

Quasi-linear PDEs (I)

Method of characteristics and
introduction to shock waves

First-order PDEs and useful notions

Consider a **first-order linear PDE**, in two variables, which can generally be expressed as:

$$A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} + C_1(x, y)u = C_0(x, y)$$

This equation is called **homogeneous** if $C_0 \equiv 0$.

More generally, functions A, B, C_1 may depend on u ; in this case, the first-order PDE of the form:

$$A(x, y, u) \frac{\partial u}{\partial x} + B(x, y, u) \frac{\partial u}{\partial y} = C(x, y, u)$$

is called **quasi-linear** (in two variables).

Remark:

Every linear PDE is also quasi-linear, because we can set:

$$C(x, y, u) = C_0(x, y) - C_1(x, y)u.$$

A prototypical -physically significant- example:

Transport (advection) equation

$$u_t + c u_x = 0$$

Linear & homogeneous PDEs

$$u_t + c(x, t) u_x = 0$$

$$u_t + c(x, t) u_x = h(x, t)$$

Linear & inhomogeneous PDE

$$u_t + c(u) u_x = 0$$

Quasi-linear PDE

Remark:

$$\left. \begin{aligned} u_t + c(u) u_x = 0 &\Rightarrow c'(u) u_t + c(u) c'(u) u_x = 0 \\ c'(u) u_t &= c_t, \quad c(u) c'(u) u_x = c c_x \end{aligned} \right\} \Rightarrow$$

$$c_t + c c_x = 0 \quad \text{Hopf (Riemann / inviscid Burgers) equation}$$

Method of characteristics

Consider the Cauchy problem for the **quasi-linear PDE**:

$$a(x, t, u) u_t + b(x, t, u) u_x = c(x, t, u), \quad u(x, 0) = f(x)$$

- Introduce a curve Γ defined as: $\Gamma: \begin{cases} x = x(r), & x(0) = s \\ t = t(r), & t(0) = 0 \end{cases}$
- Assume that, on Γ : $u(x, t) = u(x(r), t(r))$, and differentiate wrt r :

$$\frac{du}{dr} = \frac{\partial u}{\partial t} \frac{dt}{dr} + \frac{\partial u}{\partial x} \frac{dx}{dr} \quad \text{Next, require: } \frac{dt}{dr} = a, \quad \frac{dx}{dr} = b$$

- Then, on Γ , the quasi-linear PDE is reduced to the ODE: $\frac{du}{dr} = c$

- The equations: $\frac{dt}{dr} = a, \quad \frac{dx}{dr} = b, \quad \frac{du}{dr} = c \Rightarrow \frac{dt}{a} = \frac{dx}{b} = \frac{du}{c}$

are called the **characteristics** of the quasi-linear PDE.

Geometrical interpretation

Consider again the Cauchy problem for the **quasi-linear PDE**:

$$a(x,t,u) u_t + b(x,t,u) u_x = c(x,t,u), \quad u(x,0) = f(x)$$

- Let $u(x,t)$ a **solution** of the PDE, and its **graph** $z = u(x,t)$, which is a **surface** in the **xtu-space**. The initial data which define a **curve** γ in the xt -plane (e.g., if $u(x,0)=f(x)$ then γ is the x -axis, etc) provides a **space curve** Γ that **lies on the graph**.
- Let $\mathbf{F}=(a, b, c)$ the **vector field** defined by PDE's coefficients
- The **normal vector** $\mathbf{N}$ to the surface $z=u(x,t)$ is: $\mathbf{N}=(u_x, u_t, -1)$
- If $u(x,t)$ a **solution** of the PDE then

$$\mathbf{F} \cdot \mathbf{N} = au_t + bu_x - c = 0 \Leftrightarrow \mathbf{F} \perp \mathbf{N} \Leftrightarrow \mathbf{F} \text{ tangent to } z = u(x,t)$$

- ➡ **The graph of the solution can be constructed by finding the stream lines of $\mathbf{F}$ that pass through the initial curve Γ .**

1. Transport equation with const. velocity

- Simplest linear 1st-order problem:
transport (advection) equation

➤ Method of characteristics

Reduce the problem to an ODE along
some curve $\Gamma: x=x(t)$ such that $du/dt=0$

$$u_t + c u_x = 0,$$
$$u(x, 0) = F(x)$$

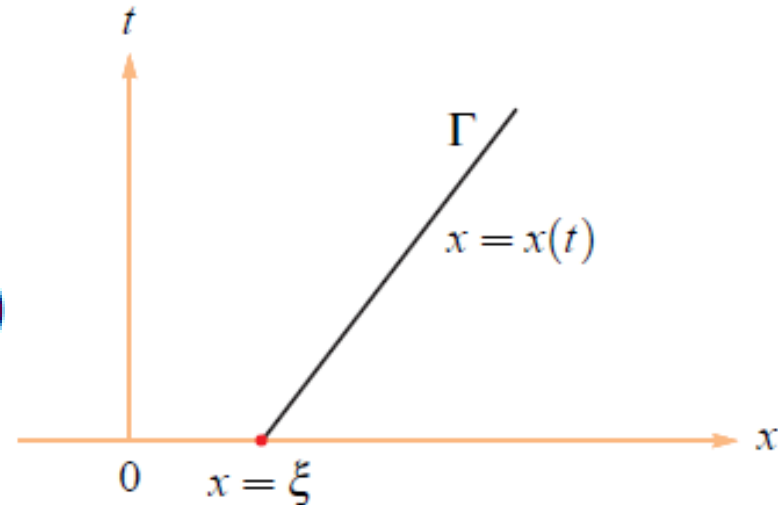
$\omega = ck \in \mathbb{R}, \omega'(k) = 0$
Non-dispersive system

