

Stellar structure and evolution of single stars – Overview

Reading material:

Undergraduate course notes on Stellar Physics

Chapters 1&3 from A. Maeder's book

Part II from Aerts notes

Δευτέρα 3-6μμ
Παρασκευή 10-11

The Mechanical Equilibrium of Stars

Hydrodynamical Equations

- ✓ Equation of continuity (expresses mass conservation) for a medium of density ρ and velocity \mathbf{v}

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0 \quad (1)$$

- ✓ Equation of Navier–Stokes, which is the equation of motion.

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{a} - \frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{v} \quad (2)$$

\mathbf{a} acceleration due to external forces,

P the pressure

ν the kinematic coefficient of viscosity

For negligible viscosity eq. (2) becomes

$$\frac{dv}{dt} = \mathbf{a} - \frac{1}{\rho} \nabla P \quad (3)$$

and for spherical symmetry, with r the distance from the center,

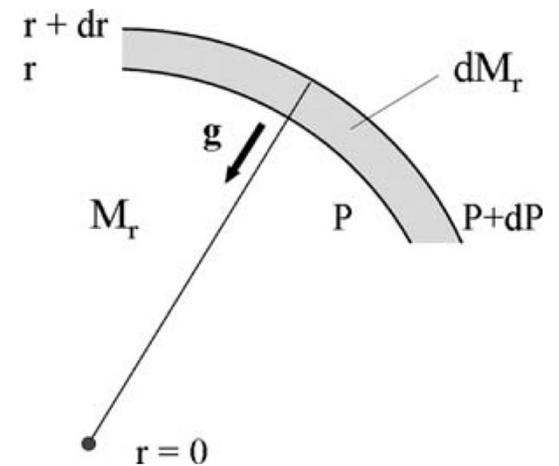
$$\frac{dv}{dt} = (\ddot{r}, 0, 0)$$

$$\nabla P = \left(\frac{\partial P}{\partial r}, 0, 0 \right)$$

$\mathbf{a} = \mathbf{g} = (-g, 0, 0)$, $g = GM_r/r^2$ where M_r the mass interior to radius r

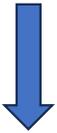
and eq (3) becomes

$$\ddot{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} - \frac{GM_r}{r^2} \quad (4) \rightarrow \text{momentum equation of hydrodynamic models}$$



Hydrostatic equilibrium

$$\ddot{r} = 0 \xrightarrow{(3,4)} \nabla P = \rho \mathbf{g} \quad \text{and} \quad \frac{1}{\rho} \frac{dP}{dr} = -\frac{GM_r}{r^2}$$



This assumption does not hold for stellar pulsations, early stages of star formation and advanced phases of evolution.

In general, the gravitational field in a star can be described by a gravitational potential Φ , which is a solution of Poisson's equation:

$$\vec{\nabla}^2 \Phi = 4\pi G \rho \Rightarrow$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) = 4\pi G \rho$$

$$\vec{g} = -\vec{\nabla} \Phi$$

Mass Conservation and Continuity Equation – in spherical symmetry

change of mass M_r in a sphere of radius r :

$$dM_r(r, t) = \left(\frac{\partial M_r}{\partial r} \right)_t 4\pi r^2 \rho dr - \left(\frac{\partial M_r}{\partial t} \right)_r 4\pi r^2 \rho v dt$$

change of mass due to a variation of radius r at a given time t

the flux of mass out of the sphere of constant r due to an outward motion with velocity $v > 0$.

$$\Sigma_{\eta\mu} \left(\frac{\partial}{\partial t} \left(\frac{\partial M_r}{\partial r} \right)_t \right)_r = \left(\frac{\partial}{\partial r} \left(\frac{\partial M_r}{\partial t} \right)_r \right)_t \Rightarrow 4\pi r^2 \frac{\partial \rho}{\partial t} = -4\pi \frac{\partial}{\partial r} (r^2 \rho v) \quad \rightarrow \text{eq. of continuity in spherical coords}$$

Eulerian vs Lagrangian description

- ✓ Eulerian description: independent variables (r, t)
- ✓ Lagrangian description: independent variables (m, t)

In the Lagrangian description, we characterize a fluid element by m , which is the mass contained in a concentric sphere at a given time t_0 .

