Burgers equation

Regularized (viscous) shock waves

Cole-Hopf transformation

Introducing the Burgers equation

Burgers' equation is a fundamental PDE occurring in various areas of applied mathematics and physics, such as **fluid mechanics**, **nonlinear acoustics**, **gas dynamics**, **traffic flow**,... It has the form:

$$u_t + uu_x = \nu u_{xx}$$

$u_t + uu_x$: **Hopf equation (nonlinear transport equation)**

νu_{xx}: **diffusion term**

For instance, for an incompressible fluid, **the fluid velocity** satisfies the **Navier-Stokes (NS) equations**:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \nu \Delta \mathbf{v} - \frac{1}{\rho} \nabla p \,, \qquad \nabla \cdot \mathbf{v} = 0$$

ν: kinematic **viscocity**; ρ: **density**; p: **pressure**

In (1+1)-dimensions, and in the absence of pressure, NS equations reduce to Burgers equation [Burgers (1939)]

The linear counterpart of the Burgers equation

The linear counterpart of the Burgers equation is of the form:

$$u_t + cu_x = \nu u_{xx};$$

Complex dispersion relation: $\omega = ck - i\nu k^2 \;\Rightarrow$

plane waves $\propto \exp[i(kx - \omega t)]$, are of the form:

$$u = u_0 \exp[ik(x - ct)] \exp(-\nu k^2 t)$$

right-going traveling wave

decaying amplitude

short waves attenuate faster than long ones

For $c = 0$ (diffusion equation), plane waves are:

$$\propto \exp(ikx) \exp(-\nu k^2 t)$$

These are not traveling waves – they oscillate in x and decay in t

Burgers equation – the role of diffusion

$$u_t + uu_x = \nu u_{xx}$$

- One expects that **the nonlinear term *uu_x***, will **tend to steepen the wave** **up to the formation of a shock** (and the eventual break up of the wave), while **the diffusion *νu_{xx}*** is expected to have a **smoothing out and broadening effect**.

- It is therefore reasonable to ask **whether the presence of diffusion can prevent the appearance of a discontinuous shock wave**.

- Indeed, as we will show below, there exist traveling waves, in the form of **viscous shocks** which, **for any finite ν**, **remain smooth and well-defined for all times**.

Traveling wave solutions

We seek **traveling wave solutions** of the Burgers equation

$$u_t + uu_x = \nu u_{xx}$$

of the form: $u(x,t) = u(\xi),$ $\xi = x - ct,$

where c is the **unknown velocity**. Taking into regard that:

$$u_t = -cu'(\xi),\ u_x = u'(\xi),\ \text{and}\ u_{xx} = u''(\xi)$$

we obtain the 2nd-order ODE: $-cu' + uu' - \nu u'' = 0.$

Then, noting that: $uu' = (u^2/2)'$, we integrate wrt. ξ and obtain:

$$-cu + \frac{1}{2}u^2 - \nu u' = K,$$

where K is a constant of integration

The associated dynamical system

We have thus derived the 1st-order autonomous nonlinear ODE:

$$-cu + \frac{1}{2}u^2 - \nu u' = K \qquad (1)$$

which admits **non-constant solutions** which **either tend to infinity or to one of the equilibrium points, as $t \to \pm\infty$.** Since we are interested in obtaining bounded solutions, we rewrite (1) as:

$$2\nu u' = u^2 - 2cu - 2K = 0,$$

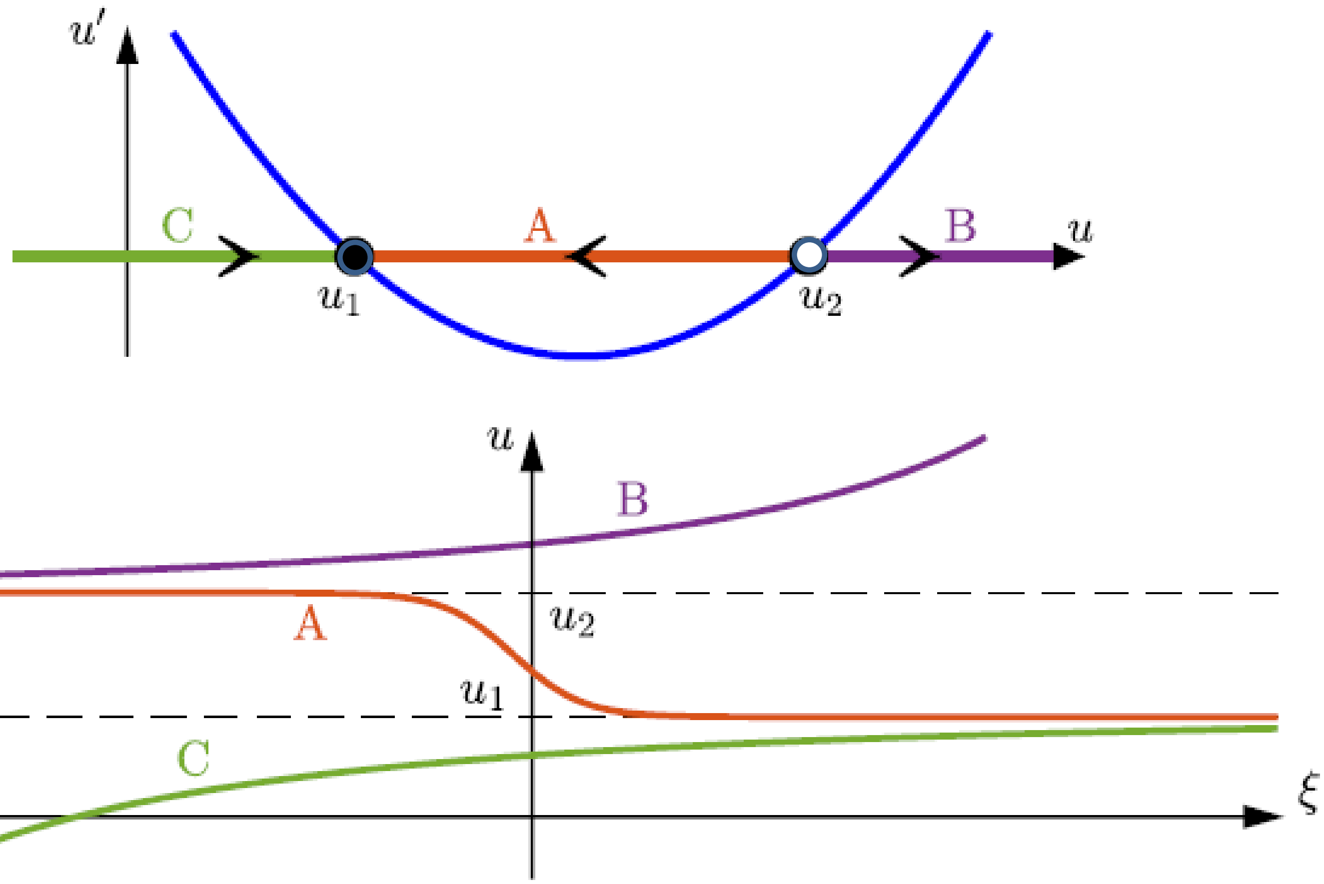
and require that the equilibrium points, i.e., **the roots $u_{1,2}$ of the quadratic polynomial** $u^2 - 2cu - 2K = 0$ **be real**. The roots are:

$$u_1 = c - \sqrt{c^2 + 2K}, \quad u_2 = c + \sqrt{c^2 + 2K},$$

We thus require that: $c^2 + 2K > 0$, and thus: $u_1 < u_2$.

