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Graphical approach to Bell's inequalities

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Following an approach proposed by Eberhard, a graphical representation of entangled particles is used to provide insight into the meaning and direct experimental consequences of local realism. The most famous Bell-type theorems, like the original inequality developed by Bell, as well as the ones presented by Clauser, Horne, Shimony, and Holt, Clauser and Horn, Eberhard, d'Espagnat, and Maccone, are then derived by means of the graphical method. The universality and efficiency of the presented method in achieving different Bell's inequalities follow from making local realism explicitly a starting point in our general reasoning. The pedagogical and illustrative nature of the approach makes it easy to see directly how all the miscellaneous inequalities are actually rooted in the local realist view on microscopic particles. https://doi.org/10.1119/5.0250670

I. INTRODUCTION

In their famous 1935 paper,[1] Einstein, Podolsky, and Rosen (EPR) suggested that quantum mechanics provides an incomplete description of reality. The EPR expectation was that a more detailed theory would replace quantum mechanics in the future, a theory that produced the same results as emerging from a statistical analysis of some "hidden variables" that characterize the microscopic constituents of matter.

Unexpectedly, in 1964, Bell presented his profound result,[2] proving that any theory that treats particles as endowed with some real properties that locally determine the measurement outcomes (the assumption called "local realism") is incompatible with the predictions of quantum mechanics. In the following years, other Bell-type theorems were developed by Clauser, Horne, Shimony, and Holt (CHSH),[3] Clauser and Horn (CH),[4] and Eberhard,[5] which proved more convenient for experimental tests. Needless to say that all experiments[6–9] since performed to verify Bell's result prove the correctness of the predictions of quantum mechanics. The most striking conclusion that follows from Bell's work is, therefore, that the reality of microscopic particles cannot be described in terms of a local realism hidden variables theory, as was expected by EPR.

In this paper, we describe a simple geometrical and universal approach following the idea presented by Eberhard[5] that enables us to derive a variety of the most famous Bell's inequalities. In addition to the Eberhard inequality,[5] the method is shown also to yield the inequalities achieved by CH,[4] CHSH,[3] d'Espagnat,[10] Maccone,[11] as well as the original one published by Bell.[2] All the original proofs of these inequalities use the local realism assumption, but this premise typically appears as the choice of specific mathematical structures rather than a specific logical step in the derivation; thus, its essential role in the reasoning may be difficult to grasp. Thanks to the geometrical representation of particles within the framework of local realism, the method presented in this paper offers a useful insight into the direct theoretical and experimental consequences of this assumption. The importance of local realism is additionally emphasized by making it a universal starting point in our method, which is capable of reproducing the diversity of Bell-type theorems from this single concept.

II. GRAPHICAL REPRESENTATION OF AN ENTANGLED PAIRS OF PARTICLES

Consider an experiment in which a pair of entangled particles p_1 and p_2 are produced at a source, and then the particles travel to two observers with a significant spacelike separation between them, p_1 to Alice and p_2 to Bob. Alice and Bob perform measurements on their particles after first choosing between one of two possible settings of their apparatuses (e.g., orientation of polarizers or Stern–Gerlach devices). The possible settings will be denoted a and a' for Alice and b and b' for Bob. What is measured is a two-valued quantity characterizing a property of each particle, such as its spin or polarization. The outcomes measured by Alice and Bob will be denoted $A = \pm 1$ and $B = \pm 1$, respectively. Performing many runs of the experiment, one can find statistical correlations between outcomes A and B.

The realism assumption treats particles as endowed with some properties, described by the so-called "hidden" variable λ, that determine uniquely[12] the outcomes A and B, irrespective of which measurement is actually carried out. The localism assumption means that the outcome A achieved by Alice does not depend on the measurement settings chosen by Bob from the two possibilities $\{b, b'\}$ and, similarly, the outcome B obtained by Bob is not influenced by the Alice's apparatus settings $\{a, a'\}$. The combination of these assumptions is called local realism.

From the locality assumption, it follows that for *the exact same* pair of particles

$$A(a,b) = A(a,b') \equiv A(a),$$
$$A(a',b) = A(a',b') \equiv A(a'), \tag{1}$$

where $A(a,b)$ denotes the outcome A that would be recorded by Alice, if her measurement setting was a while Bob's b. Analogously, for Bob's outcomes, we have

$$B(b,a) = B(b,a') \equiv B(b),$$
$$B(b',a) = B(b',a') \equiv B(b'). \quad (2)$$

Let us now follow Eberhard's argument[5] using a graphical representation. Since Alice can choose between two bases and so can Bob, we need to specify four possible results for each pair of particles in a run. For an individual run of the experiment, we will represent each possible result as a particular box in the 4×4 grid shown in Fig. 1. Each of the 16 boxes in the grid represents one of all possible pairs of outcomes $\{A = \pm 1, B = \pm 1\}$ for each different combination of Alice and Bob's chosen measurement settings, $\{a,b\}$, $\{a,b'\}$, $\{a',b\}$, and $\{a',b'\}$.