All other quantities are a function of (m, t) , e. g.

$$r = r(m, t)$$

$$\frac{\partial}{\partial m} = \frac{1}{4\pi r^2 \rho} \frac{\partial}{\partial r}$$



$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4} \quad \text{and} \quad \frac{dr}{dM_r} = \frac{1}{4\pi r^2 \rho}$$

In the stellar centre we have $m = 0$ and at the surface $m = M$

as opposed to the large variation in the radius R during a star's lifetime, the independent variable m varies, to a good approximation, over the constant interval $[0, M]$ for more than 90% of the star's lifetime.

The energetic equilibrium in Stars

Radiative transfer

equation of radiative transfer in the direction of the element of length ds

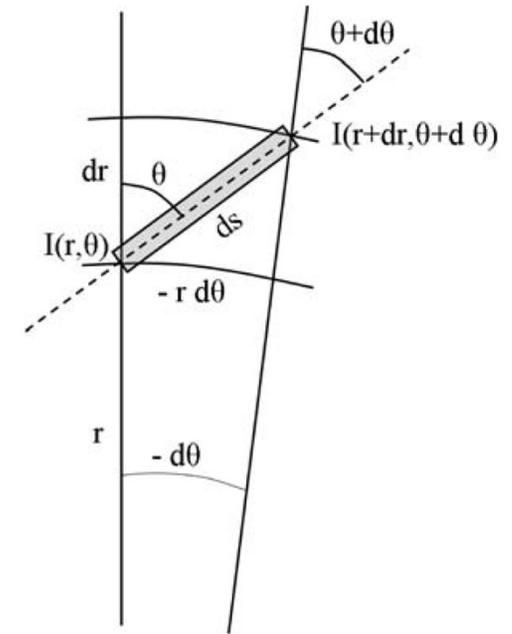
$$\frac{dI_\nu}{ds} = j_\nu \varrho - \kappa_\nu \varrho I_\nu$$

where

- I_ν specific intensity ($\text{erg s}^{-1} \text{cm}^{-2} \text{steradian}^{-1} \text{Hz}^{-1}$)
- j_ν emission coefficient ($\text{erg g}^{-1} \text{steradian}^{-1} \text{Hz}^{-1}$)
- κ_ν absorption or opacity coefficient ($\text{cm}^2 \text{g}^{-1}$)

In spherical geometry (see figure) :

$$\frac{\partial I_\nu}{\partial r} \cos \vartheta - \frac{\sin \vartheta}{r} \frac{\partial I_\nu}{\partial \vartheta} = \varrho (j_\nu - \kappa_\nu I_\nu)$$



$$\frac{d}{ds} = \frac{\partial}{\partial r} \frac{dr}{ds} + \frac{\partial}{\partial \vartheta} \frac{d\vartheta}{ds} = \frac{\partial}{\partial r} \cos \vartheta - \frac{\sin \vartheta}{r} \frac{\partial}{\partial \vartheta}$$

In a medium without emission,
 $I_\nu(s) = I_\nu(0) e^{-\kappa_\nu \varrho (s-s_0)}$



The “**mean free path**” of the photons is the length l over which the intensity of the radiation decreases by a factor of e .

$$l = \frac{1}{\kappa \varrho}$$

In stellar interiors :

➤ the average T gradient is very small e.g. for the sun

$$\left| \frac{dT}{dr} \right| \approx \frac{T_c}{R_\odot} \approx \frac{1.6 \times 10^7 \text{ K}}{7 \times 10^{10} \text{ cm}} \sim 2 \times 10^{-4} \text{ Kcm}^{-1}$$

