The Korteweg-de Vries (KdV) equation

Cnoidal waves and solitons

Introducing the KdV equation

- The **KdV equation:**

$$u_t + \underbrace{6uu_x}_{\text{quadratic nonlinearity}} + \underbrace{u_{xxx}}_{\text{long-wave dispersion}} = 0$$

is a **universal PDE** incorporating the combined effect of the **lowest-order, quadratic, nonlinearity** **(term uu_x)** and the **simplest long-wave dispersion** **(term u_{xxx})**

- The KdV describes the propagation of **weakly nonlinear long wave in dispersive media**

- It arises in numerous physical contexts, including:
  - shallow-water gravity waves
  - ion-acoustic waves in collisionless plasmas
  - internal waves in the atmosphere and ocean, …

KdV equation: lowest-order dispersion

To understand the **universal nature** of the KdV equation, consider the simple **polynomial dispersion relation**:

$$\omega = \omega(k) = \alpha_0 + \alpha_1 k + \alpha_2 k^2 + \alpha_3 k^3 + \cdots ,$$

Using: $\omega \mapsto i\partial_t, \ k \mapsto -i\partial_x$ we obtain the operator:

$$i\partial_t = \alpha_0 - i\alpha_1 k\partial_x - \alpha_2\partial_x^2 - i\alpha_3\partial_x^3 + \cdots$$

Choose: $i\partial_t = \cancel{\alpha_0} - i\alpha_1 k\partial_x - \cancel{\alpha_2\partial_x^2} - i\alpha_3\partial_x^3 + \cdots$ $a_1 = c, \ a_3 = \gamma$

and obtain: $u_t + c u_x + \gamma u_{xxx} = 0$ **Linearized KdV**

Nonlinearity can also be introduced in this linear model based on the following: a ***fundamental property of any nonlinear wave, is the field amplitude dependence of the phase velocity.***

KdV equation: lowest-order nonlinearity

In the simplest possible case, and using $v_p = c$, this dependence assumes the following **polynomial** form:

$$c = c_0(1 + \beta_1 u + \beta_2 u^2 + \cdots)$$

We can thus introduce nonlinearity in the **linearized KdV** through c:

$$\left.\begin{aligned} &u_t + cu_x + \gamma u_{xxx} = 0 \\ &c = c_0(1+\beta u) \end{aligned}\right\} \to u_t + c_0(1+\beta u)u_x + \gamma u_{xxx} = 0$$

Finally, we employ a Galilei transformation:

$$\left.\begin{aligned} &u_t + c_0 u_x + \beta u u_x + \gamma u_{xxx} = 0 \\ &x \mapsto x - c_0 t,\ \ t \mapsto t \end{aligned}\right\} \to \boxed{u_t + \beta u u_x + \gamma u_{xxx} = 0}$$

KdV equation

Traveling wave solutions

We seek **traveling wave solutions** of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

of the form: $u = f(x - ct)$

This way, we obtain the **3d-order ODE**: $-cf' + 6ff' + f''' = 0$

where primes denote derivatives with respect to $\xi = x - ct$

Then, integrating with respect to ξ yields:

$$-cf + 3f^2 + f'' = A$$

where A is **a constant of integration**.

We assume that $\boldsymbol{A < 0}$ (the case with $A > 0$ can be analyzed using the same methodology)

