Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE @ DIRECT®

ADVANCES IN Applied Mathematics

Advances in Applied Mathematics 31 (2003) 132–149

www.elsevier.com/locate/yaama

The "Jeans swindle"
A true story—mathematically speaking

Michael K.-H. Kiessling

Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854, USA

Received 10 June 2002; accepted 18 July 2002

Abstract

The century-old Jeans dispersion relation enjoys the questionable reputation that it cannot be derived in a mathematically clean manner—*as a matter of principle*. For that reason Jeans' 'derivation' of his result has become known by the (in)famous sobriquet "the Jeans swindle." The present paper rectifies the situation by giving just such a mathematically clean derivation of Jeans' dispersion relation, via a static universe with cosmological constant. The derivation not merely vindicates Jeans' analysis, it also produces proper nonlinear evolution equations which allow one to study the evolution beyond the linear regime studied by Jeans.

1. Introduction

Early in the twentieth century J.H. Jeans [8] studied the influence of Newtonian gravity on the dynamics of infinitesimal wavelike disturbances of a uniform fluid equilibrium. As is well known, in the absence of gravity (read: when gravity can be neglected) such disturbances propagate along the direction of their wave vector $\mathbf{k}$ as longitudinal sound waves with angular frequency ω, governed by the simple dispersion relation

$$\omega^2 - |\mathbf{k}|^2 c_s^2 = 0, \tag{1}$$

where c_s is the speed of sound. By resorting to some formal manipulations that have since become known in the astrophysics and cosmology communities as the "Jeans swindle"

E-mail address: miki@math.rutgers.edu.

0196-8858/03/$ – see front matter
doi:

(Binney and Tremaine [1], Börner [2]), Jeans [8] found that Newtonian gravity modifies (1) into

$$\omega^2 - |\mathbf{k}|^2 c_s^2 + k_J^2 c_s^2 = 0, \tag{2}$$

where

$$k_J^2 c_s^2 = 4\pi G \rho_0, \tag{3}$$

which defines the *Jeans wave number* k_J. In (3), ρ_0 is the constant mass density of the homogeneous fluid which supports the disturbances, and G is Newton's constant of universal gravitation. According to (2), spatially sinusoidal plane density wave disturbances now propagate only when $|\mathbf{k}| > k_J$; when $|\mathbf{k}| < k_J$, one of the roots of (2) corresponds to a mode whose amplitude grows exponentially with time. This is the celebrated *Jeans instability*.

Gravitational instabilities in a static homogeneous Newtonian universe are no longer an important topic of research in cosmology. Yet, modern monographs on astrophysics and cosmology (e.g., Fridman and Polyachenko [7], Binney and Tremaine [1], Kippenhahn and Weigert [11], Börner [2]), in their section on gravitational instabilities, usually reproduce Jeans' 'impossible derivation' of (2) together with a disclaimer that (2) cannot be backed up by a mathematically correct analysis, which is why Jeans' derivation is called a 'swindle.' A partial explanation for the curious longevity of Jeans' argument lies in the robustness of his result, combined with the relative simplicity of its 'derivation.' Indeed, the linear stability analyses of various inhomogeneous static equilibria, of stationarily rotating equilibria, and of expanding-universe solutions, which all proceed in an orderly manner but are also much more demanding, essentially confirm Jeans' conclusions. While this robustness may seem reassuring, upon closer inspection one finds the mathematical dilemma of Jeans' original problem unresolved. Evidently this is not a very satisfactory state of affairs to celebrate the centennial of Jeans' paper [8]. In view of all this it might not seem unappropriate to take yet another look at the matter.

Since (2) is such a simple dispersion relation, related by a straightforward application of Laplace and Fourier transforms to a simple system of linear partial differential equations with constant coefficients, it is clear that any mathematical problems would have to reside in the validity of those linear evolution equations, obtained by linearizing the nonlinear fluid-dynamical equations

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{4}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Phi, \tag{5}$$

$$\Delta \Phi = 4\pi G \rho \tag{6}$$

around the static reference state. Indeed, here are Binney and Tremaine [1, p. 287ff]; emphasis in the original; "DF" stands for 'distribution function,' referring to the stellar-dynamical setup; equation numbers of the present paper are inserted in [...] in the original text): “We construct our fictitious infinite homogeneous equilibrium by perpetrating what we shall call the *Jeans swindle* after Sir James Jeans, who studied this problem in 1902 (Jeans, 1929). Mathematically, the difficulty we must overcome is that if the density and pressure of the medium ρ_0, p_0 are constant, and the mean velocity $\mathbf{v}_0$ is zero, it follows from Euler’s equations (5–8) [(5)] that $\nabla\Phi_0 = 0$. On the other hand, Poisson’s equations (5–9) [(6)] requires that $\nabla^2\Phi_0 = 4\pi G\rho_0$. These two requirements are inconsistent unless $\rho_0 = 0$. Physically, there are no pressure gradients in a homogeneous medium to balance gravitational attraction. A similar inconsistency arises in an infinite homogeneous stellar system whose DF is independent of position. We remove the inconsistency by the *ad hoc* assumption that Poisson’s equation describes only the relation between the perturbed density and the perturbed potential, while the unperturbed potential is zero. This assumption constitutes the Jeans swindle; it is a swindle, of course, because in general there is no formal justification for discarding the unperturbed gravitational field.”

As we will see in this paper, however, such a “formal justification for discarding the unperturbed gravitational field” is readily supplied. In a nutshell, the difficulty is overcome by realizing that dynamically, and thus for defining an equilibrium, what counts are the forces, not the potentials. As we will show, one can set up some sensible limit with well-defined Newtonian gravitational *forces* which vanish when the mass density is a constant, ρ_0. These vanishing equilibrium forces do *not* derive from a Newtonian potential satisfying the familiar Poisson equation for ρ_0. Yet, in that same limit, Poisson’s equation does describe the relation between the perturbed density and the perturbed Newtonian potential, making obsolete any need for postulating this in an *ad hoc* manner.