The corners of the dashed red rectangle in Fig. 1 mark one possible set of outcomes that could be measured for some particular pair of particles with some particular set of hidden variables. These variables only determine the outcome for a given measurement. They do not determine which measurement settings Alice and Bob will choose. Thus, the possibilities are narrowed from 16 to 4, marked by each corner of the rectangle.

In the example shown, if Alice chooses her a basis and Bob chooses his b basis, the result will be $+1$ for Alice and -1 for Bob, which is indicated by the top-left corner of the dashed red rectangle. If Alice still chooses basis a, but Bob chooses basis b', Alice's result, according to Eq. (1), must remain the same $(+1)$, but Bob's result can be anything, and in this example, it is -1. This case is represented by the top right corner. Because Alice's result cannot change when Bob's basis changes, the line that connects these corners must be horizontal. Similar reasoning applies for the cases when Alice would switch from a to a', with the Bob setting (b or b') being frozen. Then it follows, according to the constraint (2), that the other two possible pairs of outcomes $\{A(a'), B(b)\}$ and $\{A(a'), B(b')\}$ must fall into boxes that lie directly below the top two corners of the rectangle. Additionally, due to the constraint (1), they must belong to the same row.

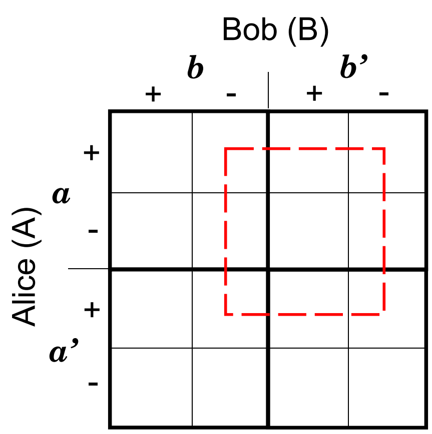

Fig. 1. The boxes of the grid represent all 16 possible outcomes $\{A = \pm 1, B = \pm 1\}$ Alice and Bob could achieve for different measurement settings. Due to the locality assumption, the outcomes one could achieve for a single pair of particles fall into boxes that form the corners of a rectangle, whose sides lie in respective rows and columns of the grid.

In other words, for a given pair of particles, the corresponding hidden variables that determine what the outcome will be for each of the four possible pairs of measurement settings that Alice and Bob could choose are denoted graphically by a shape that must be rectangular, composed of two vertical lines and two horizontal lines. This is the cornerstone of our further reasoning.

From now on, we will denote the pair of outcomes $A(a) = +1$ and $B(b) = +1$ as $\{a+, b+\}$ and similarly for the remaining 15 possible combinations of the measurement settings and values of outcomes.

III. EBERHARD INEQUALITY

When Alice and Bob do multiple runs of the experiment, randomly choosing their measurement settings each time, they might get any of the 16 possible results shown in the grid. After many runs, they will compile 16 numbers, one for each possible result giving the number of times it was obtained. According to the local realism assumption we are now exploiting, these numbers are mutually related and satisfy some inequalities, first indicated by Bell.

Let us write an inequality relationship for $n(a+, b+)$, the number of pairs with the outcome $\{a+, b+\}$. This outcome is indicated by the circle in Fig. 2. What results would the same pairs of particles produce, if Alice and/or Bob chose a different basis to use for their measurements?

From the independence assumption expressed in Eqs. (1) and (2), it follows that the set of such pairs consists of only four different subsets that are represented by the four colored rectangles marked inside the grid. The top-left corner of each rectangle lies in the same little box indicated by the circle, but the bottom-right corner of each of the four rectangles could be in any of the four bottom-right little boxes. All the corners of each rectangle represent the outcomes that would be achieved for a given subset at different measurement settings chosen by Alice and Bob.