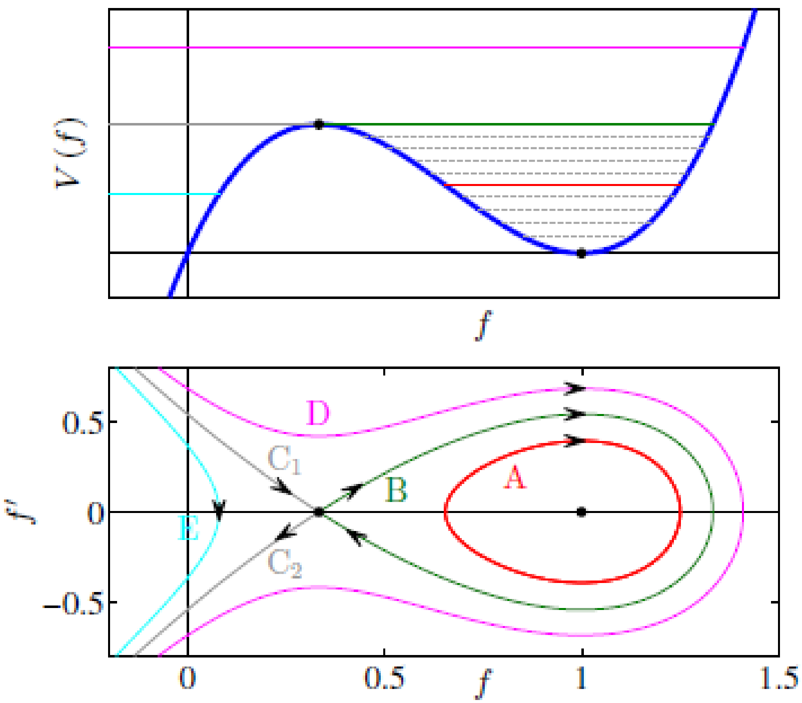
The associated dynamical system

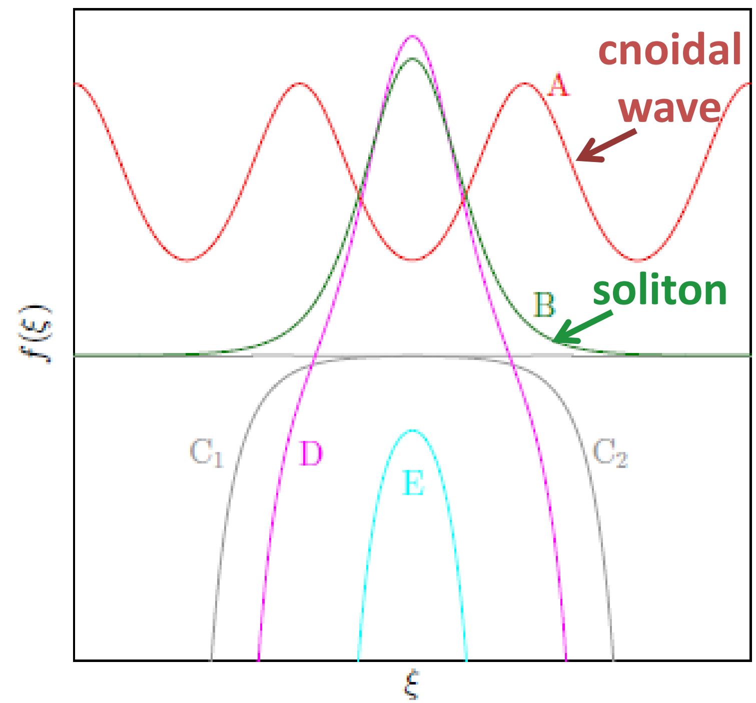
We have thus obtained the following **equation of motion**:

$$f'' = A + cf - 3f^2$$

which yields the **effective Newtonian potential**:

$$V(f) = -Af - \frac{c}{2}f^2 + f^3$$





The bounded solutions

The equation of motion can also be expressed as: $\frac{1}{2}(f')^2 + V(f) = E$

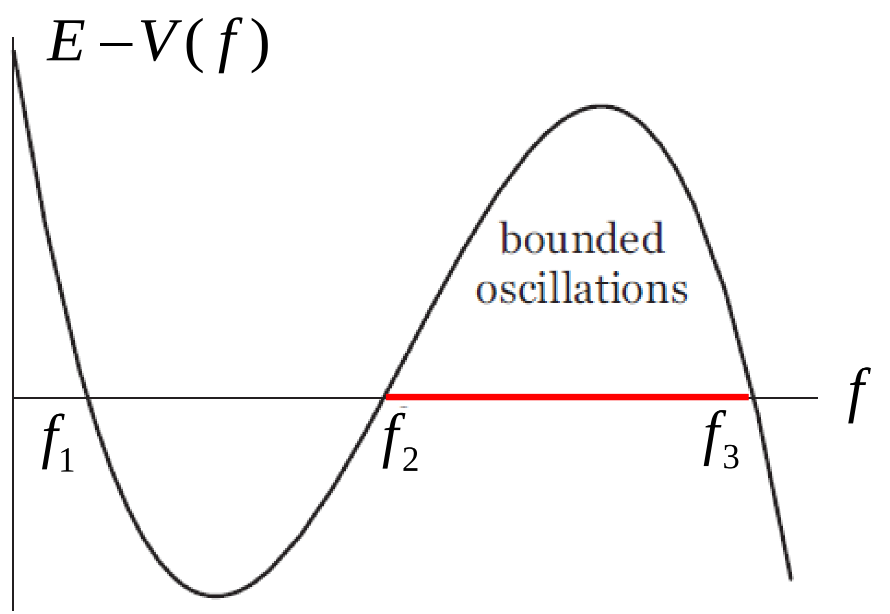
Then, denoting $f_{1,2,3}$ the roots of the **cubic polynomial** $E - V(f)$

we have: $(f')^2 = 2(E - V(f)) = 2(f_1 - f)(f_2 - f)(f_3 - f)$

$$\Rightarrow \int \frac{df}{\sqrt{2(f_1 - f)(f_2 - f)(f_3 - f)}} = \pm(\xi - x_0)$$