There are actually several equivalent ways for setting up such a limit. The perhaps simplest, and at the same time physically appealing one will be presented in this paper. More precisely, we recall that Einstein [5], to pave the way for the introduction of the cosmological constant into general relativity, in fact first showed how the cosmological constant solves the simpler nonrelativistic problem faced by Jeans. Hence, all that needs to be done to vindicate Jeans’ ‘swindle’ is to discuss such a nonrelativistic universe with cosmological constant and subsequently make it purely Newtonian by taking the limit of vanishing cosmological constant. As we will show in this note, the limit of vanishing cosmological constant exists in a proper sense, relegating the “Jeans swindle” into the realm of myth.

In the next section we briefly summarize the main features of nonrelativistic gravity with a cosmological constant; a brief appendix shows how it emerges from general relativity with cosmological constant. In Section 3 we consider the fluid-dynamical setup, presenting Euler’s nonlinear equations of fluid motion with cosmological constant and their limit for vanishing cosmological constant; the derivation of (2) after linearization is then standard (Chandrasekhar [3]). We actually show that it does not matter whether one first computes the dispersion relations and then takes the limit of vanishing cosmological constant, or the other way round. In Section 4 we present the encounterless stellar dynamical version (a.k.a. Vlasov theory for self-gravitating systems) and validate the analogous Jeans’ dispersion relation which can be found, for instance, in the monographs by Fridman and Polyachenko [7] and Binney and Tremaine [1]. In Section 5 we briefly explain why the reappearance of the Jeans criterion in different equilibrium geometries, in stationarily rotating configurations, and in homogeneously expanding solutions cannot be invoked to justify the "Jeans swindle." The conclusions of the paper are presented in Section 6.

2. Nonrelativistic gravity with cosmological constant and its Newtonian limit

To accommodate a *static* homogeneous universe, Einstein [5], first discussing a nonrelativistic setting, replaced the familiar Poisson equation (6) for the Newtonian potential Φ by the inhomogeneous Helmholtz equation

$$\Delta\Psi - \kappa^2\Psi = 4\pi G\rho \tag{7}$$

for what we will refer to as the Einsteinian potential Ψ. In (7), κ^2 is the cosmological constant. Since Einstein was ultimately interested in applying general relativity to static cosmology, he remarked (Einstein [5]) that (7) should not be taken too seriously in itself; yet (7) does obtain from Einstein's general relativistic equations with cosmological constant in the nonrelativistic limit, as we will briefly show in the appendix.

If the mass density ρ is locally sufficiently well behaved (for the sake of concreteness, let ρ be bounded), then (7) is solved by

$$\Psi(\mathbf{x}) = -G\int_{\mathbb{R}^3} \frac{\mathrm{e}^{-\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\rho(\mathbf{y})\,\mathrm{d}^3y. \tag{8}$$

We take (8) to define the behavior of Ψ at infinity. Note that the effect of a cosmological constant is to *screen* the gravitational interactions with an attenuation rate κ. In the limit of vanishing cosmological constant, (8) *formally* reduces to

$$\Phi(\mathbf{x}) = -G\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{y}|}\rho(\mathbf{y})\,\mathrm{d}^3y. \tag{9}$$

However, integral (9) makes sense only when the mass density function $\rho(\mathbf{x})$ is globally sufficiently integrable; for instance, finite mass $\int_{\mathbb{R}^3}\rho(\mathbf{x})\,\mathrm{d}^3x = M$ will do. Whenever the solution Ψ to Helmholtz's equation (7) given in (8) converges to a proper Φ given by (9), the Helmholtz equation (7) for Ψ goes over into Poisson's (6) for Φ. Of course, our interest is precisely in those situations where (9) does *not* make sense.

In this vein, consider now a monotone sequence of densities ρ, having finite mass, which converges (pointwise, say) to a constant mass density $\rho_0 > 0$. Then (8) converges (likewise pointwise) to a constant limit as well, given by

$$\Psi_0 = -G\rho_0\int_{\mathbb{R}^3} \frac{\mathrm{e}^{-\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}^3y = -4\pi G\rho_0\frac{1}{\kappa^2}, \tag{10}$$

while the Newtonian potential diverges, $\Phi \to -\infty$, as $\rho \to \rho_0$. Notice that we also have $\Psi_0 \to -\infty$ when $\kappa \to 0$, as is obvious from (10).

The divergences of Φ as $\rho \to \rho_0$, and of Ψ_0 as $\kappa \to 0$, are not yet bad news, for we know what counts are not the potentials but the forces derived from them, viz. their gradients. We will be in an acceptable Newtonian gravity configuration if we can guarantee that, as $\rho \to \rho_0$, the force field $\nabla\Phi$ converges to a gravitational force field that depends only on ρ_0 but not on the limiting procedure $\rho \to \rho_0$. Alternatively, we will also be in an acceptable Newtonian gravity configuration if we can guarantee that $\nabla\Psi_0$ converges to a gravitational force field that depends only on ρ_0 but not on the limiting procedure $\kappa \to 0$.

However, as is well known, $\nabla\Phi$, whenever it converges, does not just depend on the limit density ρ_0 but on the particular limiting sequence $\rho \to \rho_0$. (Simply consider a sequence of balls of radius R and center x_0 in which $\rho = \rho_0$, while ρ vanishes outside the balls. As $R \to \infty$, $\rho \to \rho_0$ everywhere, but the gravitational field will always point toward x_0. Since x_0 is arbitrary, the point is made.) Such a limiting procedure is therefore not a viable possibility to define Newtonian self-gravity in a homogeneous infinite system. In particular, $\nabla\Phi$ never converges to zero identically, no matter which sequence $\rho \to \rho_0$ is considered.

The existence of the constant Einsteinian potential Ψ_0 for $\rho = \rho_0$ on the other hand implies that $\nabla\Psi_0 = \mathbf{0}$ identically. The important point, for us, is that gravitational forces with cosmological constant cancel themselves out in a homogeneous universe, not "only with an appropriately chosen cosmological constant" (Börner [2, p. 320]), but for *all* values of κ. Hence, Newtonian gravitational forces in such an infinite, homogeneous and isotropic medium can now be *properly* defined by simply taking the limit $\kappa \to 0$ of the (vanishing) Einsteinian gravitational forces with cosmological constant.

Totally self-balanced gravitational forces in an infinite, homogeneous and isotropic system guarantee that such a system is automatically in equilibrium. This of course was Einstein's main motivation for introducing the cosmological constant (Einstein [5]); the fact that equilibrium obtains also in the limit of vanishing cosmological constant is a simple corollary, albeit not contemplated by Einstein.

