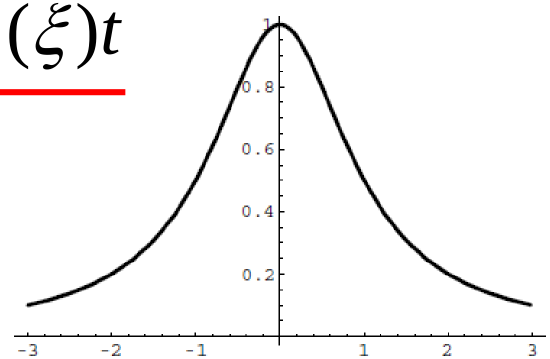
Quasi-linear PDEs (II)

Shock waves emerging from localized initial data

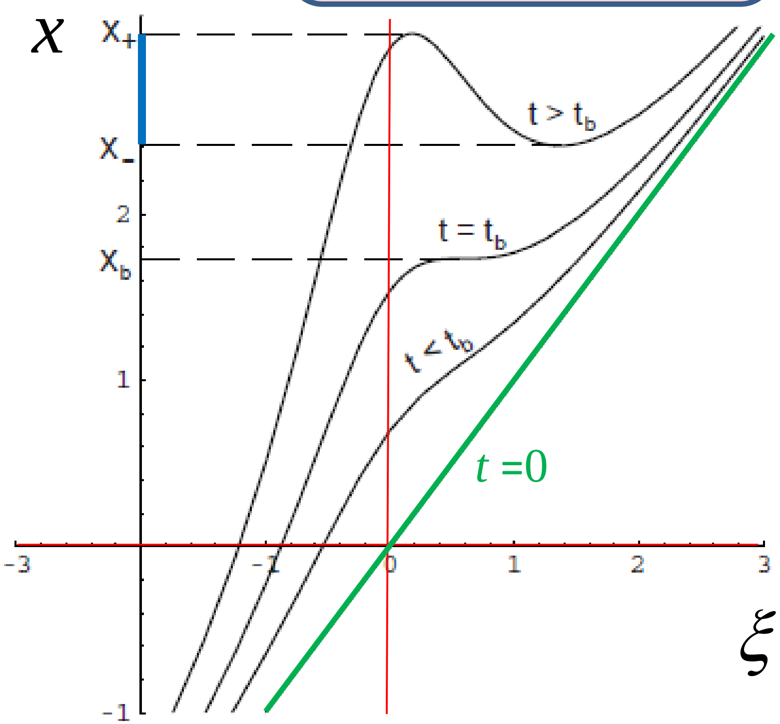
Localized initial condition - Example I

Consider the IVP: $u_t + u\,u_x = 0, \quad u(x,0) = f(x) = \dfrac{1}{x^2+1}$

We have found: $\left.\begin{aligned} x &= ut + \xi \\ u &= f(\xi) \end{aligned}\right\} \Rightarrow x = \xi + f(\xi)t$

