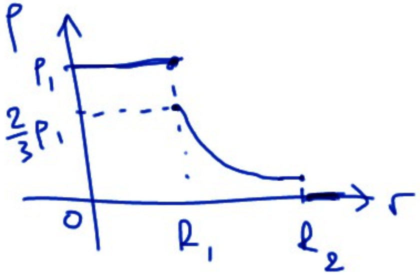
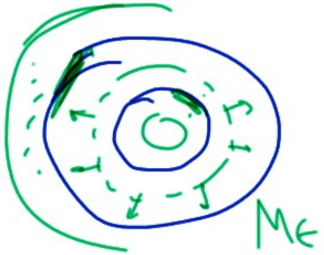
Σφαιρικά συμμετρική κατανομή μάζας με

$$\rho = \begin{cases} \rho_1 , & r < R_1 \\ \frac{2}{3}\rho_1 R_1 / r , & R_1 < r < R_2 \\ 0 , & r > R_2 \end{cases}$$



(α) $g = ;$



$$\left(g = -\frac{G M_{\varepsilon\gamma\kappa}}{r^2} \right)$$

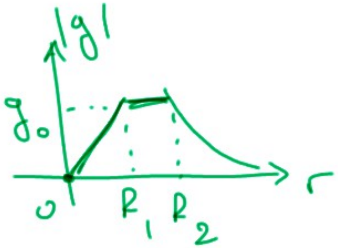
$$\oiint \bar{g} \cdot d\bar{S} = -4\pi G M_{\varepsilon\gamma\kappa}$$

$\bar{g} = g(r)\,\hat{r}$ δίνει $g \cancel{4\pi} r^2 = -\cancel{4\pi} G \int_0^r \rho \, 4\pi r^2 dr$

- $r < R_1 : \quad g r^2 = -G\rho_1 \frac{4\pi r^3}{3} \Leftrightarrow \bar{g} = -\frac{4\pi G \rho_1}{3} r \hat{r}$
- $R_1 < r < R_2 : \quad g r^2 = -G \left[\int_0^{R_1} \rho_1 4\pi r^2 dr + \int_{R_1}^{r} \frac{2}{3}\rho_1 \frac{R_1}{r} 4\pi r^2 dr \right] \Rightarrow \bar{g} = -\frac{4\pi G \rho_1 R_1}{3} \hat{r}$
- $r > R_2 : \quad g r^2 = -G \left[\int_0^{R_1} \cdots + \int_{R_1}^{R_2} \cdots \right] \Rightarrow \bar{g} = -4\pi \frac{G\rho_1 R_1 R_2^2}{3 r^2} \hat{r}$

Δηλ.

$$\bar{g} = \begin{cases} -g_0\, r/R_1\ \hat{r}, & r \le R_1 \\ -g_0\ \hat{r}, & R_1 < r < R_2 \\ -g_0\, R_2^2/r^2\ \hat{r}, & r > R_2 \end{cases} \qquad g_0 = 4\pi G \rho_1 R_1/3$$



(β) Πόση ενέργεια ξοδεύουμε για να απομακρύνουμε το κέλυφος $R_1 < r < R_2$;

Η δυναμική ενέργεια της αρχικής κατανομής είναι $V_1 = \frac{1}{2}\iiint \rho \Phi\, d^3\bar{r}$

$\bar{g} = -\nabla \Phi \qquad \Phi = -\int_\infty^r \bar{g}\cdot d\bar{r} \quad \ldots$

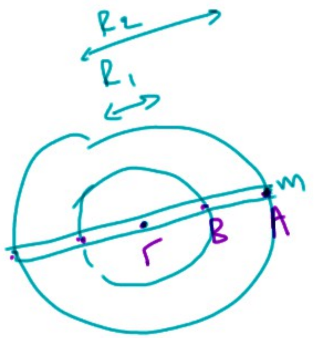
ή ισοδύναμα $V_1 = -\iiint \frac{g^2}{8\pi G} \underbrace{d^3\bar{r}}_{4\pi r^2 dr} =$

$$= -\int_0^{R_1} \frac{g_0^2\, r^2/R_1^2}{8\pi G} 4\pi r^2 dr - \int_{R_1}^{R_2} \frac{g_0^2\, 4\pi r^2 dr}{8\pi G} - \int_{R_2}^{\infty} \frac{g_0^2 R_2^4/r^4}{8\pi G} 4\pi r^2 dr =$$

$$\Rightarrow V_1 = -\frac{g_0^2}{15G}\left(10 R_2^3 - R_1^3\right) R_1$$

Χωρίς το κέλυφος $V_2 = -\frac{3 g_0^2 R_1^3}{5G}$ (θέτω $R_2 = R_1$)