➤ The average opacity coefficient is

$\kappa \sim 1 - 10^2 \text{ cm}^2 \text{g}^{-1}$ and the mean density is $\sim 1 \text{ gcm}^{-3}$

➔ mean free path for photons

$$\ell = 1/(\kappa \rho) \sim 0.01 - 1 \text{ cm}$$

(larger in outer layers)

Relative change of T over a mean free path very low:

$$\frac{\Delta T}{T} = \frac{\ell \left| \frac{dT}{dr} \right|}{T} \leq \frac{10^{-4}}{10^7} \sim 10^{-11}$$

✓

LTE assumption justified

✓

$I_\nu = B_\nu(T)$ (Planck's law)

✓

$j_\nu = \kappa_\nu B_\nu(T)$ (Kirchhoff's law)

✓ Density of radiation energy $u = aT^4$

✓ Radiation Pressure $P_{\text{rad}} = \frac{1}{3} aT^4$

Multiply $\frac{\partial I_\nu}{\partial r} \cos \vartheta - \frac{\sin \vartheta}{r} \frac{\partial I_\nu}{\partial \vartheta} = \rho(j_\nu - \kappa_\nu I_\nu)$ by $\cos \theta \, dv d\Omega$ and integrate over all ν , and all Ω

$$\int_{\Omega} \frac{\partial I}{\partial r} \cos^2 \vartheta \, d\Omega - \int_{\Omega} \frac{\partial I}{r \partial \vartheta} \cos \vartheta \sin \vartheta \, d\Omega = \int_{\Omega} \rho j \cos \vartheta \, d\Omega - \int_{\Omega} \rho \kappa I \cos \vartheta \, d\Omega$$

\downarrow $c \partial P_{\text{rad}} / \partial r$ \downarrow 0 \downarrow 0 \downarrow $-\rho \kappa F$
 Emission mechanisms isotropic

(LTE $\frac{\partial I}{\partial \vartheta} = \frac{\partial B(T)}{\partial \vartheta} = 0$)



$$\frac{dP_{\text{rad}}}{dr} + \frac{\rho \kappa F}{c} = 0$$

$$P_{\text{rad}} = \frac{1}{3} a T^4 \Rightarrow dP_{\text{rad}}/dr = (4/3) a T^3 (dT/dr)$$

$$F = L_r / (4\pi r^2)$$

$$\frac{L_r}{4\pi r^2} = - \frac{4acT^3}{3\kappa\rho} \frac{dT}{dr}$$

Rosseland mean opacity

$$\frac{1}{\kappa} = \frac{1}{dB/dT} \int_0^{\infty} \frac{1}{\kappa_{\lambda}} \frac{dB_{\lambda}}{dT} d\lambda \quad \left(\text{or} = \frac{1}{dB/dT} \int_0^{\infty} \frac{1}{\kappa_{\nu}} \frac{dB_{\nu}}{dT} d\nu \right)$$

with $\frac{dB}{dT} = \int_0^{\infty} \frac{dB_{\nu}}{dT} d\nu = \frac{ac}{\pi} T^3$

- The “Rosseland mean opacity” is used in the expression of the radiative flux in stellar interiors.
- This opacity gives a higher weight to the higher frequencies
- If the monochromatic opacity at wavelength λ behaves like, $\kappa_{\lambda} \sim \lambda^{\alpha}$
the Rosseland mean opacity behaves like $\kappa \sim T^{-\alpha}$

Energy Conservation

- volume V of gas with a surface Σ and a mass M
- dm a small element of mass in this volume
- $d\sigma$ a small element of the surface Σ
- q energy provided to the system by units of mass and time.
- ε energy produced per unit of time and mass (energy generation rate) ($\text{erg } g^{-1} s^{-1}$)