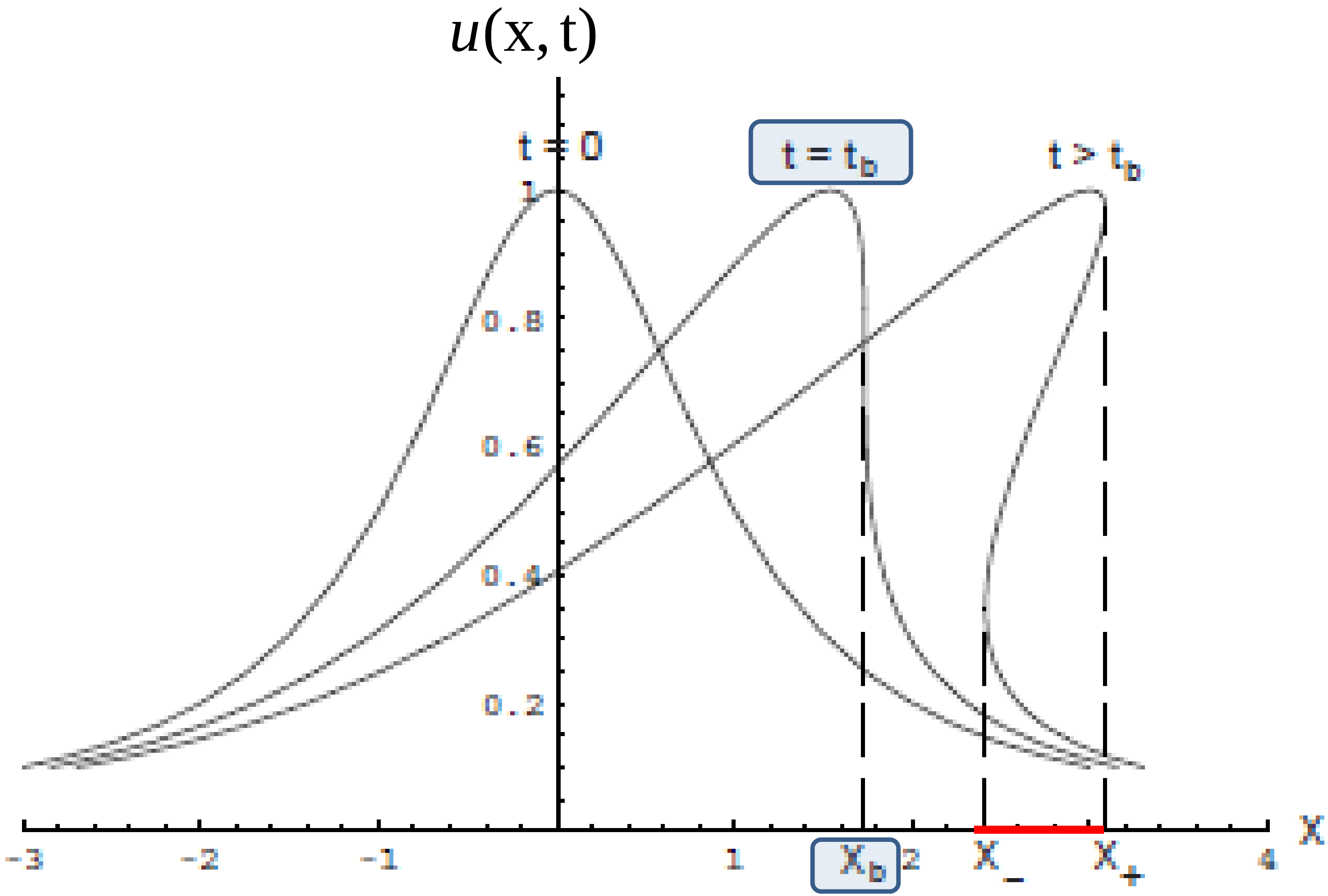


and thus: $x = \xi + \dfrac{1}{\xi^2+1}\,t$



- ***t* =0:** identical map
- As $t\uparrow$ the region of the plot $x(\xi)$ corresponding to points ξ where $f'<0$ **flattens**
- At $t = t_b$ the graph acquires a point with a **horizontal tangent line**
- For $t > t_b$ there exists a region $x_- < x < x_+$ where each x corresponds to 3 ξ-values and thus to **three $f(\xi) \equiv u(x,t)$ values!**

Example I (cont.) – the shock wave



Example I (cont.) – the notion of the envelope

The boundary between multi- and single-valued regions in the xt-plane can be found by noticing that, at the relevant points, the plot of x(ξ) has a horizontal tangent line:

$$x = \xi + f(\xi)t \Rightarrow \boxed{\frac{dx}{d\xi} = 1 + f'(\xi)t = 0}$$

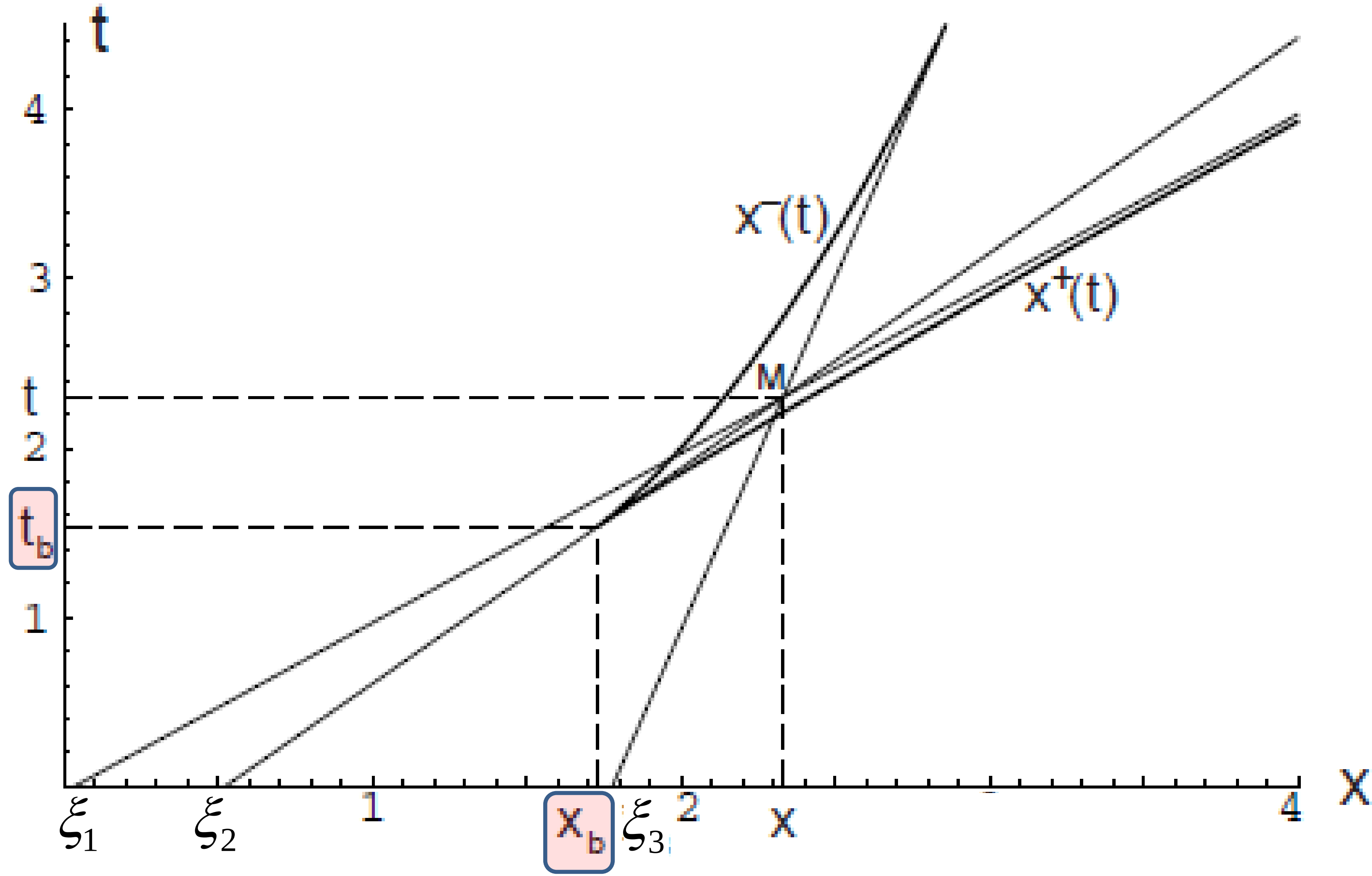
The boundary can be determined by the elimination of ξ in the equations: $1 + f'(\xi)t = 0, \quad x = f(\xi)t + \xi$