$$\frac{du(x(t), t)}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial t} \frac{dt}{dt} = \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} \right) = 0 \quad \frac{dx}{dt} = c$$

$$\Rightarrow \frac{dx}{dt} = c \Rightarrow x(t) = ct + \xi$$

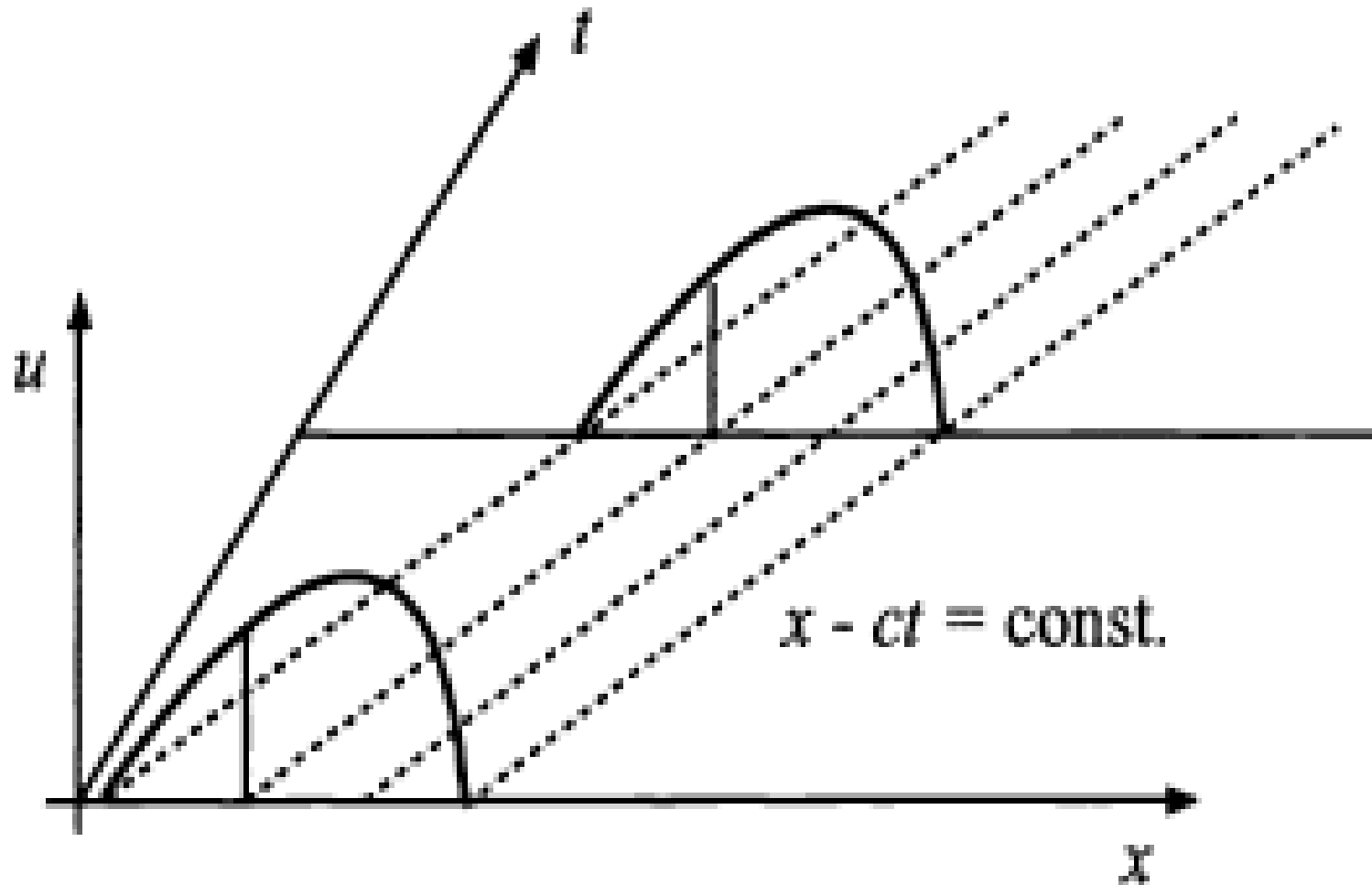
$$\Rightarrow \frac{du}{dt} = 0 \Rightarrow u(x, t) = u(\xi, 0) = F(\xi)$$

General solution: $u(x, t) = F(x - ct)$



Solution of the transport equation

The IC, $F(x)$, is simply translated **without changing shape**



2. Transport equation with non const. velocity

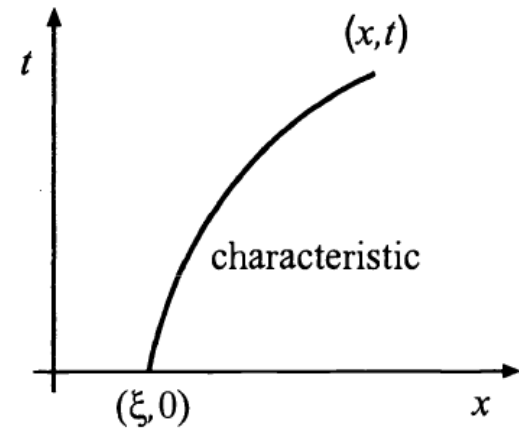
Consider the Cauchy problem:

$$u_t + 2t u_x = 0 \quad (c = 2t)$$

$$u(x, 0) = f(x) = \exp(-x^2)$$

On $\Gamma: x=x(t)$ we have $du/dt=0$:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} \frac{dt}{dt} + \frac{\partial u}{\partial x} \frac{dx}{dt} = u_t + \frac{dx}{dt} u_x = 0$$



