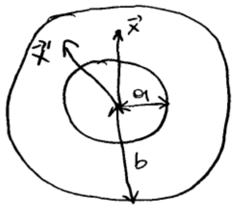
– Green function in spherical coordinates 7

Consider the space:



$\theta \in (0,\pi)$, $\phi \in (0, 2\pi)$, $a \le r \le b$.

We want to solve:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r G) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial G}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 G}{\partial\phi^2} = -4\pi \frac{\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')}{r^2 \sin\theta}$$

Based on spherical harmonics we expand

$$G = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} G_\ell(r,r')\, Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi')$$

Using the completeness relation for the $Y_{\ell m}$'s we get from the diff. eq. for G:

$$\frac{1}{r}\frac{d^2}{dr^2}(r G_\ell) - \frac{\ell(\ell+1)}{r^2} G_\ell = -\frac{4\pi}{r^2}\delta(r-r')$$

with solution:

$$G_\ell = \begin{cases} A r^\ell + B r^{-(\ell+1)} & r<r' \\ C' r^\ell + D r^{-(\ell+1)} & r>r' \end{cases}$$

Vanishing of $G_\ell \big|_{r=a,b}$ gives:

$$G_\ell = \left\{ \begin{array}{ll} A\left(r^\ell - \frac{a^{2\ell+1}}{r^{\ell+1}}\right), & r<r' \\ C\left(\frac{1}{r^{\ell+1}} - \frac{r^\ell}{b^{2\ell+1}}\right), & r>r' \end{array} \right\}.$$

Demanding $G_\ell(r,r') = G_\ell(r',r)$.

$$G_\ell = \tilde{A}\left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right)$$

Entering this into the diff. eq. for G_ℓ

$$\frac{d(rG_\ell)}{dr}\Big|_{r=r'+\varepsilon} - \frac{d(rG_\ell)}{dr}\Big|_{r=r'-\varepsilon} = -\frac{4\pi}{r'}$$

Now: $$\frac{d(rG_\ell)}{dr}\Big|_{r=r'+\varepsilon} = -\hat{A}\left(r'^\ell - \frac{a^{2\ell+1}}{r'^{\ell+1}}\right)\left(\ell\frac{1}{r'^{\ell+1}} + (\ell+1)\frac{r'^\ell}{b^{2\ell+1}}\right)$$

$$= \tilde{A}\left[\ell\frac{a^{2\ell+1}}{r'^{2\ell+2}} + (\ell+1)\left(\frac{a}{b}\right)^{2\ell+1}\frac{1}{r'} - \frac{\ell}{r'} - (\ell+1)\frac{r'^{2\ell}}{b^{2\ell+1}}\right]$$

and

$$\frac{d(rG_\ell)}{dr}\Big|_{r=r'-\varepsilon} = \hat{A}\left(\ell\frac{a^{2\ell+1}}{r'^{\ell+1}} + (\ell+1)r'^\ell\right)\left(\frac{1}{r'^{\ell+1}} - \frac{r'^\ell}{b^{2\ell+1}}\right)$$

$$= \hat{A}\left[\ell\frac{a^{2\ell+1}}{r'^{2\ell+2}} - \ell\frac{a^{2\ell+1}}{r' b^{2\ell+1}} + \frac{\ell+1}{r'} - (\ell+1)\frac{r'^{2\ell}}{b^{2\ell+1}}\right]$$

$$\therefore \left.\frac{d(rG_\ell)}{dr}\right|_{r=r'+\varepsilon} - \left.\frac{d(rG_\ell)}{dr}\right|_{r=r'-\varepsilon} =$$

$$= \tilde{A}\,\frac{1}{r'}\left(\left(\frac{a}{b}\right)^{2\ell+1} - 1\right)(2\ell+1) = -\frac{4\pi}{r'}$$

$$\rightarrow \tilde{A} = \frac{4\pi}{(2\ell+1)\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]}$$

$\therefore$ The full Green function is:
(In the interior $a \le r \le b$ of a hollow sphere)

$$\boxed{G(\vec{x},\vec{x}\,') = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}(\theta,\phi)\,Y^*_{\ell m}(\theta',\phi')}{(2\ell+1)\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]} \otimes \frac{r_<^{\ell}}{r_>^{\ell+1}}\left(1-\left(\frac{a}{r_<}\right)^{2\ell+1}\right)\left(1-\left(\frac{r_>}{b}\right)^{2\ell+1}\right)}$$

- For $b \to \infty$ one obtain

$$G(\vec{x},\vec{x}\,') = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}(\theta,\phi)\,Y^*_{\ell m}(\theta',\phi')}{(2\ell+1)}$$

$$\left[\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{\ell+1}\right]$$

This shoulb be the Green function obtain by the method of images outside a sphere.

To show that: let also $a = \infty$. Then

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta,\phi) Y^*_{lm}(\theta',\phi')}{2l+1} \frac{r_<^l}{r_>^{l+1}}$$

Green function in free space. This is the contribution of the "real charge"

The image charge: It contributes the term

$$-\left(\frac{r^2 r'^2}{a^2} + a^2 - 2rr'\cos\gamma\right)^{-1/2}.$$

In above formula $r \to \frac{rr'}{a}$, $r' \to a$. But then $r_< \to a$, $r_> \to \frac{rr'}{a}$

$$\therefore \frac{r_<^l}{r_>^{l+1}} \to \left(\frac{a}{rr'}\right)^{l+1} a^l = \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}.$$

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- For $a \to 0$ we get the Green function inside a sphere of radius b.

$$G(\vec{x},\vec{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta,\phi) Y^*_{lm}(\theta',\phi')}{2l+1}$$

$$\otimes \frac{r_<^l}{r_>^{l+1}} \left(1 - \left(\frac{r_>}{b}\right)^{2l+1}\right)$$