The resulting curve(s) in the xt-plane is the **envelope*** of the characteristics, i.e., here,

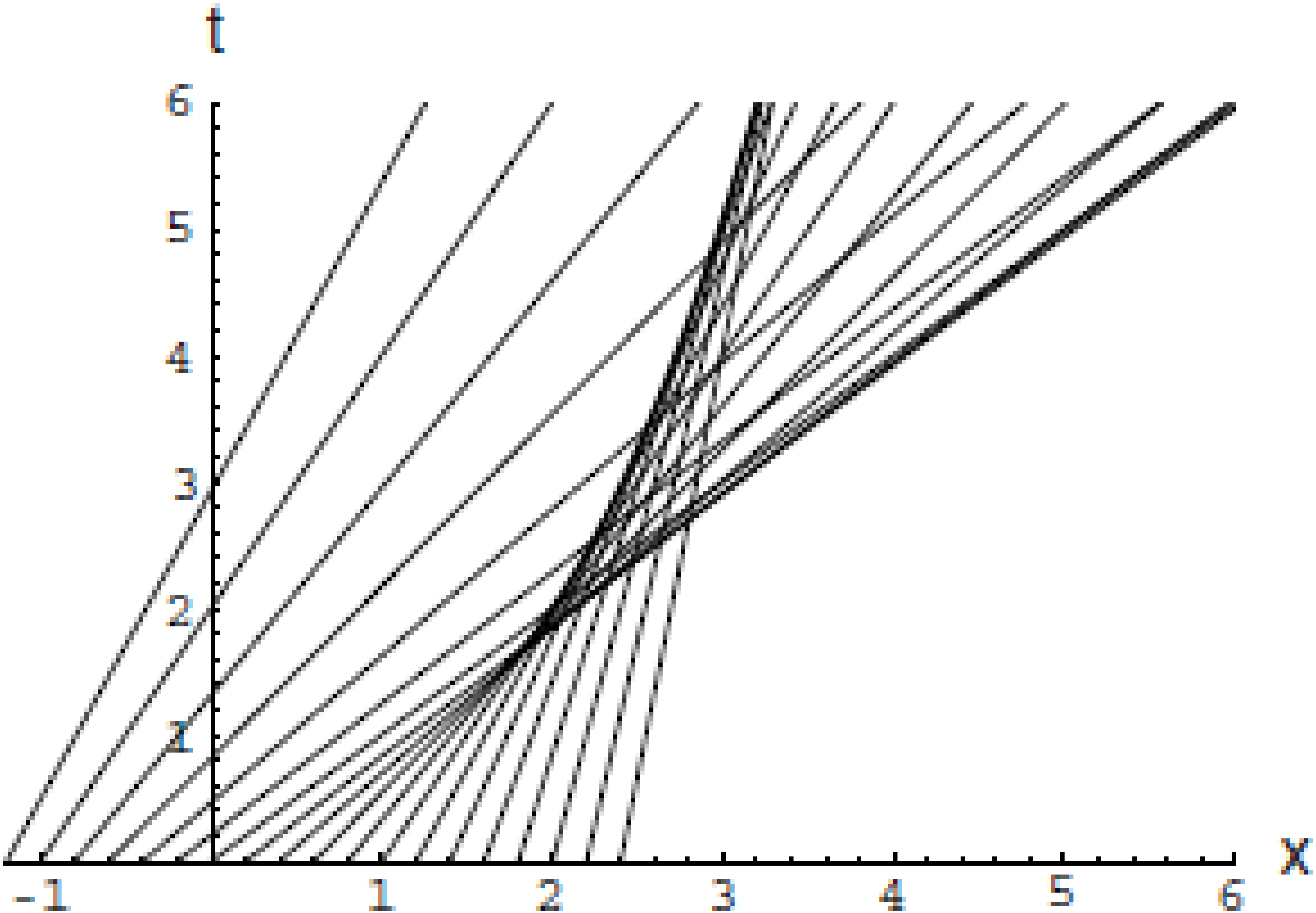
$$x^{\pm} = \xi - f(\xi)/f'(\xi), \quad 1 + f'(\xi)t = 0$$

*** Envelope is a curve that touches and is tangent to a family of curves**

Example I (cont.) – characteristics and envelope



Example I (cont.) – characteristics



Example I (cont.) – breaking time

Breaking time: $$t_b = \min_{\xi>0}\left\{ t(\xi) = -1/f'(\xi) \;\middle|\; f'(\xi) < 0 \right\}$$

Here:

$$f(\xi) = \frac{1}{\xi^2+1} \Rightarrow f'(\xi) = -\frac{2\xi}{(\xi^2+1)^2} \Rightarrow -\frac{1}{f'(\xi)} = \frac{(\xi^2+1)^2}{2\xi}$$