Energy conservation:

$$\int_0^M \frac{dq}{dt} dm = \int_0^M \varepsilon dm - \oint_{\Sigma} \mathbf{F} \cdot d\sigma \Rightarrow \int_0^V \frac{dq}{dt} \rho dV = \int_0^V \rho \varepsilon dV - \int_0^V \nabla \cdot \mathbf{F} dV \Rightarrow \frac{dq}{dt} = \varepsilon - \frac{1}{\rho} \nabla \cdot \mathbf{F}$$

- First law of thermodynamics

$dq = dU + PdV$ where all quantities are defined per unit mass, so $V = \frac{1}{\rho}$

and

$$\frac{dq}{dt} = \frac{dU}{dt} - \frac{P}{\rho^2} \frac{d\rho}{dt} = \varepsilon - \frac{1}{\rho} \nabla \cdot \mathbf{F}$$

Energy Conservation

$$\frac{dq}{dt} = \frac{dU}{dt} - \frac{P}{\rho^2} \frac{d\rho}{dt} = \varepsilon - \frac{1}{\rho} \nabla \cdot \mathbf{F}$$

dq/dt is the rate with which energy is provided to the system during a change of structure, typically by an expansion or a contraction

$$\varepsilon_{\text{grav}} \equiv - \frac{dq}{dt}$$

$\varepsilon_{\text{grav}}$ is the energy produced by the system, since there is a negative sign

$\varepsilon_{\text{grav}} > 0$ energy is provided by the system – contraction

$\varepsilon_{\text{grav}} < 0$ energy is absorbed by the system – expansion

$$\frac{1}{\rho} \nabla \cdot \mathbf{F} = \varepsilon + \varepsilon_{\text{grav}}$$

ε is generally the sum of various nuclear processes

$$\varepsilon = \underbrace{\varepsilon_{\text{nucl},\gamma} + \varepsilon_{\text{nucl},\nu}}_{\text{energy production}} - \underbrace{(\varepsilon_{\text{nucl},\nu} + \varepsilon_{\text{pair,photo,plasma},\nu})}_{\text{energy escape}}$$

$\varepsilon_{\text{nucl},\gamma}$: nuclear energy production rate in the form of γ -rays

$\varepsilon_{\text{nucl},\nu}$: nuclear energy production rate in the form of neutrinos

$\varepsilon_{\text{pair,photo,plasma},\nu}$: neutrinos in stars are also produced by various other processes (photo-neutrinos, pair-neutrinos, plasma-neutrinos, etc.)

- γ -rays participate in radiative transfer of energy
- neutrinos escape

For a spherical star

$$\mathbf{F} = \left(\frac{L_r}{4\pi r^2}, 0, 0 \right)$$

$$\frac{1}{\rho} \nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{1}{\rho} \frac{\partial}{\partial r} \left(r^2 \frac{L_r}{4\pi r^2} \right) = \frac{1}{4\pi r^2 \rho} \frac{\partial}{\partial r} L_r = \frac{\partial L_r}{\partial M_r}$$

$$\frac{\partial L_r}{\partial M_r} = \varepsilon + \varepsilon_{\text{grav}}$$

Basic Set of equations of stellar structure for a spherica, static, radiative star

- $\frac{dP}{dr} = -\rho \frac{GM_r}{r^2}$

- $\frac{dM_r}{dr} = 4\pi r^2 \rho$

- $\frac{L_r}{4\pi r^2} = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dr}$

- $\frac{\partial L_r}{\partial M_r} = \varepsilon + \varepsilon_{\text{grav}}$

Thermodynamic equilibrium in stars

- **mechanical equilibrium** is reached when in each point the pressure force is compensated by the sum of all other acting forces. → **hydrostatic equilibrium**.
- **mechanical and thermal equilibrium**. For an adiabatically enclosed volume ($\Delta Q=0$ with the surroundings) when the mechanical equilibrium is accompanied by a **single temperature in the volume**
- **Chemical equilibrium**: for constant density and temperature the relative concentrations of the reacting particles remain in equilibrium.
- **Thermodynamic equilibrium**. chemical and thermal equilibrium → the system does not change anymore.
- **Local thermodynamic equilibrium** a star can be divided in a large number of layers, which can each be taken fairly thin so that these enclose the characteristics of equilibrium in the sense of classical thermodynamics.