Fixed points and phase plane

Dynamical system: $$\frac{du}{d\xi} = \frac{1}{2\nu}(u - u_1)(u - u_2)$$



Bounded solutions of the Burgers equation occur for: $u_1 < u < u_2$

The traveling shock wave solution

Integrating the ODE: $\frac{du}{d\xi} = \frac{1}{2\nu}(u - u_1)(u - u_2)$ we obtain:

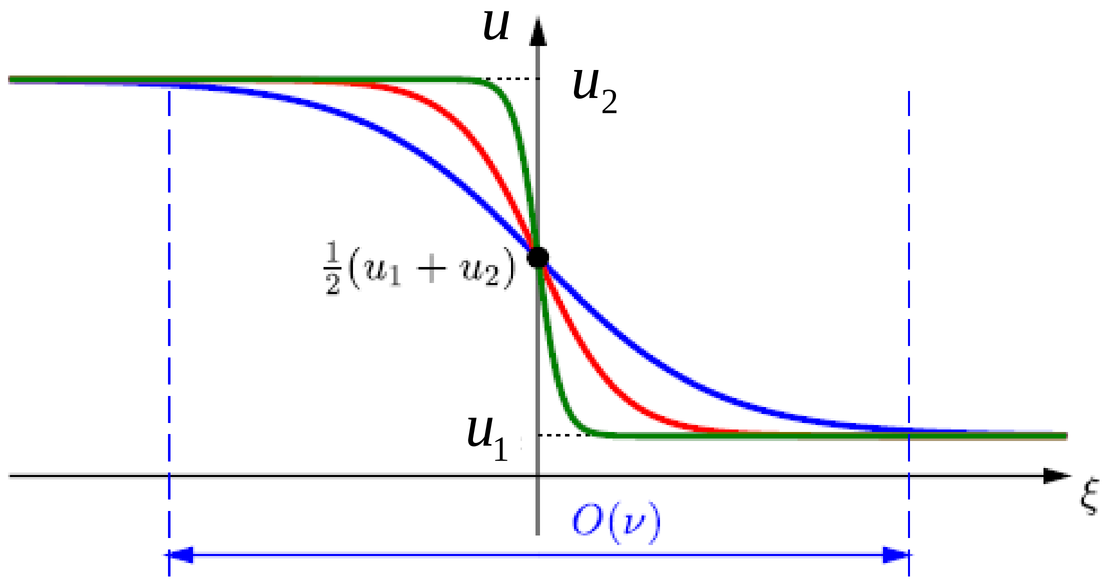
$$\int \frac{du}{(u - u_1)(u - u_2)} = \frac{1}{u_2 - u_1} \ln\left(\frac{u_2 - u}{u - u_1}\right) = \frac{1}{2\nu}(\xi - \delta)$$

where δ is a constant of integration. Then, solving the above equation for u, we obtain:

$$\begin{aligned} u(\xi) &= \frac{u_2 + u_1 \exp[\alpha(\xi - \delta)]}{1 + \exp[\alpha(\xi - \delta)]} \\ &= u_1 + \frac{u_2 - u_1}{1 + \exp[\alpha(\xi - \delta)]} \\ &= \frac{1}{2}(u_2 + u_1) - \frac{1}{2}(u_2 - u_1) \tanh\left(\frac{u_2 - u_1}{4\nu}(\xi - \delta)\right) \end{aligned}$$

where $\alpha = \frac{1}{2\nu}(u_2 - u_1) > 0.$

Structure of the traveling shock wave



- The shock wave (SW) asymptotes the equilibrium points:

$$\lim_{\xi \to -\infty} u(\xi) = u_2, \qquad \lim_{\xi \to +\infty} u(\xi) = u_1.$$

- Using: $u_1 = c - \sqrt{c^2 + 2K}, \quad u_2 = c + \sqrt{c^2 + 2K},$ we obtain:

$c = \dfrac{1}{2}(u_1 + u_2)$ velocity of the SW consistent with **RH condition!**

- "Shock thickness" $\nu/(u_2 - u_1)$: **diffusion prevents breaks up!**

The Cole-Hopf transformation

<u>Background</u>

- It was devised independently by **Eberhard Hopf (German) (1950)** and **Julian Cole (American) (1951)** [and also, even earlier, by **V. Florin (Russian) (1948)**].