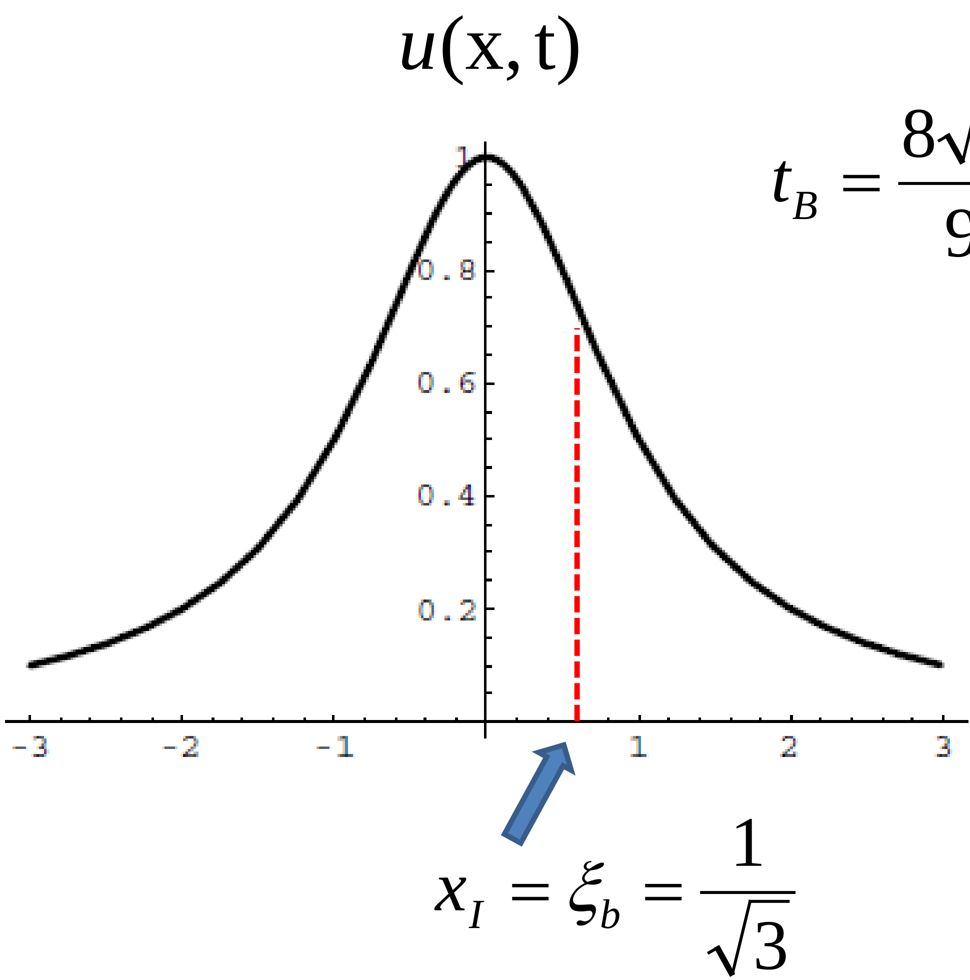
$$\min: \quad \frac{dt(\xi)}{d\xi} = 0 \Rightarrow \frac{d}{d\xi}\left(\frac{(\xi^2+1)^2}{2\xi}\right) = 0 \Rightarrow \xi_b = \frac{1}{\sqrt{3}}$$

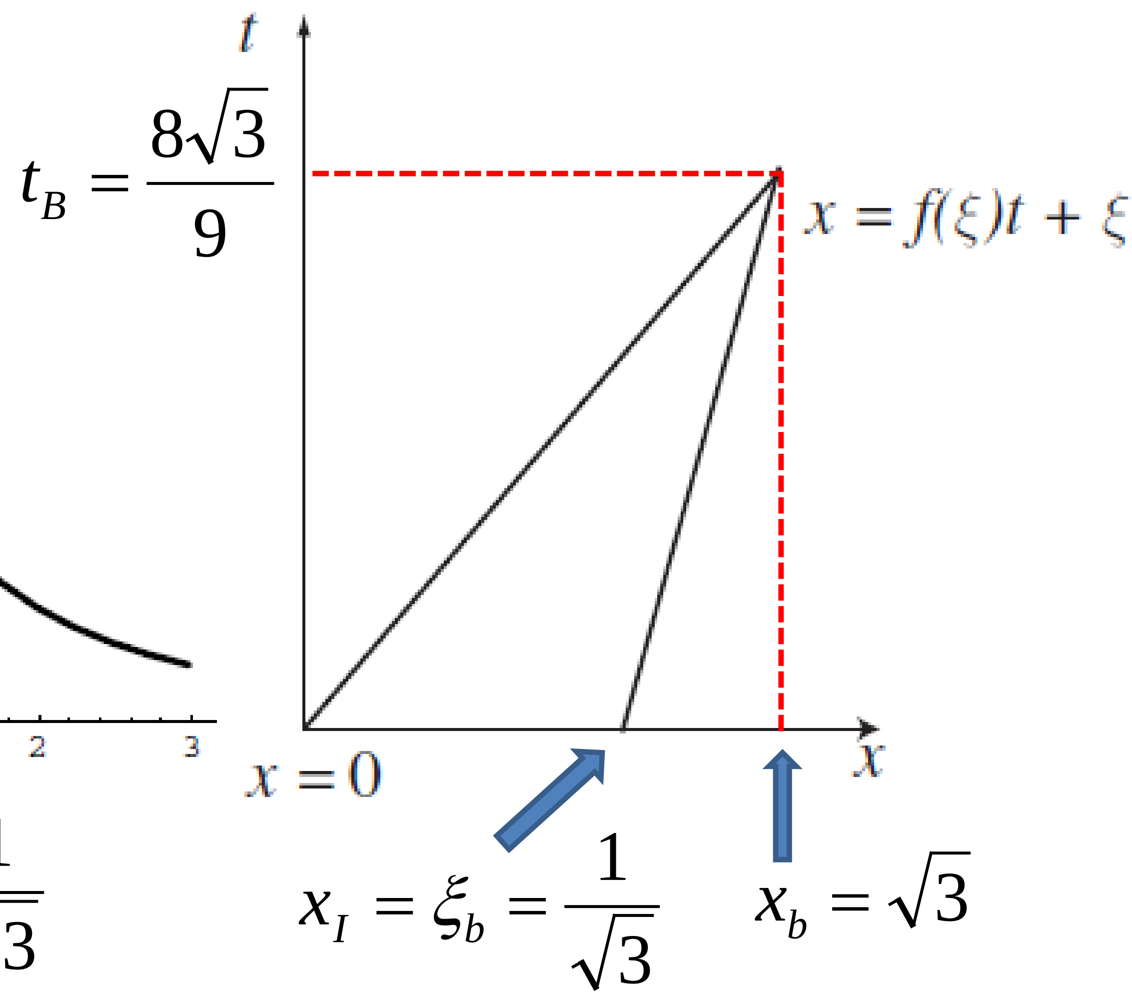
$$t_b = \left.\frac{(\xi^2+1)^2}{2\xi}\right|_{\xi_b=\frac{1}{\sqrt{3}}} \Rightarrow t_b = \frac{8\sqrt{3}}{9}$$