Reminder of thermodynamics relations

➤ first law of thermodynamics

→ relation between the added heat dq , the internal energy u and the specific volume $v = 1/\rho$ (each defined per unit mass)

$$dq = du + Pdv$$

➤ When a process like compression or expansion occurs quickly, it will be approximately **adiabatic** because the increase or decrease of heat occurs very slowly

$$dq = 0 \Rightarrow du = -dw$$

Compression $dw < 0 \Rightarrow du > 0 \quad (\rightarrow dT > 0)$
Expansion $dw > 0 \Rightarrow du < 0 \quad (\rightarrow dT < 0)$

➤ **Entropy of a system** → solely defined for states of thermodynamic equilibrium.

$$ds \equiv dq/T \quad (\text{defined here per unit mass})$$

we can only determine the variation of the entropy from the first law $du = Tds - Pdv$ (which holds if composition remains constant)

Specific heats

$$c_\alpha \equiv \left(\frac{\partial q}{\partial T} \right)_\alpha$$

c_α is the amount of heat a system has to absorb so that the temperature rises with one unit.

$$\begin{cases} c_P \equiv \left(\frac{dq}{dT} \right)_P = \left(\frac{\partial u}{\partial T} \right)_P + P \left(\frac{\partial v}{\partial T} \right)_P \\ c_v \equiv \left(\frac{dq}{dT} \right)_v = \left(\frac{\partial u}{\partial T} \right)_v \end{cases}$$

Here we have assumed that the chemical composition is constant. If not we need to add the variation in the mean molecular weight: $\rho = \rho(P, T, \mu)$

$$\varphi \equiv \left(\frac{\partial \ln \rho}{\partial \ln \mu} \right)_{P, T}$$

And the eos will become: $\frac{d\rho}{\rho} = \alpha \frac{dP}{P} - \delta \frac{dT}{T} + \varphi \frac{d\mu}{\mu}$

Relation between c_P and c_v

Start with general eos $\rho = \rho(P, T)$ and $u = u(P, T)$

Define $\alpha \equiv \left(\frac{\partial \ln \rho}{\partial \ln P} \right)_T = -\frac{P}{v} \left(\frac{\partial v}{\partial P} \right)_T$

$\delta \equiv -\left(\frac{\partial \ln \rho}{\partial \ln T} \right)_P = \frac{T}{v} \left(\frac{\partial v}{\partial T} \right)_P$

$$\frac{d\rho}{\rho} = \alpha \frac{dP}{P} - \delta \frac{dT}{T} \quad \text{eos}$$

Left as exercise – see Aaerts →

$$c_P - c_v = \frac{P\delta^2}{T\rho\alpha}$$

From first law

$$u = u(T, v)$$

$$dq = du + Pdv = \left(\frac{\partial u}{\partial T}\right)_v dT + \left[\left(\frac{\partial u}{\partial v}\right)_T + P\right] dv$$

} 
left as exercise – see Aerts

$$\Rightarrow dq = c_v dT - \frac{P\delta}{\rho\alpha} \left(\alpha \frac{dP}{P} - \delta \frac{dT}{T}\right) = \left(c_v + \frac{P\delta^2}{T\rho\alpha}\right) dT - \frac{\delta}{\rho} dP. \quad \Rightarrow \quad dq = c_P dT - \frac{\delta}{\rho} dP$$

For adiabatic changes $ds = \frac{dq}{T} = 0$

We define $\nabla_{\text{ad}} \equiv \left(\frac{\partial \ln T}{\partial \ln P}\right)_s = \left(\frac{P dT}{T dP}\right)_s = \frac{P\delta}{T\rho c_P}$

left as exercise – see Aerts

∇_{ad} defines the temperature variation perceived by the particles in a mass element of a system when this element suffers a pressure variation as a result of adiabatic expansion (no heat exchange with the surroundings)

→ mass elements which are heated deeply in the star, rise, because, due to their lower density, they are lighter than their surroundings.