E. Hopf

J. D. Cole

- It is a ***<u>remarkable</u>*** **nonlinear transformation** that reduces the **Burgers equation** to the **linear diffusion equation.** This way, the nonlinear Burgers equation can be explicitly solved.

- The Cole-Hopf transformation is a **milestone in the field of nonlinear PDEs**, and has inspired —among others— important developments in the **theory of solitons**

Introducing the Cole-Hopf transformation (I)

- Express the Burgers equation in a **conservation law** form:

$$u_t = \left(\nu u_x - \frac{1}{2}u^2\right)_x \qquad (1)$$

- Introduce the **"potential" function *U(x, t)***, which is actually the **antiderivative of *u*(*x*, *t*)**. The potential function $U(x,t)$ satisfies:

$$U_x = u, \quad U_t = \nu u_x - \frac{1}{2}u^2 \qquad (2)$$

[To see this, use the compatibility condition $u_{xt} = u_{tx}$ and (1)]

- Combine Eqs. (2) to derive the **Hamilton-Jacobi equation**:

$$U_t + \frac{1}{2}U_x^2 = \nu U_{xx}$$

- Introduce the **Cole-Hopf relation**:

$$U = -2\nu \ln \Phi$$

Introducing the Cole-Hopf transformation (II)

To this end, substitute the **Cole-Hopf relation**: $U = -2\nu \ln \Phi$

into the **Hamilton-Jacobi equation**: $U_t + \frac{1}{2}U_x^2 = \nu U_{xx}$

Then, observing that:

$$U_t = -2\nu\frac{\Phi_t}{\Phi}, \quad U_x = -2\nu\frac{\Phi_x}{\Phi}, \quad U_x^2 = 4\nu^2\frac{\Phi_x^2}{\Phi^2}, \quad U_{xx} = -2\nu\frac{\Phi\Phi_{xx} - \Phi_x^2}{\Phi^2}$$

the **Hamilton-Jacobi** equation transforms into the **linear diffusion equation for Φ**:

$$\Phi_t = \nu\Phi_{xx}$$

Thus, If $\Phi(x, t)$ is any nonzero solution of the **diffusion equation**

$$\Phi_t = \nu\Phi_{xx}$$

then $u(x,t) = \frac{\partial}{\partial x}\left[-2\nu \ln \Phi(x,t)\right] = -2\nu\frac{\Phi_x}{\Phi}$

satisfies the **Burgers equation:** $u_t + uu_x = \nu u_{xx}$

Burgers equation vs. diffusion equation

Consider the following Cauchy problem for the Burgers equation:

$$u_t + u u_x = \nu u_{xx},$$
$$u(x,0) = u_0(x), \quad -\infty < x < +\infty$$

Then, employing the Cole-Hopf relation, we can determine $\Phi_0(x)$ from $u_0(x)$, i.e., the **initial condition for the diffusion equation** from the one for the above Burgers equation:

$$u_0(x) = -2\nu \frac{\partial}{\partial x}\left[\ln \Phi_0(x)\right] \Rightarrow \Phi_0(x) = \exp\left[-\frac{1}{2\nu}\int_0^x u_0(x')dx'\right]$$

Thus, thanks to the **Cole-Hopf transformation**, instead of solving **the nonlinear problem**, we only need to solve the **linear problem**:

$$\Phi_t = \nu \Phi_{xx},$$
$$\Phi(x,0) = \Phi_0(x) = \exp\left[-\frac{1}{2\nu}\int_0^x u_0(x')dx'\right], \quad -\infty < x < +\infty$$

Solution of the Burgers equation (I)

The general solution of the Cauchy problem for the diffusion equation can be expressed in terms of the **convolution integral**:

$$\Phi(x,t) = \int_{-\infty}^{+\infty} G(x-y)\Phi_0(y)dy$$

where: $G(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left(-\frac{x^2}{4\nu t}\right)$ is the **fundamental solution** (or **Green's function**) of the diffusion equation.