$$x_b = f(\xi_b)t_b + \xi_b = \frac{t_b}{\xi_b^2+1} + \xi_b \Rightarrow x_b = \sqrt{3}$$

Example I (cont.) – wave breaking points

The value of ξ_b is equal to the inflection point x_I of $f(x)$, i.e., $f''(\xi_b) = 0$

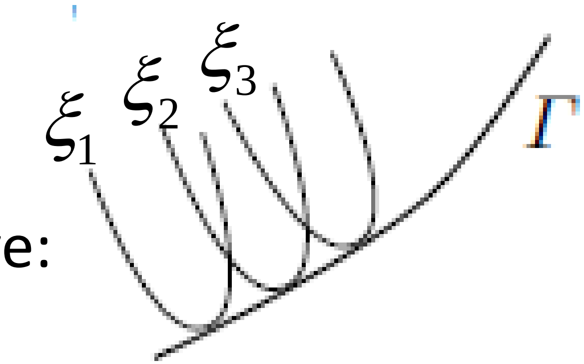




A more detailed look at the envelope

As noted above, the **envelope** can generally be determined by the elimination of ξ in the equations:

$$x = f(\xi)t + \xi, \quad 1 + f'(\xi)t = 0$$



Indeed, from a geometry point of view, we have:

Characteristics are given by $x = f(\xi)t + \xi$

or $G(x,t,\xi) = 0$ and, thus, for the envelope $\xi = \xi(x,t)$

$$dG = \underbrace{G_x dx + G_t dt}_{0} + G_\xi d\xi = 0 \quad \text{along the envelope}$$

0 (because the envelope is **tangent** to the family)

Hence: $\begin{cases} G(x,t,\xi) = 0 \\ G_\xi(x,t,\xi) = 0 \end{cases} \Rightarrow \begin{cases} x - \xi - f(\xi)t = 0 \\ 1 + f'(\xi)t = 0 \end{cases}$

The envelope and a tractable example

Alternatively, from an analysis point of view, we have:

Consider two characteristics, ξ and $\xi + \delta\xi$, that intersect at (x, t).

Then: $x = \xi + f(\xi)t$ and $x = \xi + \delta\xi + f(\xi + \delta\xi)t$

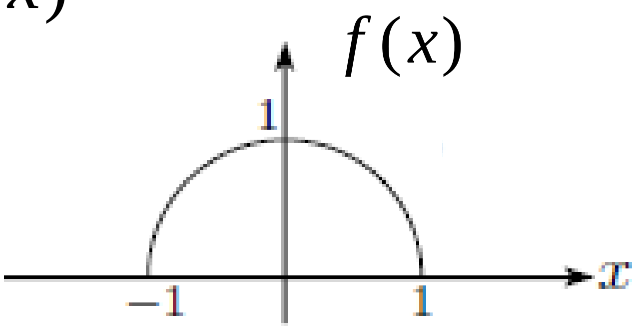
and in the limit $\delta\xi \to 0$ we obtain: $x = f(\xi)t + \xi, \; 1 + f'(\xi)t = 0$

In some cases the envelope can be found analytically

Example: Consider the IVP:

$$u_t + u u_x = 0, \quad u(x,0) = f(x)$$

$$f(x) = \begin{cases} 1 - x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$



The envelope and a tractable example – cont.

We will use the equations: $x = \xi + f(\xi)t$ and $1 + f'(\xi)t = 0$

to determine the **breaking time** and the **envelope**.