This leads to: $\frac{dx}{dt} = 2t$, $x(0) = \xi \Rightarrow x(t) = t^2 + \xi$

Also, on Γ :

$$du/dt = 0 \Rightarrow u(x, t) = \text{const.}$$

$$t = 0: u(x, t) = u(\xi, 0)$$

$$\text{IC: } u(x, 0) = f(x) \Rightarrow u(\xi, 0) = f(\xi)$$

$$\Rightarrow \underline{u(x, t) = f(\xi) = \exp[-(x - t^2)^2]}$$

A twist: characteristics and boundary-value problems

Consider the BVP:
$$\begin{cases} x^2 u_t + u_x + tu = 0, & x > 0, \quad t \in \mathbb{R} \\ \text{BC at } x = 0: & u(0, t) = f(t) \end{cases}$$

On $\Gamma: t=t(x)$ we have:
$$\frac{du}{dx} = \frac{\partial u}{\partial t} \frac{dt}{dx} + \frac{\partial u}{\partial x} = \frac{dt}{dx} u_t + u_x$$

and we choose: $\frac{dt}{dx} = x^2 \Rightarrow t = \frac{1}{3} x^3 + \tau \longrightarrow$ our ξ in this case

On $\Gamma: t=t(x)$ we have:
$$\frac{du}{dx} = -tu \Rightarrow \frac{du}{dx} = -\left(\frac{1}{3} x^3 + \tau\right) u \quad \text{and thus:}$$

$$\frac{du}{u} = -\left(\frac{1}{3} x^3 + \tau\right) dx \Rightarrow \int_{f(\tau)}^u \frac{du}{u} = \int_0^x -\left(\frac{1}{3} x^3 + \tau\right) dx \Rightarrow$$

$$u = f(\tau) \exp\left(-\frac{1}{12} x^4 - \tau x\right) = f\left(t - \frac{1}{3} x^3\right) \exp\left(-\frac{1}{4} x^4 - xt\right)$$

The Hopf equation

$$u_t + u u_x = 0, \quad u(x, 0) = f(x), \quad x \in \mathbb{R}, \quad t > 0$$

Physical origin:

➤ **Motion of a fluid / gas dynamics** is governed by the system:

- **Continuity equation:** $\rho_t + \nabla(\rho \mathbf{v}) = 0,$

- **Euler equation:** $\mathbf{v}_t + (\mathbf{v} \nabla) \mathbf{v} = -(1/\rho) \nabla p$

together with a “equation of state”: $p = p(\rho)$

➤ Kinetic theory of gases – ideal gas: $p = NkT = (\rho/m)kT$

➤ In the **limiting case:** $m \rightarrow \infty$ the pressure vanishes: $p = 0$

Thus, for “**heavy particles**” the gas dynamics equations decouple:

$$\mathbf{v}_t + (\mathbf{v} \nabla) \mathbf{v} = 0, \quad \rho_t + \nabla(\rho \mathbf{v}) = 0$$

➤ In the case of **1D flow**, the velocity field is governed by:

$$v_t + v v_x = 0 \quad \textbf{Hopf (Riemann / inviscid Burgers) equation}$$

Hopf equation – method of characteristics

$$u_t + u u_x = 0, \quad u(x,0) = f(x), \quad x \in \mathbb{R}, \quad t > 0$$

On $\Gamma: x=x(t)$ we have: $du/dt=0$: $\frac{du}{dt} = \frac{\partial u}{\partial t} \frac{dt}{dt} + \frac{\partial u}{\partial x} \frac{dx}{dt} = u_t + \frac{dx}{dt} u_x = 0$

As before:

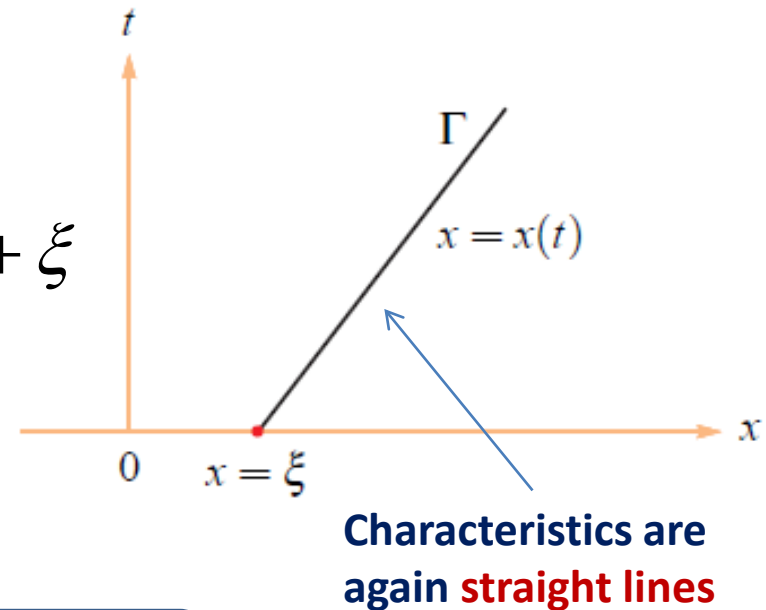
$$du/dt = 0 \Rightarrow u(x,t) = \text{const.}$$