Our real interest, however, is not in the infinite homogeneous self-gravitating equilibrium itself, but in the Newtonian evolution of initial configurations which deviate somewhat from such an equilibrium state, say by the displacement of only a finite amount of mass from the uniformly distributed state. We could be more general, but this is certainly a reasonably interesting class of mass densities to study. We now show that our definition of Newtonian forces extends unproblematically to such nonuniform mass density functions.

Writing $\rho(\mathbf{x}) = \rho_0 + \sigma(\mathbf{x})$, the density disturbance $\sigma(\mathbf{x})$ must be sufficiently integrable, satisfy

$$\int_{\mathbb{R}^3} \sigma(\mathbf{x})\, \mathrm{d}^3x = 0, \tag{11}$$

and be bounded below by $-\rho_0$, for $\rho_0 + \sigma(\mathbf{x})$ is a mass density and, therefore, must not be negative. For technical convenience, we actually demand that σ be smooth and decay rapidly to zero at spatial infinity. The Einsteinian potential Ψ for such a mass density $\rho(\mathbf{x}) = \rho_0 + \sigma(\mathbf{x})$ is readily computed. By the linearity of the integral formula (8), we have

$$\Psi(\mathbf{x}) = \Psi_0 + \psi(\mathbf{x}), \tag{12}$$

where $\Psi_0 = -4\pi G\rho_0/\kappa^2$ as before, and where

$$\psi(\mathbf{x}) = -G\int_{\mathbb{R}^3} \frac{e^{-\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\sigma(\mathbf{y})\,d^3y, \tag{13}$$

which solves the inhomogeneous Helmholtz equation

$$\Delta\psi - \kappa^2\psi = 4\pi G\sigma. \tag{14}$$

The forces are proportional to the gradient of Ψ; but $\nabla\Psi_0$ vanishes, hence

$$\nabla\Psi(\mathbf{x}) = \nabla\psi(\mathbf{x}). \tag{15}$$

Since $\sigma \neq$ constant, it follows that $\nabla\psi(\mathbf{x}) \neq \mathbf{0}$ in general. The important point now is that because of the finite amount of mass involved in the density disturbance σ, the Newtonian limit $\kappa \to 0$ of $\nabla\psi$ exists and is given by

$$\lim_{\kappa\to 0} \nabla\psi(\mathbf{x}) = \nabla\phi(\mathbf{x}), \tag{16}$$

where

$$\phi(\mathbf{x}) = -G\int_{\mathbb{R}^3} \frac{1}{|\mathbf{x}-\mathbf{y}|}\sigma(\mathbf{y})\,d^3y \tag{17}$$

is the Newtonian potential of the density disturbance σ. Clearly, (17) solves the Poisson equation

$$\Delta\phi = 4\pi G\sigma. \tag{18}$$

Thus we have extended our definition of Newtonian forces unproblematically to the nonuniform mass density functions $\rho_0 + \sigma$ declared above.

Since all the problems with the notion of Newtonian gravitational forces in a spatially *asymptotically* homogeneous and isotropic nonrelativistic universe, which were at the heart of the "Jeans swindle," have evaporated in a mathematically clean way, we may now proceed to the dynamical implementation of our scheme.

3. Fluid dynamics

3.1. The nonlinear evolution equations

In this section we consider the Euler evolution of an inviscid fluid with nonrelativistic Einsteinian self-gravitation. The dynamical variables of the model are the fluid mass density ρ and fluid velocity $\mathbf{u}$. The evolution equations for these dynamical variables comprise the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{19}$$

and Euler's force balance equation

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Psi. \tag{20}$$

The Einsteinian gravitational potential Ψ is coupled to ρ by the inhomogeneous Helmholtz equation

$$\Delta \Psi - \kappa^2 \Psi = 4\pi G \rho, \tag{21}$$

re-displayed here to have the basic equations grouped together. The pressure p is related to ρ by an equation of state, which we choose (for simplicity) to be Boyle's law of the classical perfect gas at constant temperature T_0,

$$p = \rho c_s^2, \tag{22}$$

where

$$c_s = \sqrt{\frac{k_B T_0}{m}} \tag{23}$$

is the speed of isothermal sound. The dynamical variables need to be supplemented by initial conditions at some initial time, say $t_0 = 0$. Moreover, these equations have to be supplemented by asymptotic conditions at spatial infinity. We demand that asymptotically at spatial infinity all the system variables approach the values of the stationary, infinite, homogeneous and isotropic equilibrium fluid in which the Einsteinian gravitational forces balance themselves. It is a trivial matter to verify that the set of constant variables, $\rho(\mathbf{x}) = \rho_0$, $p(\mathbf{x}) = p_0 = \rho_0 c_s^2$, $\mathbf{u}(\mathbf{x}) = \mathbf{u}_0 = \mathbf{0}$, and $\Psi(\mathbf{x}) = \Psi_0 = -4\pi G \rho_0/\kappa^2$ for all $\mathbf{x}$, forms such a stationary solution of (19)–(21).

To inquire into the dynamics in the mathematical neighborhood of this constant equilibrium solution, we write $\rho(\mathbf{x}, t) = \rho_0 + \sigma(\mathbf{x}, t)$ and demand that the initial deviation $\sigma(\mathbf{x}, 0)$ is smooth, rapidly decaying to zero at spatial infinity, and satisfies

$$\int_{\mathbb{R}^3} \sigma(\mathbf{x}, 0)\, \mathrm{d}^3x = 0. \tag{24}$$

Then $\int_{\mathbb{R}^3} \sigma(\mathbf{x}, t)\, \mathrm{d}^3x = 0$ for all $t \in (0, \tau)$, where τ is the mathematical lifespan of the solution. Pressure and Einsteinian potential are written accordingly, i.e., $p(\mathbf{x}, t) = p_0 + \sigma(\mathbf{x}, t) c_s^2$, and $\Psi(\mathbf{x}, t) = \Psi_0 + \psi(\mathbf{x}, t)$. We also write $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_0 + \mathbf{w}(\mathbf{x}, t)$. (Although for our choice of reference equilibrium we have $\mathbf{u}_0 = \mathbf{0}$ and therefore $\mathbf{w}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t)$, we prefer to introduce a new symbol for the deviation from the equilibrium velocity field, simply as a reminder that more general equilibrium velocity fields can be handled for which $\mathbf{w}(\mathbf{x},t) \neq \mathbf{u}(\mathbf{x},t)$.) Inserting the above representation of the system variables into our fluid-dynamical equations, and already implementing our equation of state into the force balance equation, as well as using the fact that derivatives of constant functions vanish and that Ψ_0 terms cancel versus ρ_0 terms from the inhomogeneous Helmholtz equation, we obtain the dynamical equations for the unknowns σ and $\mathbf{w}$,