Breaking time

$$|\xi| \le 1: \; f(\xi) = 1 - \xi^2 \Rightarrow f'(\xi) = -2\xi \text{ and } f'(\xi) < 0 \text{ for } \xi \in (0,1)$$

$$\text{For } t_B : 1 + f'(\xi)t_B = 0 \Rightarrow t_B = \min_{0<\xi<1}\left\{-\frac{1}{f'(\xi)}\right\} \Rightarrow \boxed{t_B = \frac{1}{2}}$$

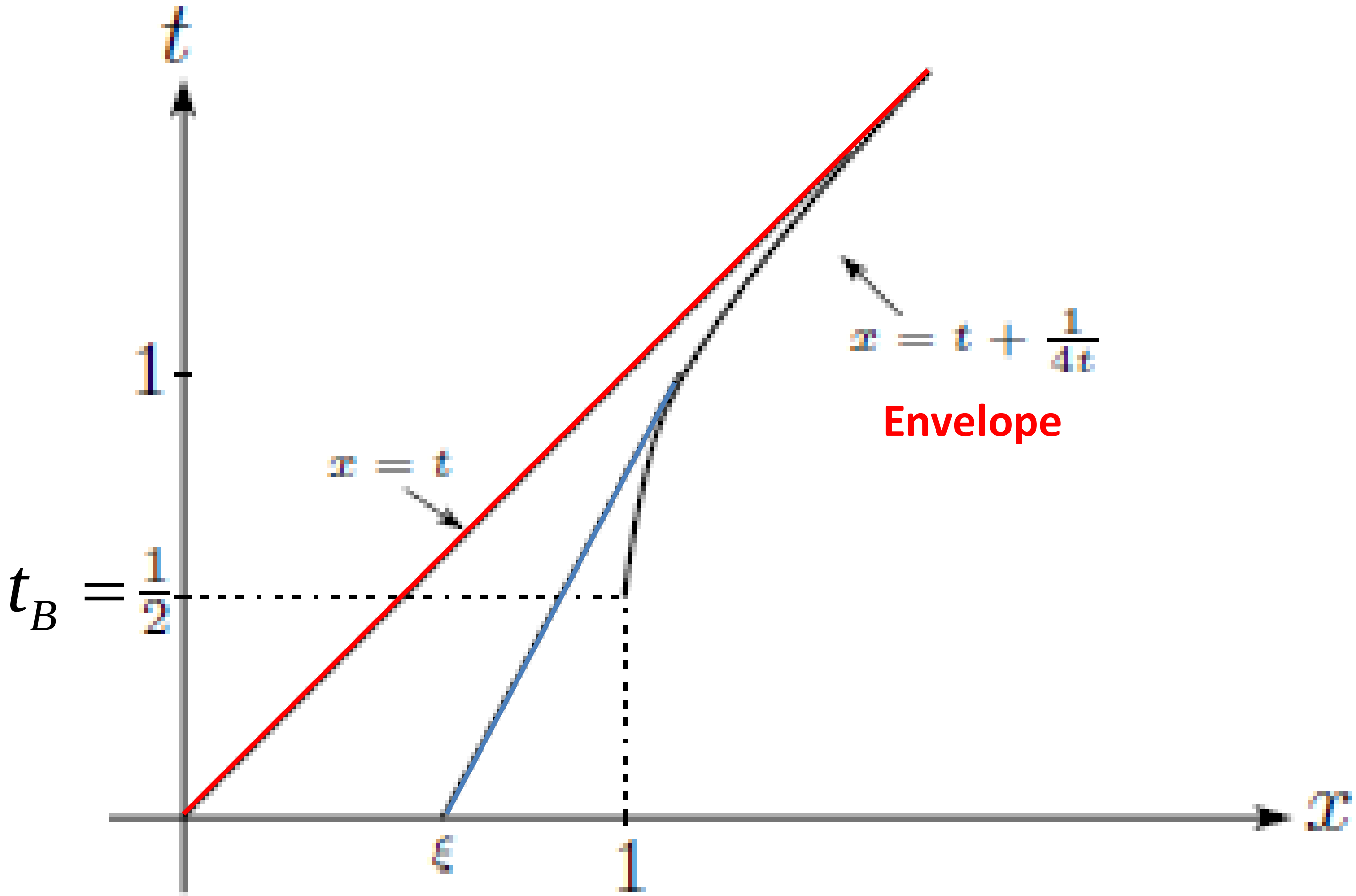
Envelope

$$\left.\begin{array}{l} 1 + f'(\xi)t = 0 \Rightarrow 1 - 2\xi t = 0 \Rightarrow \xi = \dfrac{1}{2t} \\ x = f(\xi)t + \xi \Rightarrow x = (1-\xi^2)t + \xi \end{array}\right\} \Rightarrow \boxed{x = \left[1 - \left(\frac{1}{2t}\right)^2\right]t + \frac{1}{2t}}$$

The envelope and a tractable example – cont.

To this end, the **envelope** is given by:

$$x(t) = t + \frac{1}{4t}$$



Localized initial condition - Example II

Consider the IVP: $u_t + u\,u_x = 0, \quad u(x,0) = \boxed{f(x) = \exp(-x^2)}$

Breaking time: $\boxed{t_b = \min_{\xi>0}\left\{ t(\xi) = -1/\,f'(\xi) \;\middle|\; f'(\xi) < 0 \right\}}$

Here: $f(\xi) = e^{-\xi^2} \Rightarrow f'(\xi) = -2\xi e^{-\xi^2} \Rightarrow -\dfrac{1}{f'(\xi)} = \dfrac{e^{\xi^2}}{2\xi}$