Then, we use the general solution in the form:

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-y)^2}{4\nu t}\right] \Phi_0(y)dy$$

and find its derivative with respect to x:

$$\Phi_x(x,t) = -\frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-y)^2}{4\nu t}\right] \left(\frac{x-y}{2\nu t}\right) \Phi_0(y)dy$$

Solution of the Burgers equation (II)

Finally, we can **construct the solution $u(x, t)$ of the Burgers equation** by means of the **Cole-Hopf transformation**:

$$u(x,t) = -2\nu \frac{\Phi_x}{\Phi} = \frac{\int_{-\infty}^{+\infty} \frac{x-y}{t} \exp\left[-\frac{(x-y)^2}{4\nu t}\right] \Phi_0(y) dy}{\int_{-\infty}^{+\infty} \exp\left[-\frac{(x-y)^2}{4\nu t}\right] \Phi_0(y) dy}$$

We can also use the equation: $\Phi_0(x) = \exp\left[-\frac{1}{2\nu} \int_0^x u_0(x')dx'\right]$

and rewrite the solution of the Burgers equation as follows:

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{(x-y)}{t} \exp\left[-\frac{1}{2\nu} F(x,y,t)\right] dy}{\int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2\nu} F(x,y,t)\right] dy}$$

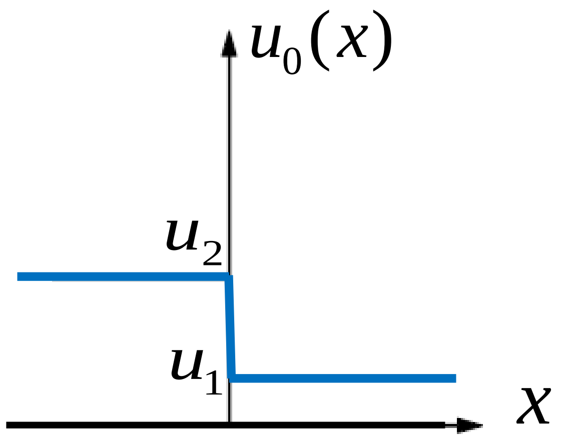
where $F(x,y,t)$ is given by: $F(x,y,t) = \frac{(x-y)^2}{2t} + \int_0^y u_0(y')dy'$

Riemann problem* for the Burgers equation

Consider the following IVP for the Burgers equation:

$$u_t + uu_x = \nu u_{xx},$$
$$u(x,0) = u_0(x),$$

where $u_0(x)$ reads: $u_0(x) = \begin{cases} u_2 & \text{for } x < 0 \\ u_1 & \text{for } x > 0 \end{cases}$



This initial condition has the form of a step-like shock, which will evolve to a genuine shock wave **in the inviscid limit of $\nu = 0$**

We wish to employ the results of the analysis above, and find the solution of this Riemann problem for the Burgers equation

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* Recall that a **Riemann problem** is an initial value problem for a hyperbolic PDE (or a system thereof) in which the initial data is piecewise constant with a discontinuity.

Solution via the Cole-Hopf transformation (I)

Recall that: $$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{(x-y)}{t} \exp\left[-\frac{1}{2\nu}F(x,y,t)\right] dy}{\int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2\nu}F(x,y,t)\right] dy}$$

We can find: $$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{x-y}{t} R(y) \exp\left[\frac{-(x-y)^2}{4\nu t}\right] dy}{\int_{-\infty}^{+\infty} R(y) \exp\left[\frac{-(x-y)^2}{4\nu t}\right] dy} \qquad (1)$$

where R is given by: $$R(x) = \begin{cases} \exp\left(-\frac{u_2 x}{2\nu}\right) & \text{for } x < 0, \\ \exp\left(-\frac{u_1 x}{2\nu}\right) & \text{for } x > 0. \end{cases}$$