$$\partial_t \sigma + \rho_0 \nabla \cdot \mathbf{w} + \nabla \cdot (\sigma \mathbf{w}) = 0, \tag{25}$$

$$\partial_t \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{w} = -\frac{c_s^2}{\rho_0 + \sigma} \nabla \sigma - \nabla \psi, \tag{26}$$

coupled to ψ via

$$\Delta \psi - \kappa^2 \psi = 4\pi G \sigma. \tag{27}$$

All deviation variables are equipped with the asymptotic conditions that they vanish asymptotically as $|\mathbf{x}| \to \infty$.

At this point already we can let $\kappa \to 0$ in (25)–(27), thereby obtaining the nonlinear dynamical equations for the evolution of the disturbances of an infinitely extended fluid with Newtonian gravity. The continuity equation remains unchanged,

$$\partial_t \sigma + \rho_0 \nabla \cdot \mathbf{w} + \nabla \cdot (\sigma \mathbf{w}) = 0, \tag{28}$$

while Euler's force balance equation becomes

$$\partial_t \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{w} = -\frac{c_s^2}{\rho_0 + \sigma} \nabla \sigma - \nabla \phi \tag{29}$$

and the inhomogeneous Helmholtz equation turns into Poisson's equation

$$\Delta \phi = 4\pi G \sigma. \tag{30}$$

The deviation variables continue to be equipped with the asymptotic conditions that they vanish as $|\mathbf{x}| \to \infty$. Notice that no linearization has been invoked so far.

3.2. The linearized evolution equations

To linearize (25)–(27), we write

$$\sigma = \sigma_1 + \sigma_2 + \cdots, \qquad \mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 + \cdots, \qquad \psi = \psi_1 + \psi_2 + \cdots, \tag{31}$$

where the index $k = 1, 2, 3, \ldots$ indicates the 'level of smallness.' Thus, σ_2 is treated as one level smaller than σ_1; and $\sigma_1 \nabla \psi_1$ is at the same level of smallness as $\rho_0 \nabla \psi_2$; etc. We are only interested in the first level of the hierarchy. Hence, retaining only level 1 terms in (25)–(27), we obtain for the quantities at level 1,

$$\partial_t \sigma_1 + \rho_0 \nabla \cdot \mathbf{w}_1 = 0, \tag{32}$$

$$\rho_0 \partial_t \mathbf{w}_1 = -c_s^2 \nabla \sigma_1 - \rho_0 \nabla \psi_1, \tag{33}$$

$$\Delta \psi_1 - \kappa^2 \psi_1 = 4\pi G \sigma_1, \tag{34}$$

supplemented by initial conditions for σ_1, $\mathbf{w}_1$, and the asymptotic vanishing conditions at infinity for σ_1, $\mathbf{w}_1$, ψ_1.

In the same manner we linearize (28)–(30), using the expansions (31) for σ and $\mathbf{w}$ as well as

$$\phi = \phi_1 + \phi_2 + \cdots. \tag{35}$$

Retaining only level 1 terms in (28)–(30), we obtain

$$\partial_t \sigma_1 + \rho_0 \nabla \cdot \mathbf{w}_1 = 0, \tag{36}$$

$$\rho_0 \partial_t \mathbf{w}_1 = -c_s^2 \nabla \sigma_1 - \rho_0 \nabla \phi_1, \tag{37}$$

$$\Delta \phi_1 = 4\pi G \sigma_1, \tag{38}$$

supplemented by initial conditions for σ_1, $\mathbf{w}_1$, and the asymptotic vanishing conditions at infinity for σ_1, $\mathbf{w}_1$, and ϕ_1. We remark that the same set of linearized equations obtains if, instead of taking the limit $\kappa \to 0$ of (25)–(27) first and then linearizing the equations (28)–(30), we first linearize the equations (25)–(27) to obtain (32)–(34) and then take the limit $\kappa \to 0$ of (32)–(34).

The linearized equations (36)–(38) are precisely the linear dynamical equations studied by Jeans, only this time we have derived them without mathematical 'swindle.' This completes our "formal justification for discarding the unperturbed gravitational field."

3.3. The dispersion relations

The solution of these linearized equations is found in the standard way using Fourier transforms in space and Laplace transforms in time, denoted by $\widehat{\ }$ and $\widetilde{\ }$, respectively. For the linearized equations with Newtonian gravity this procedure is discussed in various monographs, in particular also by Chandrasekhar [3], Fridman and Polyachenko [7], Binney and Tremaine [1], Kippenhahn and Weigert [11], Börner [2]. Of course there is no added difficulty to do the same in the presence of a cosmological constant; however, the final result features an interesting and apparently new aspect that is worth mentioning: *the cosmological constant can suppress the Jeans instability.*

For the density perturbation we find from (32)–(34),

$$\widetilde{\widehat{\sigma}}_1(\mathbf{k}, \omega) = \frac{\omega \widehat{\sigma}_1(\mathbf{k}, 0) - \rho_0 \mathbf{k} \cdot \widehat{\mathbf{w}}_1(\mathbf{k}, 0)}{c_s^2(|\mathbf{k}|^2 + \kappa^2) - 4\pi G \rho_0 - \omega^2}. \tag{39}$$

We read off the (modified) Jeans dispersion relation for the isothermal disturbances of a static isothermal fluid universe with cosmological constant as

$$\omega^2 - |\mathbf{k}|^2 c_s^2 + \left(k_J^2 - \kappa^2\right)c_s^2 = 0. \tag{40}$$

In (40), k_J is the Jeans wave number defined in (3), with c_s given in (23). Note that *no* (linear) gravitational instability occurs if $\kappa \geqslant k_J$. The borderline case $\kappa = k_J$ is particularly curious, for in that case (40) coincides exactly with the classical dispersion relation (1) for sound waves (here for isothermal wave motion).

