Ασκήσεις

9/4/21

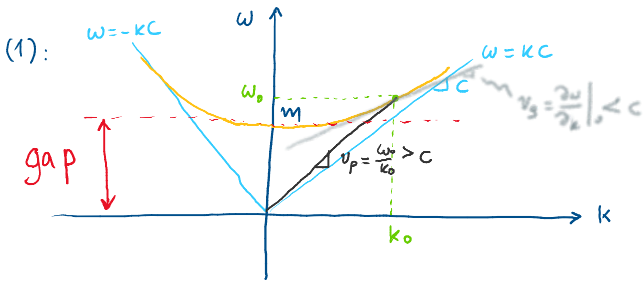
$x \mapsto x + x_0\,,\; t \mapsto t + t_0$

$$\left.\begin{aligned} &u_{tt} - c^2 u_{xx} + m^2 u = 0 \\ &u = u_0 e^{i(kx - \omega t)} \end{aligned}\right\} \Rightarrow \omega^2 = c^2 k^2 + m^2 \Rightarrow \omega = \pm\sqrt{c^2k^2 + m^2} \quad (1)$$

$\frac{\partial}{\partial x} \mapsto ik\,,\; \frac{\partial}{\partial t} \mapsto -i\omega$: ΜΔΕ $\longmapsto$ Σχ. Διασποράς

$k \mapsto -i\frac{\partial}{\partial x}$; $\omega \mapsto i\frac{\partial}{\partial t}$: Σχ. Διασποράς $\longrightarrow$ ΜΔΕ

(1):



(1): $v_p = \frac{\omega}{k} = \pm c\sqrt{1 + \frac{m^2}{c^2k^2}} \qquad |v_p| > c$

$v_g = \omega'(k)$: $\quad \omega^2 = c^2k^2 + m^2 \Rightarrow \cancel{2}\omega\omega' = \cancel{2}c^2 k$

$\Rightarrow v_g = \frac{k}{\omega}c^2 \rightarrow v_g = \frac{c^2}{v_p} \Rightarrow v_p v_g = c^2$

$v_g = \pm \frac{c^{\cancel{2}}}{\cancel{c}\sqrt{1 + m^2/c^2k^2}} = \pm \frac{c}{\sqrt{1 + m^2/c^2k^2}} \Rightarrow |v_g| < c$

$\omega = \sqrt{c^2k^2 + m^2} = ck\sqrt{1 + \frac{m^2}{c^2k^2}} \xrightarrow[\frac{m^2}{c^2k^2} \to 0]{} ck$

$\omega = m\sqrt{1 + \frac{c^2k^2}{m^2}} \xrightarrow[\frac{c^2k^2}{m^2} \ll 1]{} m\left(1 + \frac{c^2k^2}{2m^2}\right) \approx m + \frac{c^2k^2}{2m}$

---

(1): $f(x)$: $\hat{f}(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$

$u_{tt} - c^2 u_{xx} + m^2 u = 0 \Rightarrow \int_{-\infty}^{+\infty} \left(u_{tt} e^{-ikx} - c^2 e^{-ikx} \overset{(*)}{u_{xx}} + m^2 e^{-ikx} u\right) dx = 0$

$\Rightarrow \hat{u}_{tt} + \omega^2(k)\hat{u} = 0 \Rightarrow \hat{u}(k,t) = C_1(k) e^{-i\omega(k)t} + C_2(k) e^{+i\omega(k)t} \quad (2)$

(*) : Αν $f(x) \longrightarrow \hat{f}(k)$ τότε $\frac{\partial^n f}{\partial x^n} \longrightarrow (ik)^n \hat{f}(k)$

ΑΣ: $\quad u(x,0) = f(x) \longrightarrow \hat{u}(k,0) = \hat{f}(k)$

$\quad\quad u_t(x,0) = g(x) \longrightarrow \hat{u}_t(k,0) = \hat{g}(k)$

$$(2): \left.\begin{aligned} &\hat{u}(k,0) = C_1(k) + C_2(k) = \hat{f}(k) \\ &u_t(k,0) = -i\omega C_1(k) + i\omega(k) C_2(k) = \hat{g}(k) \end{aligned}\right\} \Rightarrow$$