When Alice and Bob choose measurement settings a and b and find the outcome $\{a+, b+\}$, we do not know which of the four subsets of possible pairs the particular measured pair of particles belongs to. What is crucial in our reasoning is that for each of the four possible subsets, we can point out some *other sets* that would produce the same outcome for a different measurement setting. For example, the particles described by the green rectangle (as well as the blue one) in Fig. 2 are also members of the set of particles giving the

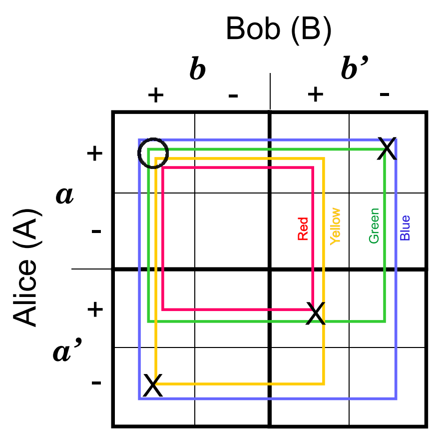

Fig. 2. The pairs of particles giving the outcomes $\{a+, b+\}$ separate into four subsets depicted by the four rectangles.

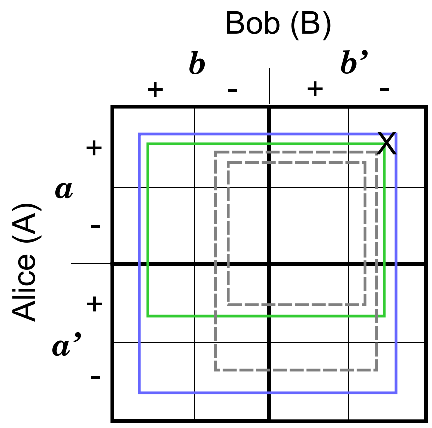

Fig. 3. The four rectangles hooked with one of their corners in the X-marked box represent the four subsets that completely cover the set of particles giving outcomes $\{a+, b'-\}$. Among these rectangles are the green and blue ones (the solid lines) introduced in Fig. 2.

result $\{a+, b'-\}$, as shown in Fig. 3. We denote then this fact by inserting "X" in the box representing the outcome $\{a+, b'-\}$ in the grid shown in Fig. 2. In the same way, we find some other sets containing the subsets represented by the red and yellow rectangles and put the X mark in the respective boxes, indicated in Fig. 2. Why are these and only these three other little boxes chosen? We must choose so that there is an "X" for each possible other set of measurement settings, and so that at least one corner of each of the four colored rectangles is marked by "X."

Having first begun with measurement settings a and b, we placed our first "X" within one of the four little boxes for measurement settings a and b'. The little box $\{a+, b'-\}$ contains a corner for the green and blue rectangles. Next, we consider measurement settings a' and b', putting our "X" in $\{a'+, b'+\}$, which contains a red corner. The yellow rectangle has not yet been included; thus, for the final a' and b, we choose $\{a'-, b+\}$ because it contains a yellow corner.

Having done that, we conclude that any pair from the set of pairs giving $\{a+, b+\}$ must belong to the set yielding $\{a+, b'-\}$ and/or to the set for which one gets $\{a'-, b+\}$ and/or to the set for which one obtains $\{a'+, b'+\}$. It means that the number of pairs $n(a+, b+)$ yielding $\{a+, b+\}$ is no greater than the sum of the numbers of pairs from the above-mentioned sets defined by the boxes with the "X" mark. We have then

$$n(a+, b+) \le n(a+, b'-) + n(a'-, b+) + n(a'+, b'+). \tag{3}$$

This is the Eberhard inequality,[5] where in his original notation outcome $\{o\}$ identified with our $\{+1\}$ and outcomes $\{u, e\}$ are lumped into our single outcome $\{-1\}$. Under some conditions, the Eberhard inequality appears to be violated by experimentally confirmed predictions of quantum mechanics.

By making different choices of where to put the X-marks in the grid, but caring always about covering all four rectangles, we can derive other inequalities involving $n(a+, b+)$. Some of these inequalities may appear to be trivial, like it is for the case when we put (only) two X marks in the boxes $\{a+, b'+\}$ and $\{a+, b'-\}$. Alternatively, when introducing more than three X marks, we would unnecessarily enlarge the right side of the inequality (3), which makes them less sensitive to undergo some possible discrepancies with the quantum mechanical predictions. Certainly, we could also put the circle in other squares and draw our four colored rectangles from there to derive similar Eberhard-like inequalities. Having understood the described above procedure, constructing different Bell-type inequalities is very simple and almost automatic.