Our goal is of course the opposite parameter regime, where $\kappa \to 0$. By simply taking the limit of vanishing cosmological constant in (40) we now obtain the original Jeans dispersion relation (2) for the disturbances of an infinite, homogeneous static fluid universe with isothermal equation of state and Newtonian gravitational interactions which, as mentioned earlier, coincides with the one obtained directly from (36)–(38).

4. Stellar dynamics

4.1. The nonlinear evolution equations

In a stellar-dynamical setting, the dynamical variable of the model is the density-of-stars function $f(\mathbf{x}, \mathbf{v}, t)$ on the one-'particle' phase space $\mathbb{R}^3 \times \mathbb{R}^3$ at time $t \in \mathbb{R}$. It satisfies the encounterless Boltzmann kinetic equation

$$\partial_t f + \mathbf{v} \cdot \nabla f - \nabla \Psi \cdot \partial_{\mathbf{v}} f = 0, \tag{41}$$

coupled, in a universe with cosmological constant, to the inhomogeneous Helmholtz equation for the Einsteinian gravitational potential $\Psi(\mathbf{x}, t)$,

$$\Delta\Psi - \kappa^2\Psi = 4\pi G \int_{\mathbb{R}^3} f \, \mathrm{d}^3 v. \tag{42}$$

We will refer to the system of Eqs. (41) and (42) as the 'Vlasov–Helmholtz equations.'

The static, spatially homogeneous and isotropic universe now corresponds to a phase space density function f_0 that is constant in physical space, with mass density ρ_0, but which is a Maxwellian in velocity space, with constant temperature T_0. Thus, f_0 is given by $f_0(\mathbf{v}) = \rho_0(2\pi c_s^2)^{-3/2} \exp(-0.5|\mathbf{v}|^2/c_s^2)$, with the Helmholtz potential given as before by $\Psi = \Psi_0 = -4\pi G\rho_0/\kappa^2$.

The dynamical equations for the evolution of deviations from the stationary solution are obtained by writing $f(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{v}) + g(\mathbf{x}, \mathbf{v}, t)$ and $\Psi(\mathbf{x}, t) = \Psi_0 + \psi(\mathbf{x}, t)$, and requiring g and ψ to vanish at spatial and velocital infinity, and g to integrate to zero over phase space. The evolution equations for the unknowns g and ψ read

$$\partial_t g + \mathbf{v} \cdot \nabla g - \nabla \psi \cdot \partial_{\mathbf{v}} g = \nabla \psi \cdot \partial_{\mathbf{v}} f_0, \tag{43}$$

$$\Delta \psi - \kappa^2 \psi = 4\pi G \int_{\mathbb{R}^3} g \, \mathrm{d}^3 v. \tag{44}$$

Taking the limit $\kappa \to 0$ gives the nonlinear Vlasov–Poisson equations of an infinitely extended, asymptotically (in space) uniform encounterless stellar-dynamical system with Newtonian gravity,

$$\partial_t g + \mathbf{v} \cdot \nabla g - \nabla \phi \cdot \partial_{\mathbf{v}} g = \nabla \phi \cdot \partial_{\mathbf{v}} f_0, \tag{45}$$

$$\Delta \phi = 4\pi G \int_{\mathbb{R}^3} g \, \mathrm{d}^3 v. \tag{46}$$

4.2. The linearized evolution equations

Expanding with respect to the levels of smallness,

$$g = g_1 + g_2 + \cdots, \qquad \psi = \psi_1 + \psi_2 + \cdots, \tag{47}$$

and retaining only level 1 terms, we find the linearized Vlasov–Helmholtz equations,

$$\partial_t g_1 + \mathbf{v} \cdot \nabla g_1 = \nabla \psi_1 \cdot \partial_{\mathbf{v}} f_0, \tag{48}$$

$$\Delta \psi_1 - \kappa^2 \psi_1 = 4\pi G \int_{\mathbb{R}^3} g_1 \, \mathrm{d}^3 v. \tag{49}$$

Taking the limit $\kappa \to 0$, gives the linearized Vlasov–Poisson equations,

$$\partial_t g_1 + \mathbf{v} \cdot \nabla g_1 = \nabla \phi_1 \cdot \partial_{\mathbf{v}} f_0, \tag{50}$$

$$\Delta \phi_1 = 4\pi G \int_{\mathbb{R}^3} g_1 \, \mathrm{d}^3 v. \tag{51}$$

Alternately we obtain the linearized Vlasov–Poisson equations by expanding the nonlinear Vlasov–Poisson equations with respect to the levels of smallness in g and ϕ and retaining only level 1 terms. Again, we have found the linear evolution equations for Newtonian gravity without invoking a ‘swindle,’ or anything illegitimate of that sort.

4.3. The dispersion relations

The solution of the linearized Vlasov–Poisson equations (50), (51) in terms of Fourier and Laplace transformation is again standard, though one has to be somewhat careful with analytic continuations to derive the stellar dynamical Jeans dispersion relation from the linearized Vlasov–Poisson equations, see, e.g., Fridman and Polyachenko [7], Binney and Tremaine [1]. In complete analogy one finds the solution of the linearized Vlasov–Helmholtz equations (48), (49), which converge to solutions of (50), (51) in the limit $\kappa \to 0$ and therefore provide a slightly different means of deriving the Jeans dispersion relation. As in the fluid-dynamical setting, it turns out that a cosmological constant can suppress the Jeans instability. It suffices to summarize the main steps.

