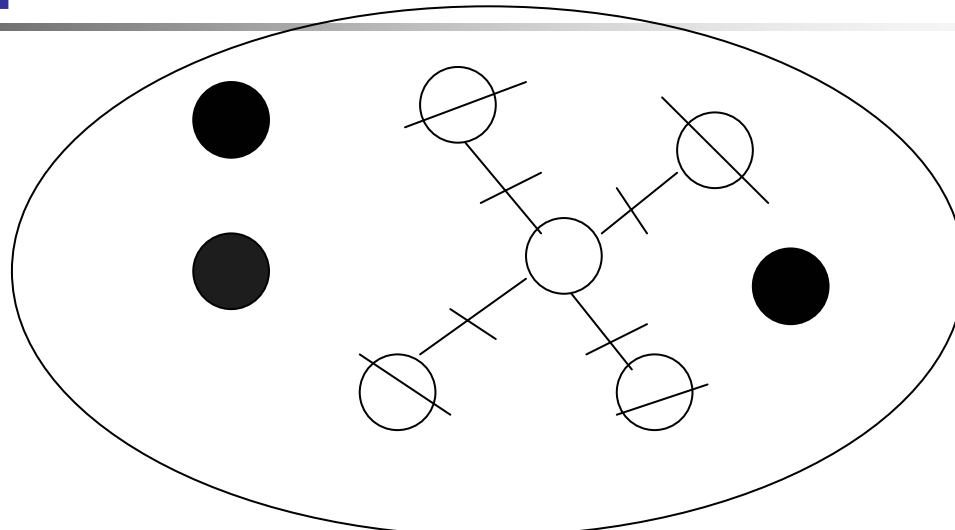
we remove at most $a - 1$ vertices

for obtaining a maximal Independent set

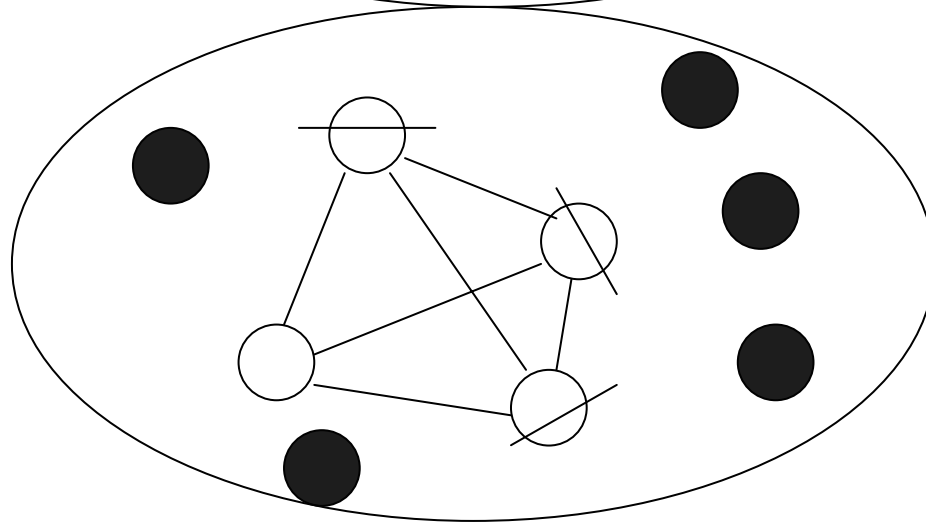
Example

ex

- $a=5$
4 edges
4 vertices



- $a=7$
6 edges
3 vertices





LS Drawbacks

- Local optimum
- “good“ neighborhoods
- exploration strategies
- Performances guarantee ?
- Parallelization ?



Conclusions

- Some new results on the performance of Local Search and its variants
- The k-densest and the k-lightest problems
- A constructive result for the max. Ind. Set Pr.
(Th. MEYER)



FUTURE WORK

- ?!
- Proposed bound for QAP is it attained?
- TSP with 2-OPT is it in PLS - complete?
- Other problems with guaranteed LS
- Parallelization of PBs in PLS
- " Good " neighborhood structures
- Scheduling scheme in SA