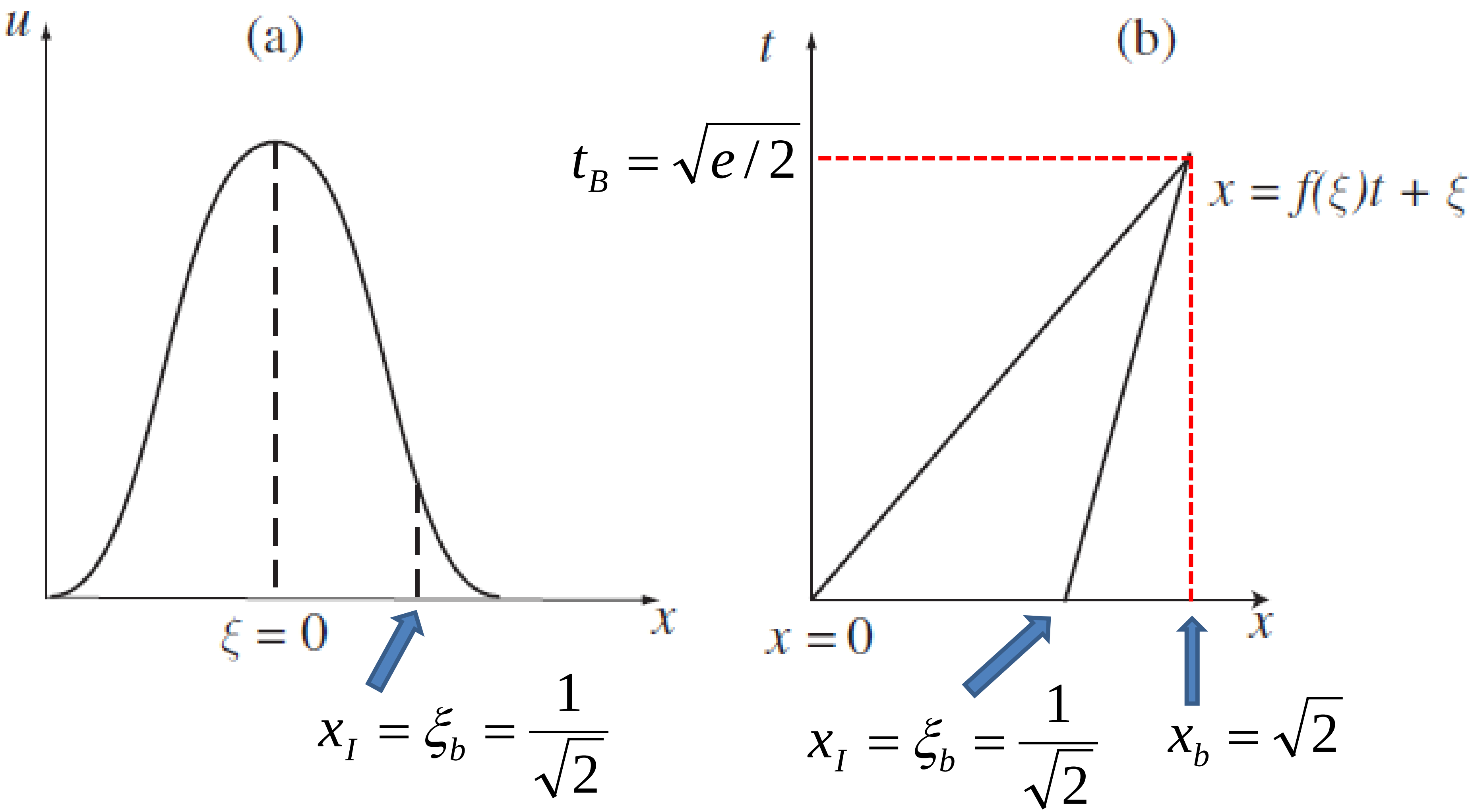
$$\min: \ \frac{dt(\xi)}{d\xi} = 0 \Rightarrow \frac{d}{d\xi}\left(\frac{e^{\xi^2}}{2\xi}\right) = 0 \Rightarrow \boxed{\xi_b = \frac{1}{\sqrt{2}}}$$

Thus: $t_b = \left.\dfrac{e^{\xi^2}}{2\xi}\right|_{\xi_b = \frac{1}{\sqrt{2}}} \Rightarrow \boxed{t_b = \sqrt{\dfrac{e}{2}} \approx 1.16}$

$$x_b = f(\xi_b)t_b + \xi_b = e^{-\xi^2} t_b + \xi_b \Rightarrow \boxed{x_b = \sqrt{2}}$$

Example II (cont.) – wave breaking points

The value of ξ_b is equal to the inflection point x_I of $f(x)$, i.e., $f''(\xi_b) = 0$



The role of dissipation - Example I

Consider the IVP: $u_t + u\,u_x = -u,\quad u(x,0) = f(x) = u_0 \exp(-x^2)$

Here, the Hopf equation incorporates a **linear** **dissipative term**: indeed, in the absence of nonlinearity, the solution is $\propto \exp(-t)$

Question: Which values of u_0 give rise to wave breaking?

On Γ: $x=x(t)$ we have: $\dfrac{dx}{dt} = u,\quad x(0) = \xi;\quad \dfrac{du}{dt} = -u,\quad u(0) = u_0 e^{-\xi^2}$

- The 2nd eq. leads to: $\left.\begin{array}{l} \dfrac{du}{dt} = -u \Rightarrow u = Ae^{-t} \\ u(0) = u_0 e^{-\xi^2} \end{array}\right\} \Rightarrow u = u_0 e^{-\xi^2} e^{-t}$
- Thus, the 1st eq. leads to:

$$\frac{dx}{dt} = u = u_0 e^{-\xi^2} e^{-t} \Rightarrow \int_\xi^x dx = u_0 e^{-\xi^2} \int_0^t e^{-t} dt \Rightarrow x(t) = u_0 e^{-\xi^2}(1-e^{-t}) + \xi$$

The role of dissipation - Example I (cont.)