→ They end up in the higher layers where the density is lower and they therefore expand.

→ The expansion of the mass elements causes a decrease in temperature of the gas. ∇_{ad} is the value of this temperature change.

→ The pressure as well as the temperature decrease moving outwardly. The value of the decrease in pressure is given by the equation of hydrostatic equilibrium and once this value is determined, we can compute ∇_{ad}

Stability

- Up to now we assumed strict spherical symmetry → all functions are constant over concentric spheres.
- Small fluctuations occur, e.g. the thermal motion of gas particles.
 - ✓ local disturbances negligible if they do not grow into macroscopic, non-spherical, local motions.
 - ✓ we maintain spherical symmetry in the basic equations if we consider the variables as **accurate average values** over the concentric spheres.
- BUT microscopic motions can have a large impact on the stellar structure.
 - They can “mix” stellar material,
 - They transport energy. Hot fluid elements will rise, while cool fluid elements will sink. This energy transport mechanism is called **convection**.
- Whether or not convection occurs in a certain stellar layer depends on whether small fluctuations remain small, or are able to grow → it is a question of **stability**.

Dynamical instability

- We will assume that moving fluid elements do not have a sufficient amount of time to exchange a substantial fraction of their heat with their surroundings. → **adiabatic process**
- Consider the situation where physical quantities such as temperature, density, etc. are not constant at the edge of a concentric sphere inside a star, but have small local fluctuations.
- element (e) for which a quantity A is different than in the surroundings (s). We define $DA = A_e - A_s$.
- E.g. element with $DT > 0$. Pressure will be equalized with the surroundings at the sound speed (\gg than element motion) so $DP = 0$ (we assume subsonic speeds for the “element”)
- For ideal gas $\rho \propto P/T$, so $DT > 0 \Rightarrow D\rho < 0 \rightarrow$ buoyancy \rightarrow element will move **radially** upwards
- To test the stability of a layer with respect to local temperature fluctuations, one can thus equivalently take a radial displacement $\Delta r > 0$ as the initial perturbation of the element
- Consider a fluid element that is in equilibrium with its surroundings at its original position r , but that is lifted by a perturbation to a position $r + \Delta r$

$$D\rho = \left[\left(\frac{d\rho}{dr} \right)_e - \left(\frac{d\rho}{dr} \right)_s \right] \Delta r$$

$\frac{d\rho}{dr} \Big|_e$ change of density of the element due to its rise

$\frac{d\rho}{dr} \Big|_s$ density gradient of surroundings

→ additional radial force due to buoyancy

(per unit mass) $\vec{K} = K_r \hat{r} = -\left(\frac{g D\rho}{\rho}\right) \hat{r}$

$D\rho < 0 \rightarrow K_r > 0 \rightarrow$ upward motion \rightarrow unstable (it will continue rising)

$D\rho > 0 \rightarrow K_r < 0 \rightarrow$ downward motion \rightarrow will return to initial position \rightarrow stable

➤ Condition for stability

$$\left(\frac{d\rho}{dr}\right)_e - \left(\frac{d\rho}{dr}\right)_s > 0.$$

→ Not very useful, need to express it in terms of temperature gradient, appearing in the energy transport equation of stellar structure

→ Use the general eos $\frac{d\rho}{\rho} = \alpha \frac{dP}{P} - \delta \frac{dT}{T} + \varphi \frac{d\mu}{\mu}$ and rewrite the stability criterion as:

$$\rightarrow \left(\frac{\alpha dP}{P dr}\right)_e - \left(\frac{\delta dT}{T dr}\right)_e + \left(\frac{\varphi d\mu}{\mu dr}\right)_e - \left(\frac{\alpha dP}{P dr}\right)_s + \left(\frac{\delta dT}{T dr}\right)_s - \left(\frac{\varphi d\mu}{\mu dr}\right)_s > 0$$