We will now make a **change of variables**, motivated by the fact that the periodic solution lies between f_2 and f_3:

$$f = f_3 + (f_2 - f_3)\sin^2(\theta)$$
$$= f_2 + (f_3 - f_2)\cos^2(\theta)$$



Determining the bounded solutions

This way, we obtain:

$$\int \frac{\sqrt{2}d\theta}{\sqrt{(f_3 - f_1) - (f_3 - f_2)\sin^2(\theta)}} = \pm(\xi - x_0)$$

$$\Rightarrow \int \frac{d\theta}{\sqrt{1 - m\sin^2(\theta)}} = \pm\sqrt{\frac{f_3 - f_1}{2}}(\xi - x_0)$$

where $m = (f_3 - f_2)/(f_3 - f_1)$

Jacobi elliptic integral of the 1st kind

The **inverse** of the Jacobi elliptic integral of the 1st kind is termed **the Jacobi amplitude function am:**

$$\theta = \mathrm{am}\left(\pm\sqrt{\frac{f_3 - f_1}{2}}(\xi - x_0), m\right)$$

$$\sin(\theta) = \sin(\mathrm{am}(u, m)) = \mathrm{sn}(u, m)$$

$$\cos(\theta) = \cos(\mathrm{am}(u, m)) = \mathrm{cn}(u, m)$$

The cnoidal wave

Finally, the solution is found to be of the form:

$$f = f_2 + (f_3 - f_2)\cos^2(\theta) \Rightarrow$$

$$f = f_2 + (f_3 - f_2)\mathrm{cn}^2\left(\pm\frac{\sqrt{f_3 - f_1}}{2}(\xi - x_0), \underbrace{\frac{f_3 - f_2}{f_3 - f_1}}\right)$$

elliptic modulus m: $0 < m < 1$

This is the so-called **cnoidal wave**, a **periodic solution**, of period:

$$T = \int_0^{2\pi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} = 4K(m)$$

$$K(m) = \int_0^{\pi/2} d\theta / \sqrt{1 - m\sin^2\theta}$$

complete elliptic integral

Important limiting cases:

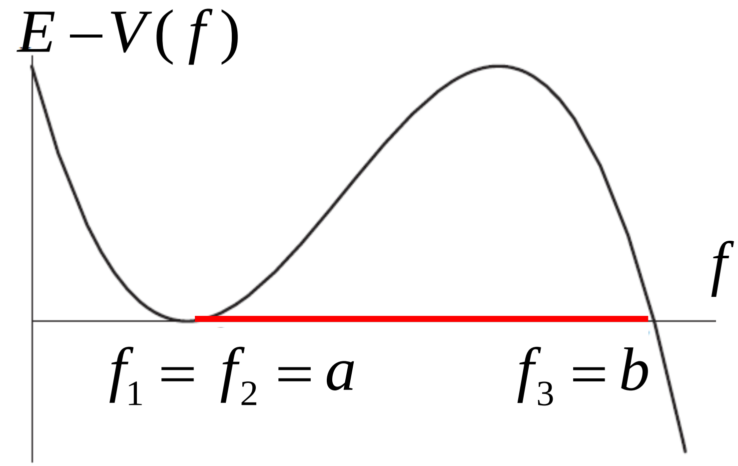
$m \to 0$: $K(0) = \pi/2 \Rightarrow$ $T = 2\pi$ and $\mathrm{cn}(u,0) = \cos(u)$

$m \to 1$: $K(1) \to \infty \Rightarrow$ $T \to \infty$ and $\mathrm{cn}(u,1) = \mathrm{sech}(u)$

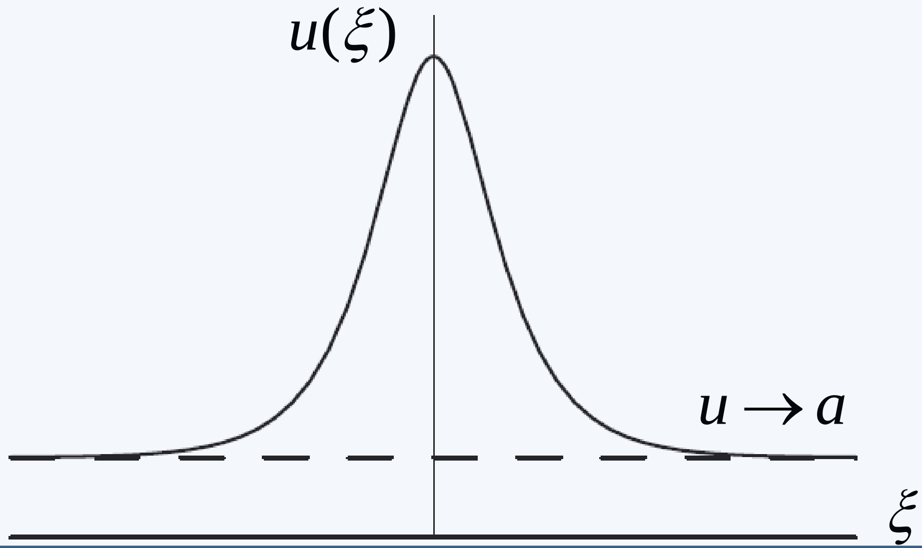
Cnoidal waves in shallow water:
off the coast of Lima, Peru

The soliton solution

The **soliton** can be found from the cnoidal wave in the **limiting case** of a **double root**, $\boldsymbol{f_1 = f_2 = a}$**:** $m = \dfrac{f_3 - f_2}{f_3 - f_1} = 1$



Then: $$u(x,t) = a + (b-a)\,\mathrm{sech}^2\left[\frac{1}{2}(b-a)\left(x - ct - x_0\right)\right]$$



KdV solitons: evolution and collisions