$$dx/dt = u, \quad x(0) = \xi \Rightarrow x(t) = ut + \xi$$

Since:

$$u(x,0) = f(x) \Rightarrow u(\xi,0) = f(\xi)$$

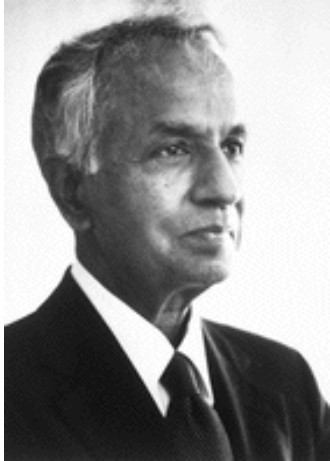
$$\left. \begin{aligned} u(x,t) &= u(\xi,0) = f(\xi) \\ \xi &= x - ut \end{aligned} \right\} \Rightarrow \boxed{u = f(x - ut)}$$



Implicit solution of the Hopf equation

An explicit solution of the Hopf equation

S. Chandrasekhar found (1943) an **explicit solution** of the following IVP:

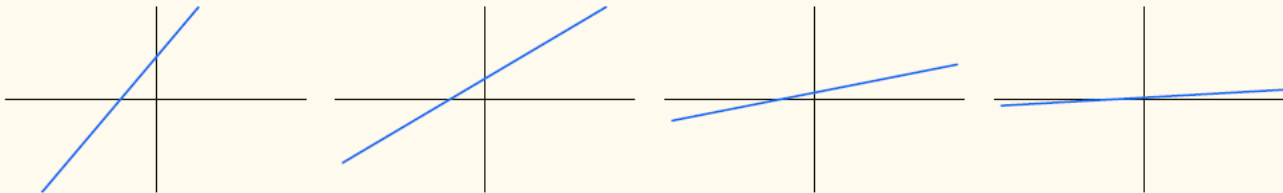


$$u_t + u u_x = 0, \quad u(x, 0) = f(x) = ax + b$$

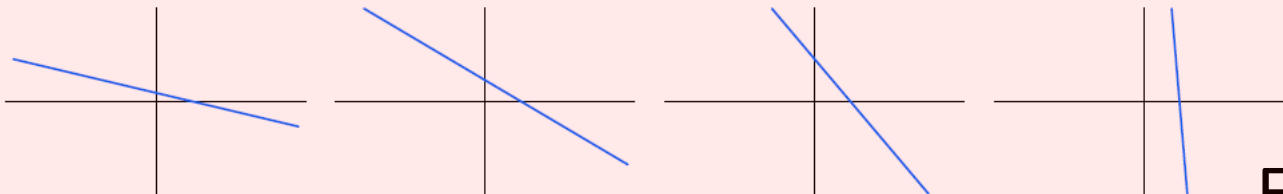
We have found: $u = f(x - ut)$

and thus: $u = a(x - ut) + b \Rightarrow u = \frac{ax + b}{1 + at}$

$a > 0$: the solution **flattens** as $t \rightarrow \infty$



Rarefaction wave

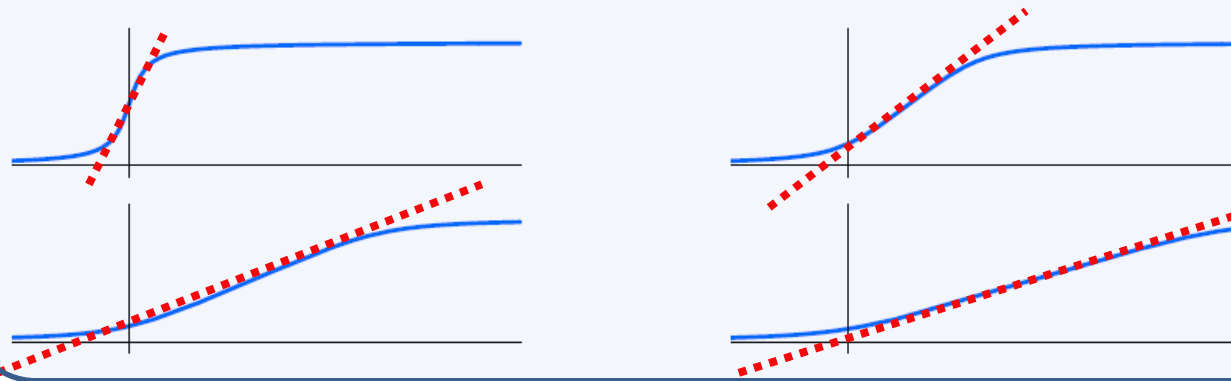


$a < 0$: the solution **steepens** as $t \rightarrow \infty$

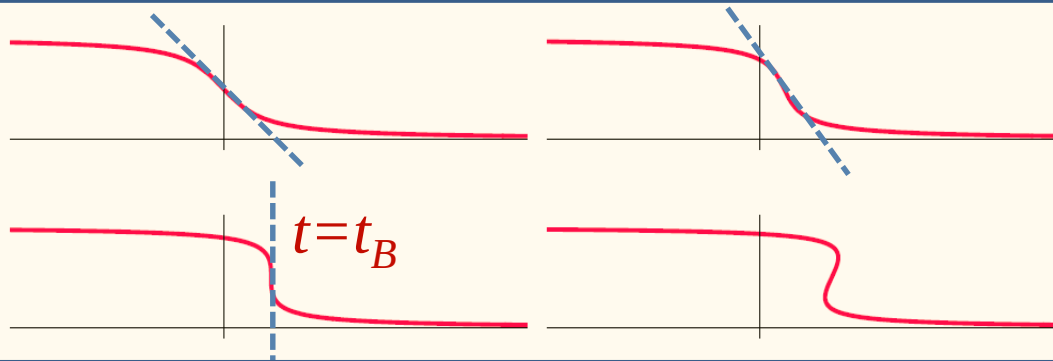
Shock wave

For $t = t_B = -1/a$ the solution **blows up**

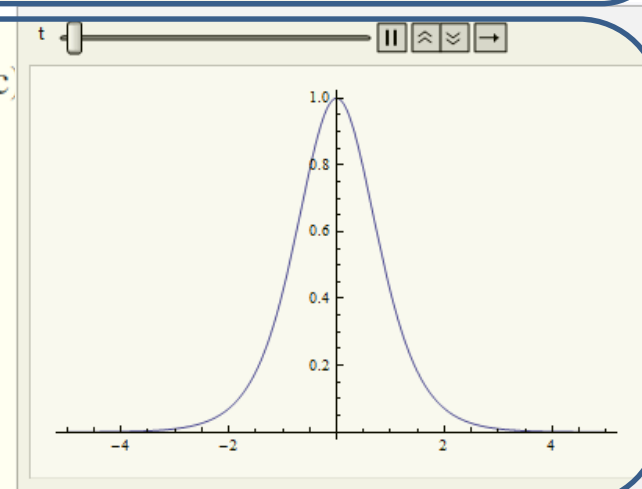
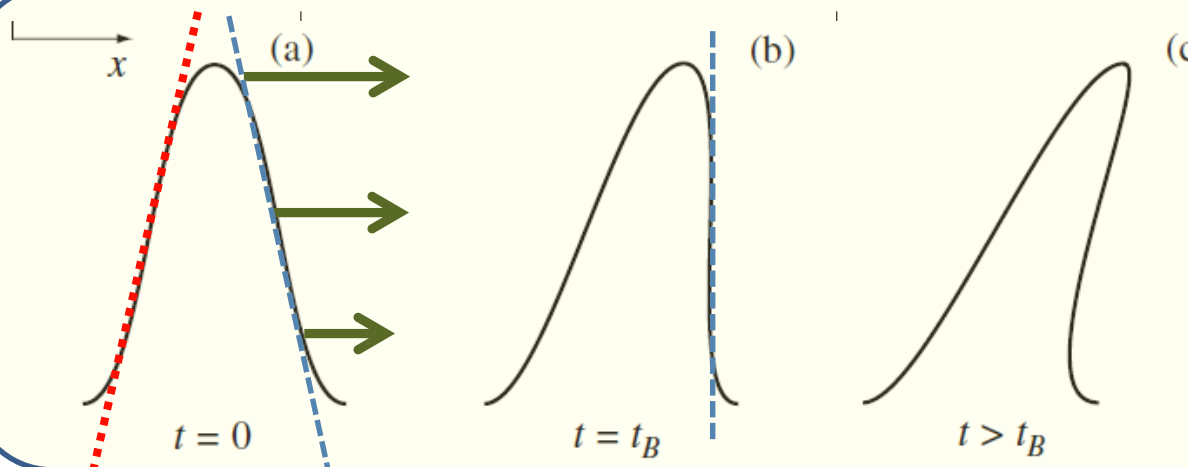
What can we learn from the explicit solution?