So, the stability criterion can be written as:

$$\left(\frac{\delta dT}{T dr}\right)_s > \left(\frac{\delta dT}{T dr}\right)_e + \left(\frac{\varphi d\mu}{\mu dr}\right)_s$$

We multiply by the pressure scale height $H_P \equiv -\frac{dr}{d \ln P} = -P \frac{dr}{dP}$

(distance over which pressure falls by factor e)

and get

$$\underbrace{\left(\frac{d \ln T}{d \ln P}\right)_s}_{\text{III}} < \underbrace{\left(\frac{d \ln T}{d \ln P}\right)_e}_{\text{III}} + \frac{\varphi}{\delta} \underbrace{\left(\frac{d \ln \mu}{d \ln P}\right)_s}_{\text{III}}$$

∇, ∇_μ : surroundings
 ∇_e : element

Let us find the **radiative temperature gradient**, which is the local logarithmic derivative of T wrspt P that would be necessary if the entire luminosity had to be transported through radiation:

$$\nabla_{\text{rad}} \equiv \left(\frac{d \ln}{d \ln} \right)_{\text{rad}}$$

$$\frac{dP_{\text{rad}}}{dr} = - \frac{\rho k F}{c} \quad , \quad P_{\text{rad}} = \frac{1}{3} a T^4 \quad , \quad F = \frac{L_r}{4\pi r^2} \quad , \quad \frac{1}{\rho} \frac{d\rho}{dr} = - \frac{GM_r}{r^2}$$