Δίνω $\Delta E = V_2 - V_1 = \frac{2 g_0^2 (R_2^3 - R_1^3)}{3G}$



Ανοίγω "πηγάδι" και αφήνω m στο ένα άκρο.
Ποια η περίοδος της κίνησης;

$$\bar{g} = \begin{cases} -g_0 \, r/R_1 \, \hat{r} , & r<R_1 \\ -g_0 \, \hat{r} , & R_1<r<R_2 \end{cases}$$

Λύση:

Για $A \to B$: $\bar{\alpha} = \bar{g} \Leftrightarrow \ddot{r} = -g_0 \Leftrightarrow \dot{r} = -g_0 t + \cancel{c} \Leftrightarrow r = -\dfrac{g_0 t^2}{2} + R_2$

$$t_{AB} \xrightarrow{r=R_1} \sqrt{\frac{2(R_2-R_1)}{g_0}}$$

Για $B \to \Gamma$: $\ddot{r} = -\dfrac{g_0}{R_1} r \Leftrightarrow \ddot{r} + w^2 r = 0$ με $w = \sqrt{g_0/R_1}$

$r = C_1 \sin(wt') + C_2 \cos(wt')$ με $r\big|_{t'=0} = R_1 \Leftrightarrow C_2 = R_1$

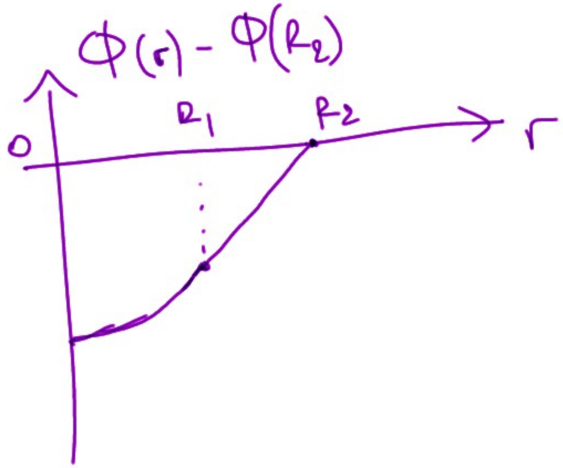
$t' = t - t_{AB}$ και $\dot{r}\big|_{t'=0} = -g_0 t_{AB} \Leftrightarrow C_1 = -\sqrt{2R_1(R_2-R_1)}$

Άρα $r\big|_{t'=t_{B\Gamma}} = 0 \Leftrightarrow 0 = -\sqrt{2R_1(R_2-R_1)} \sin(w t_{B\Gamma}) + R_1 \cos(w t_{B\Gamma}) \Leftrightarrow$

$$t_{B\Gamma} = \sqrt{R_1/g_0} \arctan\sqrt{\frac{R_1}{2(R_2-R_1)}}$$

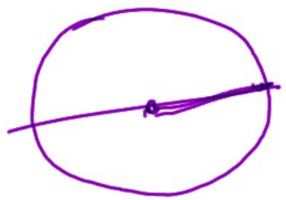
Τελικά $T = 4(t_{AB} + t_{B\Gamma})$

Αλλιώς: $\Phi(r) - \Phi(R_2) = -\int_{R_2}^{r} g\,dr = \begin{cases} g_0\left(r-R_2\right) & \text{για } R_1<r<R_2 \\ g_0 \dfrac{r^2-2R_1R_2+R_1^2}{2R_1} & \text{για } r<R_1 \end{cases}$



$$\frac{\dot r^2}{2} + \Phi(r) = \text{σταθ} = \Phi(R_2)$$

$$\frac{m\dot r^2}{2} + \underset{\substack{\uparrow \\ m\Phi}}{V} = \text{σταθ}$$



$$\dot r = -\sqrt{2\left[\Phi(R_2)-\Phi(r)\right]} \quad \text{και} \quad T = 4\int_{R_2}^{0}\frac{dr}{\dot r}$$

$$\int \frac{dr}{\sqrt{(\;)-r^2}}$$

$r=\sqrt{(\;)}\,\sin\varphi$