Positive slope
Rarefaction wave



Negative slope
Shock wave



Breaking time

Consider the general problem: $u_t + u u_x = 0$, $u(x,0) = f(x)$

We have found:
$$\left. \begin{array}{l} x = ut + \xi \\ u = f(\xi) \end{array} \right\} \Rightarrow \underline{x = \xi + f(\xi)t} \quad (1)$$

We wish to determine the **breaking time** t_B occurring when the profile of the solution develops an infinite slope:

$$\left. \begin{array}{l} u_x = \frac{\partial u}{\partial x} = \frac{du}{d\xi} \frac{\partial \xi}{\partial x} = f'(\xi) \xi_x \\ (1), \partial_x : 1 = \xi_x + f'(\xi)t \xi_x \Rightarrow \xi_x = \frac{1}{1 + f'(\xi)t} \end{array} \right\} \Rightarrow u_x = \frac{f'(\xi)}{1 + f'(\xi)t}$$

If $f'(\xi) > 0 \quad \forall x$ then the solution is **finite** $\forall t \rightarrow$ rarefaction wave

If $f'(\xi) < 0$ the solution breaks up at the **earliest critical time**:

$$t_B = \min_{\xi > 0} \left\{ -1 / f'(\xi) \mid f'(\xi) < 0 \right\}$$

What happens before the breaking time t_B

Using $x = \xi + f(\xi)t$ (1) we found: $u_x = \frac{f'(\xi)}{1 + f'(\xi)t}$ (2)