The Fourier–Laplace transformed expression for the phase space density perturbation reads, for $\Im(\omega) < 0$,

$$\widetilde{\widehat{g}}_1(\mathbf{k},\mathbf{v},\omega) = \frac{-\mathrm{i}\widehat{g}_1(\mathbf{k},\mathbf{v},0)}{D_{\mathrm{I}}(\mathbf{k},\omega)D_{\mathrm{II}}(\mathbf{k},\omega)} \tag{52}$$

with

$$D_{\mathrm{I}}(\mathbf{k},\omega) = \omega + \mathbf{k}\cdot\mathbf{v} \tag{53}$$

and

$$D_{\mathrm{II}}(\mathbf{k},\omega) = 1 - \frac{k_{\mathrm{J}}^2}{|\mathbf{k}|^2+\kappa^2}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\xi\,\mathrm{e}^{-\xi^2/2}\,\mathrm{d}\xi}{\xi + \frac{\omega}{|\mathbf{k}|c_s}}, \tag{54}$$

where k_{J} is the Jeans wave number defined in (3), with c_s given in (23). Apart from the ballistic term D_{I}, absent in fluid theory, we immediately read off the (modified) stellar-dynamical Jeans dispersion relation for $\Im(\omega) < 0$,

$$D_{\mathrm{II}}(\mathbf{k},\omega) = 0, \tag{55}$$

which has to be analytically continued to $\Im(\omega) \geqslant 0$. In particular, if there is a transition from stable to unstable behavior, the stability boundary occurs when $\Im(\omega) = 0$. Using Plemilj's formula, we find that for $\Im(\omega) = 0$ the dispersion relation can be fulfilled only if $\Re(\omega) = 0$ as well, in which case the ξ-integral in (54) equals $\sqrt{2\pi}$. Hence, the critical wave number satisfies

$$|\mathbf{k}|^2_{\mathrm{crit}} = k_{\mathrm{J}}^2 - \kappa^2. \tag{56}$$

Once again we find that *no* (linear) gravitational instability occurs if $\kappa \geqslant k_{\mathrm{J}}$. However, if $\kappa < k_{\mathrm{J}}$, (56) can be fulfilled for real $|\mathbf{k}|_{\mathrm{crit}}$ so that a linear gravitational instability exists for wave vectors satisfying

$$|\mathbf{k}| < \sqrt{k_{\mathrm{J}}^2 - \kappa^2}. \tag{57}$$

In the limit $\kappa \to 0$ we recover the celebrated Jeans criterion $|\mathbf{k}| < k_{\mathrm{J}}$ for gravitational instability in a static, homogeneous, Newtonian stellar-dynamical universe.

5. The deceptive robustness of the Jeans criterion

The linear stability analyses of a variety of conventional solutions of (4)–(6), from static inhomogeneous over stationarily rotating to expanding-universe solutions, all reproduce the essence of the original Jeans criterion. This robustness of the Jeans criterion, combined with the fact that the treatment of these more sophisticated stability problems does not suffer from the peculiarities unique to the analysis of the static homogeneous universe, has sparked the notion that the "Jeans swindle" is somehow justified. However, robustness of the result and orderly conduct of the analysis are only necessary but not sufficient ingredients for a proper vindication of the "Jeans swindle." We re-emphasize that in addition to those two criteria we must also be able to pass to the limit of the genuine 'Jeans-swindle-situation' in the sequence of 'no-Jeans-swindle-situations,' as we did in this paper. In this section we will briefly peruse various proposals based on static inhomogeneous, stationarily rotating, or expanding-universe solutions and see that none passes all three 'vindication criteria.' For the sake of the continuity of the discussion, we address only the isothermal systems.

We begin with the stability analysis of static inhomogeneous equilibria, which have the symmetry of a plane, a cylinder, or a sphere. Explicit formulas for the plane- and cylinder-symmetric isothermal self-gravitating equilibria are long known (e.g., Walker [18]). The mass density of plane-symmetric equilibria varies as

$$\rho(z) = \frac{c_s^2}{2\pi G}\frac{\kappa_\perp^2}{\cosh^2(\kappa_\perp z)}, \tag{58}$$

where z is a Cartesian coordinate with origin in the plane of symmetry, while the mass density of cylinder-symmetric equilibria varies as

$$\rho(r) = \frac{2c_s^2}{\pi G}\frac{\kappa_\perp^2}{(1+\kappa_\perp^2 r^2)^2}, \tag{59}$$

where r is the radial cylindrical coordinate; in both formulas, $\kappa_\perp$ is an arbitrary reciprocal length scale. The stability of the plane-symmetric equilibria is discussed by Spitzer [16], and those of the cylinder-symmetric ones by Nagasawa [14]. In both instances the equilibria are unstable with respect to perturbations whose wave vector $\mathbf{k}$ points along the invariant direction(s) when $|\mathbf{k}| < g k_{\mathrm{J}}(0)$, where $k_{\mathrm{J}}(0)$ is the central Jeans wave number, defined by

$$k_{\mathrm{J}}^2(0)c_s^2 = 4\pi G\rho(0), \tag{60}$$

and where g is a geometrical factor, with $g = 1/\sqrt{2}$ for the plane-symmetric equilibrium and $g \approx 0.561$ (computed numerically) in the cylinder-symmetric equilibrium. Furthermore, the time-dependence of the instability is exponential in both cases. Hence, the criteria of robustness of the Jeans result and of orderly analysis are satisfied. However, to pass to a homogeneous density function, $\rho \to \rho(0)$, we need to let $\kappa_\perp \to 0$ while keeping $\kappa_\perp^2 c_s^2$ fixed, so that $c_s \to \infty$. Clearly, $k_{\mathrm{J}}(0) \to 0$ in that limit. We see that we cannot pass to the limit of the genuine 'Jeans-swindle-situation' in these sequences of 'no-Jeans-swindle-situations.' For the same reason we cannot take spherical isothermal equilibria, for which no simple analytic expression is known (with the exception of the singular solution of Zöllner [20]), but which have been computed, tabulated and extensively discussed by Emden [6].

We next comment on the following suggestions of Binney and Tremaine [1, p. 288]: "... there are circumstances in which the swindle is justified. For example,

(i) ... [if] ... the wavelength ... is much smaller than the scale over which the equilibrium density and pressure vary ... the Jeans swindle should be valid for the analysis of small-scale instabilities.
(ii) ... a uniformly rotating, homogeneous system ... can be in static equilibrium in the rotating frame and no Jeans swindle is necessary (although the stability properties are somewhat modified from those of the nonrotating medium because of Coriolis forces ...)"