Then, upon manipulating the integrals in Eq. (1), we can find that the solution can be rewritten in the following form:

Solution via the Cole-Hopf transformation (II)

$$u(x,t) = u_1 + \frac{u_2 - u_1}{1 + \frac{\mathrm{erfc}\left(-\frac{x-u_1 t}{2\sqrt{\nu t}}\right)}{\mathrm{erfc}\left(\frac{x-u_2 t}{2\sqrt{\nu t}}\right)} \exp\left[\frac{u_2-u_1}{2\nu}(x-ct)\right]}$$

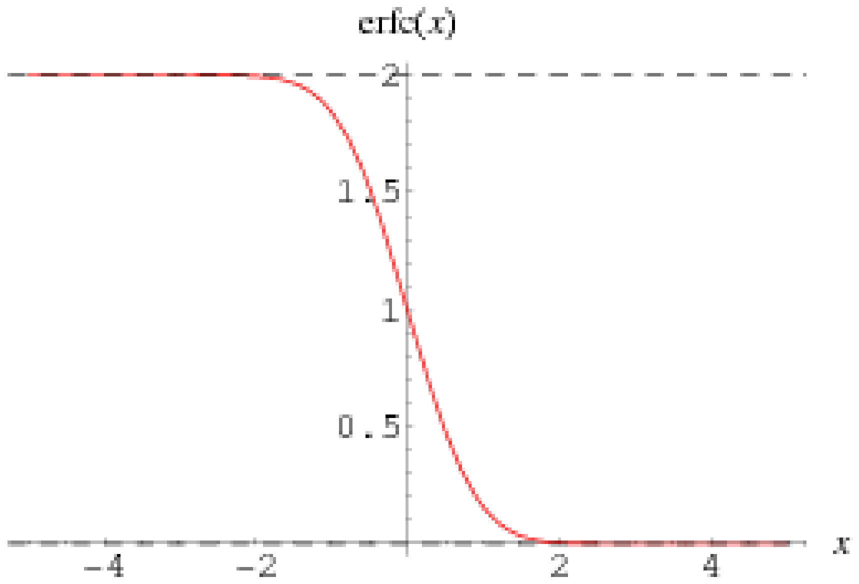
where: $c = (1/2)(u_1 + u_2)$ as per the **Rankine-Hugoniot condition!**

while erfc(x) is the **complementary error function** defined as:

$$\mathrm{erfc}(x) \equiv 1 - \mathrm{erf}(x)$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^x \exp(-x'^2)dx'$$

$$= \frac{2}{\sqrt{\pi}} \int_x^{+\infty} \exp(-x'^2)dx'$$



Asymptotic form of the solution

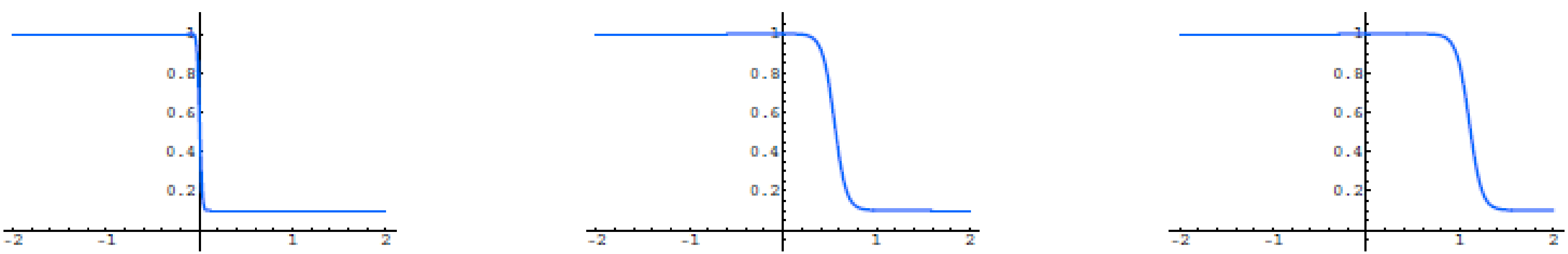
For fixed $x - ct$, the asymptotic behavior of the solution as $t \longrightarrow +\infty$ is:

$$\lim_{t \to +\infty} u(x,t) = u_1 + \frac{u_2 - u_1}{1 + \exp\left[\frac{u_2 - u_1}{2\nu}(x - ct)\right]}$$

which is identical to the equilibrium solution of the Burgers equation!