Similarly:

$$\left. \begin{aligned} u_t &= \frac{\partial u}{\partial t} = \frac{du}{d\xi} \frac{\partial \xi}{\partial t} = f'(\xi) \xi_t \\ (1), \partial_t : 0 &= \xi_t + f'(\xi)t \xi_t + f(\xi) \Rightarrow \xi_t = -\frac{f(\xi)}{1 + f'(\xi)t} \end{aligned} \right\} \Rightarrow$$

$$u_t = -\frac{f(\xi)f'(\xi)}{1 + f'(\xi)t} \quad (3)$$

Hence, from (2)-(3), and for $t < t_B$, the solution remains **single-valued** and **satisfies the Hopf equation**:

$$u_t + uu_x = -\frac{f(\xi)f'(\xi)}{1 + f'(\xi)t} + f(\xi)\frac{f'(\xi)}{1 + f'(\xi)t} = 0 !!!$$

What happens for times $t \geq t_B$:

Characteristics intersect

The slope of a characteristic passing (x^0, t^0) is: $\frac{dx}{dt} = c(f(x^0))$

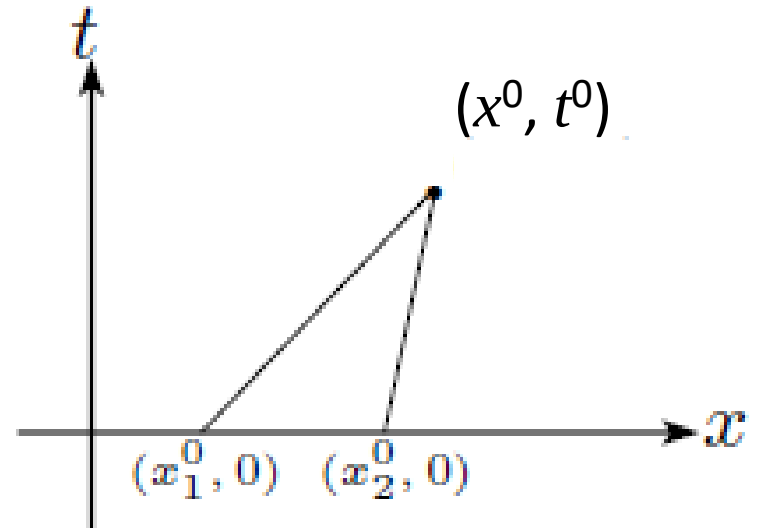
Here, $c = c(u)$ is a **strictly increasing** function:

$$u_t + c(u)u_x = 0, \quad c(u) = u \Rightarrow c'(u) = 1 > 0$$

Let f be a **strictly decreasing** function;
then, $c(f)$ is also **strictly decreasing**:

$$x_1^0 < x_2^0 \rightarrow c(f(x_1^0)) > c(f(x_2^0))$$

and, hence, characteristics passing
through $(x_1^0, 0)$, $(x_2^0, 0)$ **intersect!**

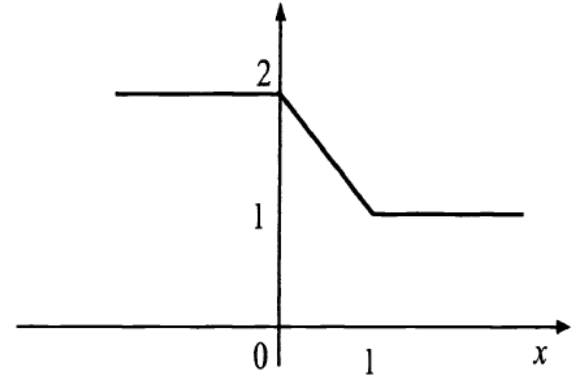


At the intersection point, the solution $u(x, t)$ becomes **multi-valued** because it takes both values $f(x_1^0)$, $f(x_2^0)$

An example

Consider the IVP: $u_t + u u_x = 0$, $u(x,0) = f(x)$, $x \in \mathbb{R}$

$$f(x) = \begin{cases} 2, & x < 0 \\ 2 - x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$



We have found:
$$\left. \begin{aligned} x &= ut + \xi \\ u &= f(\xi) \end{aligned} \right\} \Rightarrow x = \xi + f(\xi)t$$

Characteristics:

$$f(\xi) = \begin{cases} 2, & \xi < 0 \\ 2 - \xi, & 0 \leq \xi \leq 1 \\ 1, & \xi > 1 \end{cases}$$