Unfortunately, suggestion (i) is not viable because the effective scale of nonuniformity of such a self-gravitating equilibrium is precisely the effective Jeans length, a point emphasized also by Kulsrud and Mark [12] and by Fridman and Polyachenko [7]. This is manifestly evident for the case of the plane- and cylinder-symmetric equilibria, for which it follows from (58)–(60) that $k_J = \sqrt{2}\kappa_\perp$, respectively $k_J = 2\sqrt{2}\kappa_\perp$. Hence, any hypothetical "small scale instability" would have to be small in scale compared to the effective Jeans length, viz. would *not* be analyzable by (2). Obviously such a stability result would not satisfy the criterion of robustness of Jeans' result.

In the situation depicted in suggestion (ii) the criteria of robustness of the result and of orderly analysis are satisfied: the introduction of uniform rotation with angular frequency vector Ω does regularize the homogeneous gravitational problem in such a way that an analysis in the spirit of Jeans can be carried out without any 'swindle' (Chandrasekhar [3,4]), and the resulting instability criterion for wave vectors $\mathbf{k}\|\Omega$ is precisely $|\mathbf{k}|^2 c_s^2 - 4\pi G\rho_0 < 0$, in agreement with (2). (For wave vectors $\mathbf{k} \perp \Omega$ the dispersion relation is different from (2) due to the presence of Coriolis forces (Chandrasekhar [3]).) However, the angular frequency of a uniformly rotating equilibrium and the equilibrium mass density ρ_0 are related by $|\Omega|^2 = 2\pi G\rho_0$. Hence, because the mass density ρ_0 vanishes in the nonrotating limit of a uniformly rotating system, the dispersion relation of the rotating system does not go over into the dispersion relation discovered by Jeans using his 'swindle.'

We finally address the suggestion, made elsewhere, that the correct manner of defining (2) is via the 'static limit' of an expanding-universe solution with Newtonian gravity. Such a nonrelativistic analog of the expanding flat Friedman–Lemaître universe is found by solving the system of Eqs. (4)–(6) under the assumption that at any point in time $t > 0$ the density and temperature are constant while the magnitude of the density alone varies with time, diverging as $t \downarrow 0$. Note however that one also has to pick an arbitrary center of symmetry for Φ, while a true expanding universe solution does not have a center. Assuming for continuity of the discussion that the temperature is constant also during the evolution, such a solution describing a 'big bang' is easily found to be given by

$$\rho_0(t) = \frac{1}{6\pi G}\frac{1}{t^2}, \tag{61}$$

$$u_0(\mathbf{x}, t) = \frac{2}{3}\frac{\mathbf{x}}{t}, \tag{62}$$

$$\Phi_0(t) = \frac{1}{9}\frac{|\mathbf{x}|^2}{t^2}. \tag{63}$$

Incidentally, these formulas coincide with those of the Einstein–De Sitter universe, cf. Börner [2, p. 334]. The analysis of the linearized nonrelativistic evolution of infinitesimal disturbances of this homogeneous, isotropic, expanding Einstein–De Sitter universe solution "does not suffer from the "Jeans swindle," i.e., the use of a background not obeying the dynamics" (Börner [2, p. 346]). Setting $\rho_1(\mathbf{x}, t) = \rho_0(t)\alpha(t)\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}/t^{2/3}}$ (and similarly for the other perturbed variables), the linearized nonrelativistic equations around the Einstein–De Sitter universe can be reduced to the following ordinary differential equation for $\alpha(t)$:

$$\ddot{\alpha} + \frac{4}{3}\frac{1}{t}\dot{\alpha} + \left(\frac{c_s^2|\mathbf{q}|^2}{t^{4/3}} - \frac{2}{3}\frac{1}{t^2}\right)\alpha = 0, \tag{64}$$

which is identical to Eq. (11.21a) in Börner [2], though here we used isothermal rather than adiabatic perturbations. Note that the sign of the coefficient in front of α determines whether the amplitude of a density disturbance grows relative to $\rho_0(t)$. Defining the notion of *comoving* wave vectors by $\mathbf{k}(t) = \mathbf{q}t^{-2/3}$, we find that enhancement of a density disturbance relative to the background evolution occurs if

$$\left|\mathbf{k}(t)\right| < k_{\mathrm{J}}(t), \tag{65}$$

where

$$k_{\mathrm{J}}^2(t)c_s^2 = 4\pi G\rho_0(t) \tag{66}$$

defines the 'dynamical' Jeans wave number of our nonrelativistic Einstein–De Sitter universe. (We remark that in a general relativistic Friedman–Lemaître universe the dynamical Jeans wave number is given by

$$K_{\mathrm{J}}^2(t)c_s^2 = 4\pi G(\rho_0 + p_0)(t), \tag{67}$$

see Weinberg [19], Börner [2].) Once again, the Jeans criterion proves its robustness. However, the unstable disturbances do not grow exponentially in time but like a power law.[1] In particular, at early times the pressure term is overpowered by the mass

[1] The exponential time-dependence on p. 336 in Börner [2] is obtained under the explicit assumption that $R(t) \approx$ const., where R is the cosmic distance scale which depends on the universe model under consideration. This assumption can be justified in an expanding flat universe solution with cosmological constant, an Eddington–

density term, and (64) reduces to an Euler differential equation (cf., Eq. (11.22) in Börner [2]),

$$\ddot{\alpha} + \frac{4}{3}\frac{1}{t}\dot{\alpha} - \frac{2}{3}\frac{1}{t^2}\alpha = 0, \tag{68}$$

which has one growing and one decaying mode, evolving like $t^{2/3}$ and t^{-1}, respectively. Since the disturbances and the expanding homogeneous background evolve both as power laws, there is no separating time scale between them. Hence, to take a 'static limit' for the expanding universe at the same time removes the dynamics of the perturbations as well, and with it the possibility to vindicate the Jeans swindle.

6. Concluding remarks

To summarize, in this paper we have given the first clean mathematical vindication of the "Jeans swindle." We also explained why various other suggestions fail to do so.