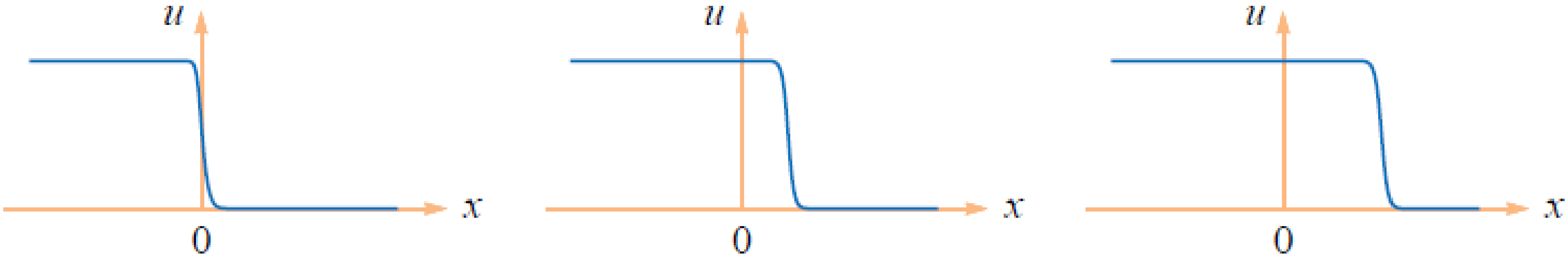
The results of the above analysis can be interpreted as follows.

The **initial discontinuity** is **eventually smoothed**, with the solution developing a continuously varying transition layer between the two asymptotic values u_2 and u_1.

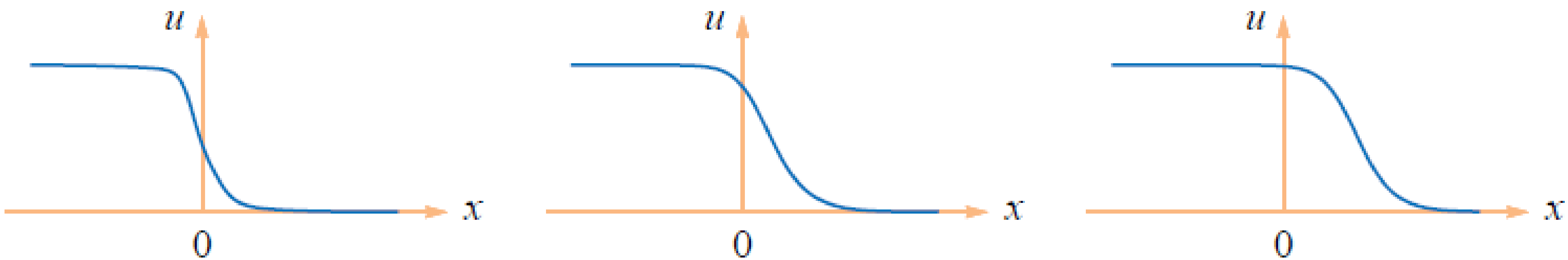


Spatial profiles of the solution, at $t = 0.01$, 1 and 2, for $u_2 = 1$, $u_1 = 0.1$ and $\nu = 0.03$

Profile of the solution



Shock wave solution of Burgers equation for low viscosity ν



Shock wave solution of Burgers equation for high viscosity ν

The smoothing effect is more pronounced for larger viscosity ν