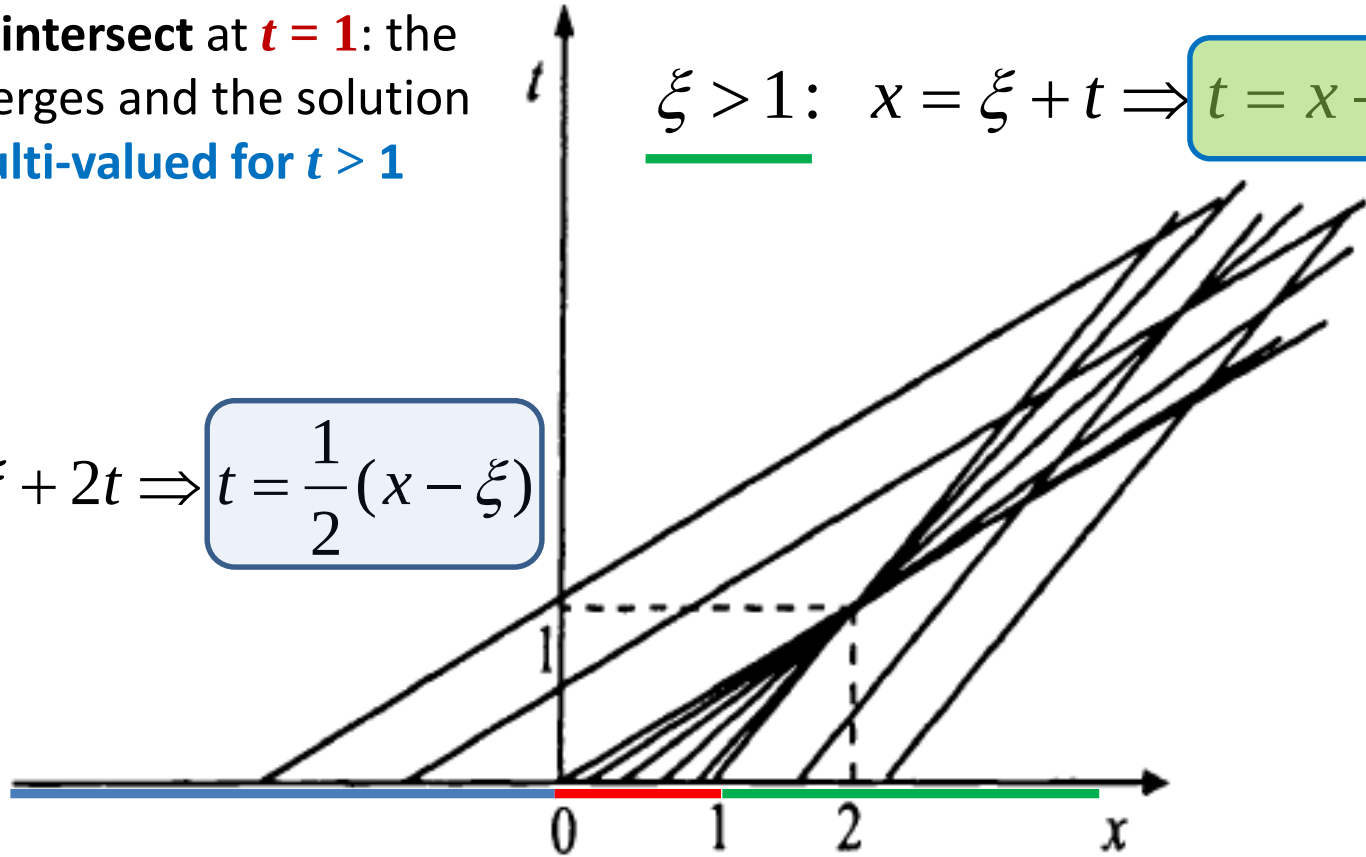
$$x = \begin{cases} \xi + 2t, & \xi < 0 \\ \xi + (2 - \xi)t, & 0 \leq \xi \leq 1 \\ \xi + t, & \xi > 1 \end{cases}$$

An example (cont.) - characteristics

Characteristics intersect at $t = 1$: the **shock wave** emerges and the solution becomes **multi-valued for $t > 1$**

$\xi > 1$: $x = \xi + t \Rightarrow t = x - \xi$

$\xi < 0$: $x = \xi + 2t \Rightarrow t = \frac{1}{2}(x - \xi)$



$0 \leq \xi \leq 1$: $x = \xi + (2 - \xi)t \Rightarrow t = \frac{x - \xi}{2 - \xi}$

An example (cont.) – breaking time

How to determine the **breaking time** t_B :

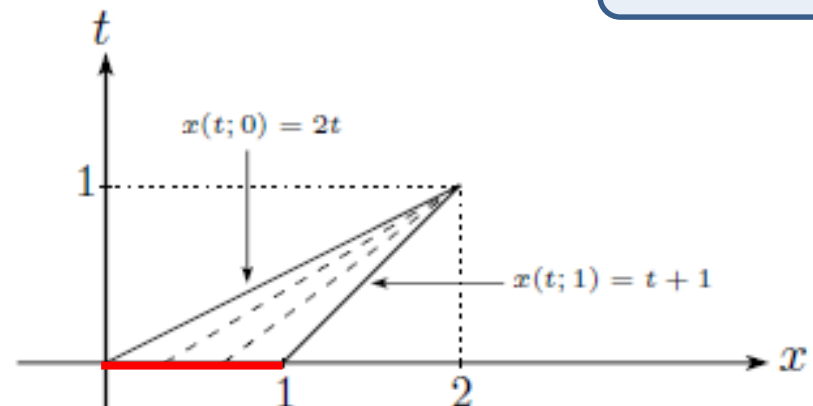
Recall that: $t_B = \min_{\xi} \left\{ -1 / f'(\xi) \mid f'(\xi) < 0 \right\}$

Here, we have: $f(\xi) = \begin{cases} 2, & \xi < 0 \\ 2 - \xi, & 0 \leq \xi \leq 1 \\ 1, & \xi > 1 \end{cases}$ and hence $t_B = 1$