$C_1(k) = \cdots\,,\; C_2 = \cdots$

οπότε

$$\hat{u}(k,t) = \frac{1}{2}\left[\hat{f}(k) - \frac{\hat{g}(k)}{i\omega(k)}\right] e^{-i\omega(k)t} + \frac{1}{2}\left[\hat{f}(k) + \frac{\hat{g}(k)}{i\omega(k)}\right] e^{+i\omega(k)t}$$

$$\hat{u}(k,t) = \underbrace{\frac{1}{2}\left[\hat{f}(k) - \frac{\hat{g}(k)}{i\omega(k)}\right]}_{C_1} e^{-i\omega(k)t} + \underbrace{\frac{1}{2}\left[\hat{f}(k) + \frac{\hat{g}(k)}{i\omega(k)}\right]}_{C_2} e^{+i\omega(k)t}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{2}\left[\hat{f}(k) - \frac{\hat{g}(k)}{i\omega(k)}\right] \exp[i(kx - \omega(k)t)]\,dk$$ right-going

$$+ \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{2}\left[\hat{f}(k) + \frac{\hat{g}(k)}{i\omega(k)}\right] \exp[i(kx + \omega(k)t)]\,dk$$ left-going

$f(x) = \delta(x)$ $\quad g(x) = 0 \Rightarrow \hat{f}(k) = 1, \quad \hat{g}(k) = 0$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{2}\left[e^{i[kx - \omega(k)t]} + e^{i[kx + \omega(k)t]}\right] dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\left[\frac{1}{2}\left(e^{-i\omega t} + e^{+i\omega t}\right)\right] dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} \cos[\omega(k)t]\,dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \cos(kx)\cos[\omega(k)t]\,dk$$

Εξ. συνέχειας: $\rho_t + (\rho v)_x = 0$ (διατήρηση μάζας)

Αν v = σταθ τότε $\rho_t + v\rho_x = 0$ (εξ. μεταφοράς)

Κυμ. εξ. 2ης τάξης:

$$u_{tt} - c^2 u_{xx} = 0 \Rightarrow (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$$

ΠΣΤ: $x^2 u_t + u_x + tu = 0$ (1)

$x > 0, \quad t \in \mathbb{R}$

ΣΣ: $x = 0$: $u(0,t) = f(t)$

Επί της Γ: $t = t(x)$

και $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dx} + \dfrac{\partial u}{\partial t}\dfrac{dt}{dx} = u_t \dfrac{dt}{dx} + u_x$

Επιλέγουμε: 1) $\left.\begin{aligned}\frac{dt}{dx} &= x^2 \\ t(0) &= \tau\end{aligned}\right\} \Rightarrow t = \frac{1}{3}x^3 + \tau$ ⟿ το αντίστοιχο ξ (*)

2) $\dfrac{du}{dx} = -tu$ (από την (1))

$$\Rightarrow \frac{du}{dx} = -\left(\frac{1}{3}x^3 + \tau\right)u \Rightarrow$$

$$\Rightarrow \frac{du}{u} = -\left(\frac{1}{3}x^3 + \tau\right)dx$$

$$\Rightarrow \int_{f(\tau)}^{u} \frac{du}{u} = \int_0^x -\left(\frac{1}{3}x^3 + \tau\right)dx$$

γιατί στο $x=0$: $t=\tau$

$$\Rightarrow \ln\frac{u}{f(\tau)} = -\frac{x^4}{12} - \tau x$$

$$\left.\begin{aligned} &\Rightarrow u = f(\tau)\exp\left[-\frac{x^4}{12} - \tau x\right] \\ &\tau = t - \frac{1}{3}x^3 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow u = f\left(t - \frac{1}{3}x^3\right)\exp\left[-\frac{x^4}{12} - \left(t - \frac{1}{3}x^3\right)x\right]$$

$$\Rightarrow \boxed{u = f\left(t - \frac{1}{3}x^3\right)\exp\left(\frac{x^4}{4} - xt\right)}$$

Να ελεγχθεί αν ικανοποιεί την εξίσωση!