For the **breaking time** we use: $\dfrac{dx}{d\xi}=0, \quad x(t)=u_0 e^{-\xi^2}(1-e^{-t})+\xi$

Here:

$$\frac{dx}{d\xi}=0 \Rightarrow u_0 e^{-\xi^2}(-2\xi)(1-e^{-t})+1=0 \Rightarrow t(\xi)=-\ln\left(1-\frac{e^{\xi^2}}{2u_0\xi}\right)$$

Wave breaking occurs if: $t_b=\min_{\xi>0}\{t(\xi)\}>0$

$$\frac{dt(\xi)}{d\xi}=0 \Rightarrow \frac{1}{1-\dfrac{e^{\xi^2}}{2u_0\xi}}\left(\frac{e^{\xi^2}(2\xi)(2u_0\xi)-e^{\xi^2}(2u_0)}{4u_0^2\xi^2}\right)=0 \Rightarrow \xi_b=\frac{1}{\sqrt{2}}$$

Hence: $t_b=t(\xi_b)\Big|_{\xi_b=\frac{1}{\sqrt{2}}}>0 \;\Rightarrow\; 0<\dfrac{e^{1/2}}{2u_0\dfrac{1}{\sqrt{2}}}<1 \;\Rightarrow\; u_0>\sqrt{\dfrac{e}{2}}$

The role of dissipation - Example II

Consider the IVP: $u_t + u\,u_x = -au, \quad u(x,0) = f(x)$

Again, the Hopf equation incorporates a **dissipative term**

Show that, for a ⍰ 0, the breaking time $t_b(a)$ is **greater** than the corresponding one, $t_b(0)$, for a=0, i.e., $\mathbf{t_b(a) > t_b(0)}$

Here: $\dfrac{dx}{dt} = u, \quad x(0) = \xi \quad (1); \quad \dfrac{du}{dt} = -au, \quad u(0) = f(\xi) \quad (2)$

- Eq. (2) leads to: $u = f(\xi)\exp(-at)$ and, thus, Eq. (1) gives:

$$\frac{dx}{dt} = u = f(\xi)e^{-at} \Rightarrow \int_{\xi}^{x} dx = f(\xi)\int_{0}^{t} e^{-at}dt \Rightarrow$$

$$x(t) = \frac{f(\xi)}{a}\left(1 - e^{-at}\right) + \xi$$

The role of dissipation - Example II (cont.)

For the **breaking time** we use: $\dfrac{dx}{d\xi}=0,\;\; x(t)=\dfrac{f(\xi)}{a}\left(1-e^{-at}\right)+\xi$

Here:

$$\frac{dx}{d\xi}=0 \Rightarrow 1+\frac{f'(\xi)}{a}\left(1-e^{-at}\right)=0 \Rightarrow at=-\ln\left(1+\frac{a}{f'(\xi)}\right) \Rightarrow$$

$$t(\xi)=-\frac{1}{a}\ln\left(1+\frac{a}{f'(\xi)}\right)$$

and breaking time: $t_b=\min_{\xi>0}\left\{t(\xi)\right\}$

$$t_b=\min_{\xi>0}\left\{-\frac{1}{a}\ln\left(1+\frac{a}{f'(\xi)}\right)\right\}$$

This equation is valid in both cases, $a=0$ and $a \neq 0$

The role of dissipation - Example II (cont.)

We have: $t_b = t_b(a) = \min_{\xi>0}\left\{-\frac{1}{a}\ln\left(1+\frac{a}{f'(\xi)}\right)\right\}$

Case I: $a = 0$:

$$t_b(0) = \lim_{a\to 0} t_b(a) = \lim_{a\to 0}\left\{-\frac{1}{a}\ln\left(1+\frac{a}{f'(\xi)}\right)\right\}$$

$$= -\lim_{a\to 0}\frac{\dfrac{1}{\left(1+\dfrac{a}{f'(\xi)}\right)f'(\xi)}}{1} \Rightarrow \boxed{t_b(0) = -\frac{1}{f'(\xi)}}$$

where we used **L'Hôpital's rule**

Well-known result from the dissipationless case

The role of dissipation - Example II (cont.)

Case II: a ⍰ 0:

$$\frac{dt}{d\xi}=0\Rightarrow\frac{1}{1+\frac{a}{f'(\xi)}}\frac{d}{d\xi}\left(\frac{1}{f(\xi)}\right)=0\Rightarrow$$

solution **independent** of a

We now compare: $t_b(a)=-\frac{1}{a}\ln\left(1+\frac{a}{f'(\xi)}\right),\quad t_b(0)=-\frac{1}{f'(\xi)}$

It remains to show that **$t_b(a) > t_b(0)$.** If this holds, then:

$$-\frac{1}{a}\ln\left(1+\frac{a}{f'(\xi)}\right)>-\frac{1}{f'(\xi)}\Rightarrow\ln\left(1+\frac{a}{f'(\xi)}\right)<\frac{a}{f'(\xi)}$$

This inequality is **valid** because: $\ln(1+x)<x,\quad x<0$

Note: $f(x)=\ln(1+x)$ is **concave** and $g(x)=x$ is its tangent