Our vindication of Jeans' result (2) is based on a simple limiting process for a 'cosmologically appealing' model of a nonrelativistic flat universe with cosmological constant for which the limit of vanishing cosmological constant is well-defined. However, as indicated in the introduction, other vindications are possible. For instance, we could study the universe not in three-dimensional flat space $\mathbb{R}^3$ but in the Einstein space $\mathbb{S}^3_R$, which is the three-dimensional analog of the surface of a conventional sphere of radius R, having constant positive curvature. The Einstein space is finite but without boundary. In the general relativistic setting one needs again the cosmological constant to obtain a static universe, but in our pre-relativistic setting we have no problem in defining a homogeneous and isotropic static universe for Newtonian gravity on $\mathbb{S}^3_R$ with fixed R, and also not in studying the dynamics in its neighborhood. Letting the radius $R \to \infty$ subsequently (as a parameter) and subtracting a dynamically mute constant from the potential, we arrive at our dynamical equations in $\mathbb{R}^3$, obtaining the Jeans dispersion relation again in an orderly manner. Another possibility, pointed out to me by Sheldon Goldstein, is to use a weaker definition of the integral for the total Newtonian force on a mass element of an infinite, asymptotically homogeneous medium, which basically amounts to a three-dimensional analog of Cauchy's principal value integral. This procedure has the advantage that it works without modifying Newtonian gravity by a cosmological constant, or without going into a curved space; however, the use of a principal value type integral itself needs some justification. In any event, it produces the same results as those reported here.

Our mathematical vindication of the "Jeans swindle" does not touch upon the question of applicability of the Jeans criterion in astrophysical and cosmological theories of star and galaxy formations. This is a different question altogether, see, e.g., Kippenhahn and Weigert [11] and Weinberg [19] for a discussion of some perplexingly unreasonable numbers predicted by the Jeans criterion. In this context it is interesting to register that

Lemaître universe, which can stay near to Einstein's static universe for arbitrarily long periods. It is not justified in the nonrelativistic expanding Einstein–De Sitter universe solution.

the Jeans criterion has a natural place in the statistical mechanics of gravitating systems by defining a spinodal line associated with the meta-stability region around a gravitational phase transition, see Kiessling [9]. We close with the remark that the phase transition data give considerably more reasonable numbers than the spinodal (Jeans) data (Stahl et al. [17], Kiessling [10]).

Acknowledgments

The author thanks Walter Appel, Sheldon Goldstein, Carlo Lancellotti, Joel Lebowitz, and David Merritt for various comments on an earlier draft. Special gratitude is extended to Doron Zeilberger for his enthusiastic support.

Appendix A

While Einstein [5] remarked that (7) should not be taken too seriously in itself, (7) does obtain from Einstein's general relativistic equations with cosmological constant

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \kappa^2 g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{69}$$

in the nonrelativistic limit. In this limit, we have

$$R_{00} - \frac{1}{2} R g_{00} \sim \Delta g_{00} \tag{70}$$

and

$$T_{00} \sim \rho c^2, \tag{71}$$

so that

$$-\Delta g_{00} + \kappa^2 g_{00} = \frac{8\pi G}{c^2} \rho. \tag{72}$$

Setting $g_{00} = 2\Psi/c^2$ we obtain (7).

We remark that Lemaître [13], setting

$$g_{00} \sim -1 - 2\frac{1}{c^2}\phi, \tag{73}$$

and making the further tacit assumption that ϕ and κ^2 are of the same level of smallness, rather than the inhomogeneous Helmholtz equation, obtained a Poisson equation for ϕ in which an effective *negative* background mass density features. This type of equation, which leads to gravitational screening of *test particle masses*, has been discussed by Spiegel [15].

References

[1] J. Binney, S. Tremaine, Galactic Dynamics, Princeton Univ. Press, Princeton, NJ, 1987.
[2] G. Börner, The Early Universe, Springer-Verlag, New York, 1992.
[3] S. Chandrasekhar, The gravitational instability of an infinite homogeneous medium when a Coriolis acceleration is acting, Vistas Astron. 1 (1954) 344–347.
[4] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Oxford Univ. Press, Oxford, 1961.
[5] A. Einstein, Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie, Sitzungsber. Preuss. Akad. Wiss. 1 (1917) 142–152.
[6] R. Emden, Gaskugeln, Teubner, Leipzig, 1907.
[7] A.M. Fridman, V.L. Polyachenko, Physics of Gravitating Systems, Vols. I and II, Springer-Verlag, New York, 1984.
[8] J.H. Jeans, The stability of a spherical nebula, Philos. Trans. Roy. Soc. London 199 (1902) 1–53.
[9] M.K.-H. Kiessling, Statistical Mechanics of isothermal classical gravitating matter, J. Stat. Phys. 55 (1989) 203–257.
[10] M.K.-H. Kiessling, Statistical mechanics of gravitational condensation and the formation of galaxies, in: D. Merritt, J.A. Sellwood, M. Valluri (Eds.), Galaxy Dynamics, in: Astr. Soc. Pac. Conf. Ser., Vol. 182, 1999, pp. 545–546.
[11] R. Kippenhahn, A. Weigert, Stellar Structure and Evolution, Springer-Verlag, Berlin, 1990.
[12] R.M. Kulsrud, J.W.K. Mark, The hose pipe instability in stellar systems, Astrophys. Space Sci. 14 (1971) 52–55.
[13] G.E. Lemaître, The cosmological constant, in: P.A. Schilpp (Ed.), Albert Einstein: Philosopher–scientist, in: The Library of Living Philosophers, Vol. 7, 1949, pp. 437–456, 1st printing 1949; 7th printing, Open Court, La Salle, Illinois, 1997.
[14] M. Nagasawa, Gravitational instability of the isothermal gas cylinder with axial magnetic field, Prog. Theor. Phys. 77 (1987) 635–652.
[15] E.A. Spiegel, Gravitational screening, in: A. Harvey (Ed.), A Festschrifft for Engelbert Schucking, Springer-Verlag, Heidelberg, 1998, p. 9, to appear, astro-ph/9801014.
[16] L. Spitzer Jr., Diffuse Matter in Space, Wiley, New York, 1968.
[17] B. Stahl, M. Kiessling, K. Schindler, Gravitational phase transition and the formation of small objects, Planet. Space Sci. 43 (1995) 271–282.
[18] G.W. Walker, Some problems illustrating the forms of nebulae, Proc. Roy. Soc. A 91 (1915) 410–420.
[19] S. Weinberg, Gravitation and Cosmology, New York, Wiley, 1972.
[20] J.C.F. Zöllner, Über die Natur der Cometen, Beiträge zur Geschichte und Theorie der Erkenntnis, Engelmann, Leipzig, 1871, 2nd edition 1872, 3rd edition 1882.