Alternatively, recall that we found: $x = \xi + (2 - \xi)t$, $0 \leq \xi \leq 1$

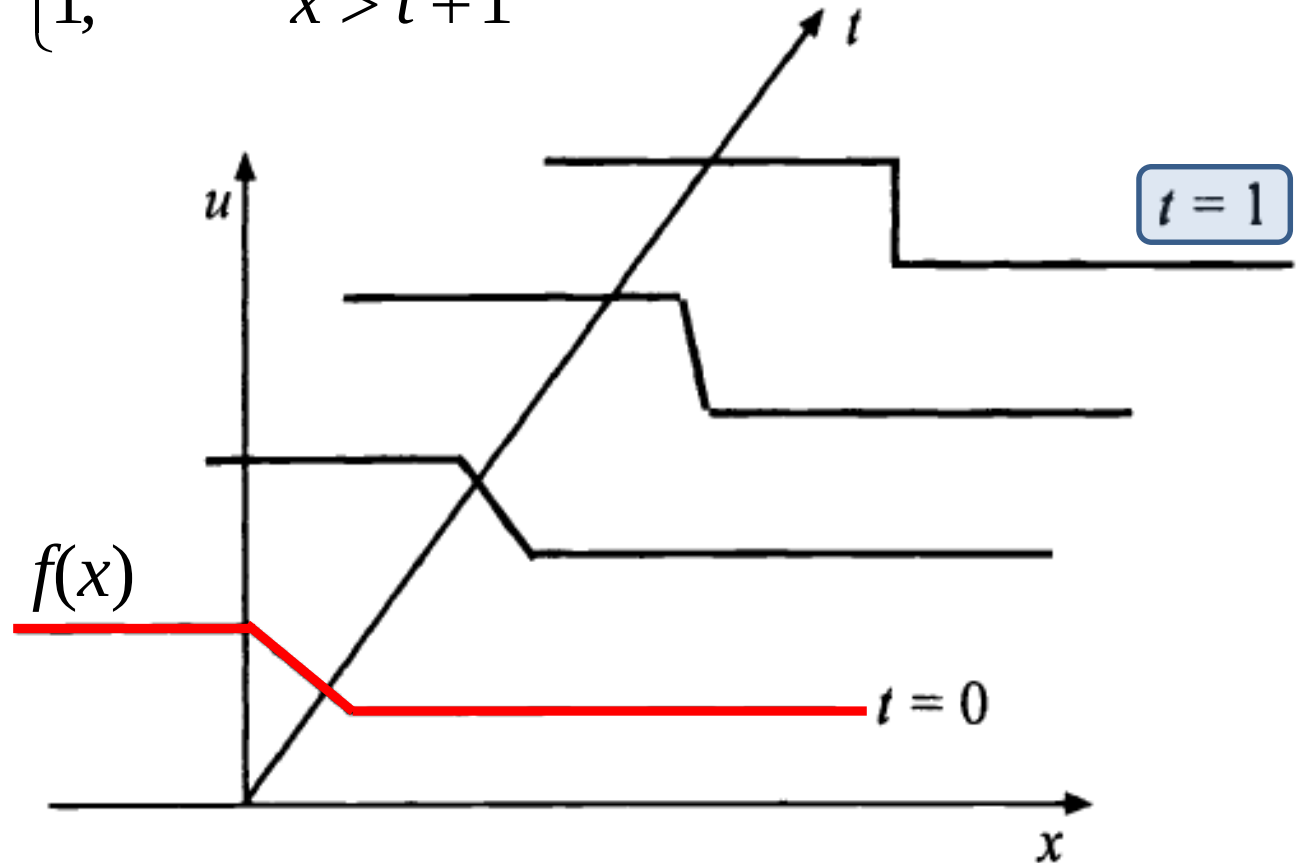
and so $\left. \begin{array}{l} \underline{\xi = 0}: x = 2t \\ \underline{\xi = 1}: x = 1 + t \end{array} \right\} \Rightarrow \text{characteristics intersect at } (x, t) = (2, 1)$

All other characteristics in this interval pass $(x, t) = (2, 1)$



An example (cont.) – shock formation

Solution: $u(x,t) = \begin{cases} 2, & x < 2t \\ \frac{2-x}{1-t}, & 2t \leq x \leq t+1 \\ 1, & x > t+1 \end{cases}$



Exercises – linear problems

1) Use the method of characteristics to solve the IVPs:

1.1) $u_t + u_x + u = 0, \quad u(x,0) = f(x) = \cos x, \quad x \in \mathbb{R}$

1.2) $u_t + 2xt u_x = u, \quad u(x,0) = f(x) = x, \quad x \in \mathbb{R}$

2) Use the method of characteristics to show that the solution of the IVP:

$$u_t + cu_x = h(x,t), \quad u(x,0) = f(x), \quad x \in \mathbb{R}$$

is: $u(x,t) = f(x - ct) + \int_0^t h(x - c(t - t'), t') dt'$

In all cases confirm, by direct substitution, that the solution you found satisfies the corresponding PDE.

Exercises – nonlinear problems

- 3)** Use the method of characteristics to solve the Hopf equation:

$$u_t + uu_x = 0, \quad u(x,0) = f(x)$$

in the following cases:

$$\begin{array}{ll} \mathbf{3.1)} & f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases} \\ \mathbf{3.2)} & f(x) = \begin{cases} 0, & x < 0 \\ -x, & 0 \leq x \leq 1 \\ -1, & x > 1 \end{cases} \end{array}$$

In both cases:

- a)** Draw the characteristics in the xt -plane
- b)** Write down the solution and draw some characteristic snapshots of the solution at different time instants
- c)** Determine the breaking time (when relevant)