IV. APPLICATIONS

The *CH inequality*[4] is a direct corollary of the Eberhard inequality (3) and one more inequality that can be derived according to the grid presented in Fig. 4, namely,

$$n(a'+, b'+) \le n(a-, b-) + n(a+, b'+) + n(a'+, b+). \tag{4}$$

Combining (3) with (4), we get (see the supplementary material)

$$\begin{aligned} -n \le\ & n(a+, b+) + n(a+, b'+) + n(a'+, b+) \\ & - n(a'+, b'+) - s(a+) - s(b+) \le 0, \end{aligned} \tag{5}$$

where n is the total number of pairs that are investigated in a run of the experiment. The symbol $s(a+)$ denotes the number of all pairs of particles for which Alice obtains outcome $A(a) = +1$, regardless of the measurement result achieved by Bob for the second particle. Similarly is defined $s(b+)$. By dividing the above inequality by the number of pairs n, we obtain the original CH inequality that is expressed in terms of the respective probabilities

$$\begin{aligned} -1 \le\ & P(a+, b+) + P(a+, b'+) + P(a'+, b+) \\ & - P(a'+, b'+) - P(a+) - P(b+) \le 0, \end{aligned} \tag{6}$$

where $P(a+, b+) = n(a+, b+)/n$, and similarly for the remaining symbols.

The *d'Espagnat inequality*[10] refers to the singlet state of entangled particles (e.g., two spin-1/2 particles) for which outcomes A and B are always opposite when Alice and Bob perform their measurements along the same direction. In this case, we will only consider three different basis choices (a, b, and c), one of which (c) is an option that is shared by Alice and Bob (i.e., now $a' = b' \equiv c$). In the d'Espagnat setup, Alice will choose between a and c, whereas Bob will choose between b and c. Using the grid depicted in Fig. 5, we obtain the d'Espagnat theorem

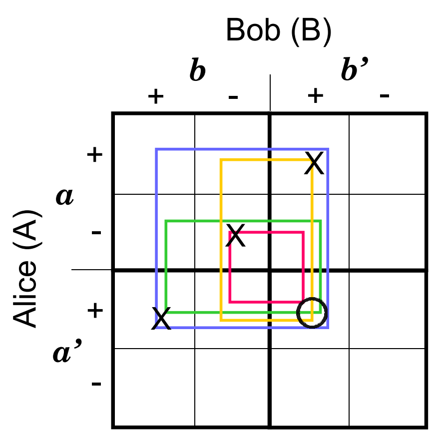

Fig. 4. The additional grid to derive the CH inequality.

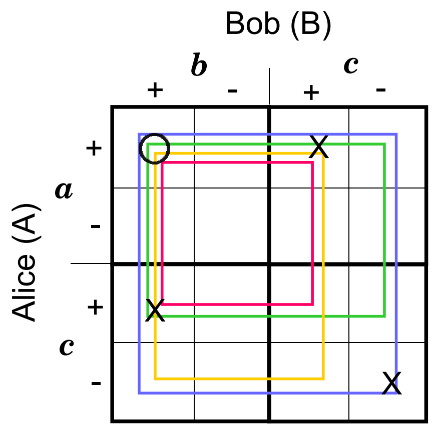

Fig. 5. The measurement settings and X-marked boxes for derivations of the d'Espagnat inequality.

$$n(a+,b+) \leq n(a+,c+) + n(c+,b+), \qquad (7)$$

where it was taken into account that $n(c-,c-) = 0$ because the particles are anti-correlated. Similarly one can obtain the *Maccone inequality*[11] referring to the Bell-state $|\Phi^+\rangle$ of entangled photons that are perfectly correlated (see the supplementary material).

The graphical method may serve us also to derive the Bell-type theorems that are expressed in terms of the expectation value of the product $A(a,\lambda)B(b,\lambda)$ that can be represented as $C(a,b) = P(a+,b+) + P(a-,b-) - P(a+,b-) - P(a-,b+)$. Using the grids from Figs. 2 and 4 but with the signs "+" and "–" interchanged, one obtains (see the supplementary material) some counterparts of (3), which combined lead to the original *CHSH inequality*,[3]

$$-2 \leq C(a,b) + C(a,b') + C(a',b) - C(a',b') \leq 2. \qquad (8)$$

In turn, the *original Bell's inequality* refers to the singlet state of particles, so one can make use of the inequality (7) and its counterparts with changed signs of the outcomes, as well as the ones with b and c interchanged (see the supplementary material). Combining the results, we get

$$|C(a,b) - C(a,c)| \leq 1 + C(b,c), \qquad (9)$$

which is the desired Bell's inequality.[2]

SUPPLEMENTARY MATERIAL

Please click on this link to access the supplementary material for the reviewer's website. Print readers can see the supplementary material at https://doi.org/10.60893/figshare.ajp.c.7558038.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

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