$$\Rightarrow \dots \Rightarrow \left(\frac{d \ln T}{d \ln P} \right)_{\text{rad}} = \frac{3k L_r \rho}{4\pi a c G M_r T^4} \quad \left(\text{left as exercise} \right)$$

~~~~~

$\nabla_{\text{rad}}$

## Stability criterion Ledoux

In a layer where the energy transport is uniquely done by radiation, we have  $\nabla = \nabla_{\text{rad}}$ .

Assuming the element moves adiabatically  $\nabla_e = \nabla_{\text{ad}}$

hence

$$\nabla < \nabla_e + \frac{\varphi}{\delta} \nabla_{\mu} \Rightarrow \nabla_{\text{rad}} < \nabla_{\text{ad}} + \frac{\varphi}{\delta} \nabla_{\mu}$$

*stabilizing term*

$$\nabla_{\text{ad}} \equiv \left( \frac{\partial \ln T}{\partial \ln P} \right)_s = \left( \frac{P}{T} \frac{dT}{dP} \right)_s = \frac{P\delta}{T\rho c_P}$$

## Schwarzschild criterion for dynamical stability

If chemical composition homogeneous ( $\nabla_{\mu} = 0$ ) the stability criterion becomes

$$\nabla_{\text{rad}} < \nabla_{\text{ad}}$$

- When the inequality sign is reversed the layer is dynamically unstable  
→ the energy transport via radiation would impose too large a temperature gradient
- When both sides in the equation are equal, there is marginal stability.

Convective motion only occurs in a star when the criteria of Ledoux or Schwarzschild are not fulfilled. This happens when:

- $L_r/M_r$  is large, i.e., when the energy production within a radius  $r$  is very large. This occurs in massive stars, and they therefore have a convective core.
- the opacity  $\kappa$  is large. This occurs in (the outer layers of) stars with low surface temperatures.
- $\nabla_{\text{ad}}$  is small. This occurs mostly in partial ionisation zones of hydrogen, in the outer layers of cool stars, because  $c_P$  becomes very large there (the absorbed heat is mostly used to further ionise the matter, not to heat it).

$$\left( \frac{d \ln T}{d \ln P} \right)_{\text{rad}} = \frac{3 \kappa L_r P}{4 G \pi a c G M_r T^4} >$$

$$\nabla_{\text{ad}} \equiv \left( \frac{\partial \ln T}{\partial \ln P} \right)_s = \left( \frac{P dT}{T dP} \right)_s = \frac{P \delta}{T \rho c_P}$$

## Mixing length theory - MLT

- Local treatment of convection
- Good approximation for regions near the stellar core.
- Assumes
  - Stars in hydrostatic equilibrium
  - time-independent convection
- The mixing length theory states that convection can be compared to heat transport by molecules.

The transporting particles are macroscopic “spheres/cells” instead of molecules, and their mean free path (“mixing length”) is **the distance over which they move before dissolving in their new environment.**

## Convective flux in the MLT

- At a level  $r$ , the average flux results from the motions of the cells (“spheres”) which have an average velocity  $\bar{v}$  and an average  $T$  excess  $\overline{\Delta T}$ .
- Both the hotter upward moving fluid elements and the cooler downward moving elements contribute to the outward transport of energy.
- The convective flux is  $F_{\text{conv}} = c_p \rho \bar{v} \overline{\Delta T}$
- Usually, the mixing-length,  $\ell = \alpha H_p$ , where  $H_p$  the pressure scale height and  $\alpha \sim 1$



$$\Delta V = A \Delta s$$

$$\Delta m = \rho \Delta V = \rho A \Delta s$$

$$F = \frac{\Delta Q \cdot \Delta m}{A \cdot \Delta t} = \frac{c_p \overline{\Delta T}}{A \cdot \Delta t} \cdot \rho \Delta V = \frac{c_p \overline{\Delta T}}{A \cdot \Delta t} \cdot \rho A \overline{\Delta s} = c_p \rho \bar{v} \overline{\Delta T}$$

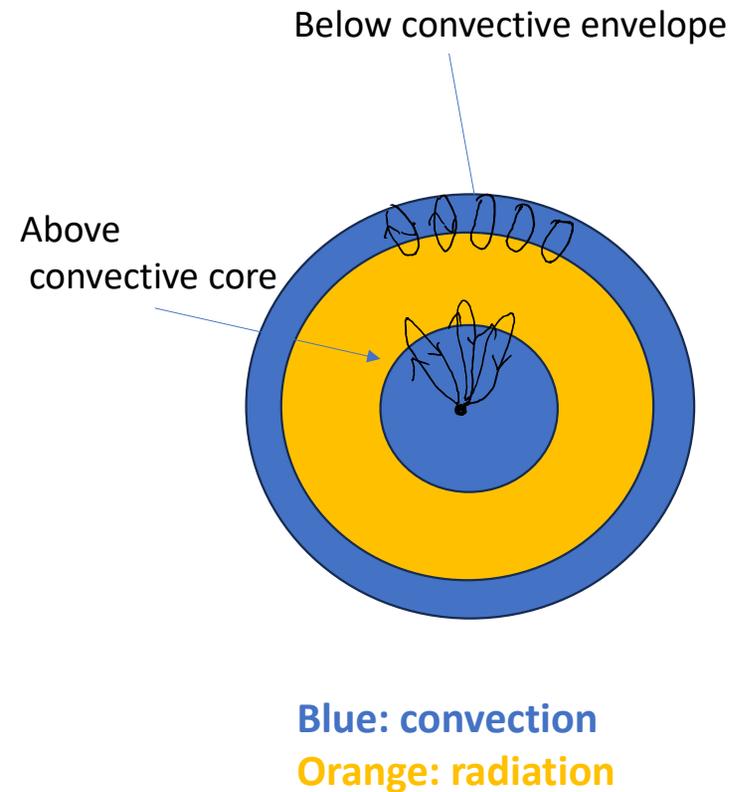
## Convective overshooting

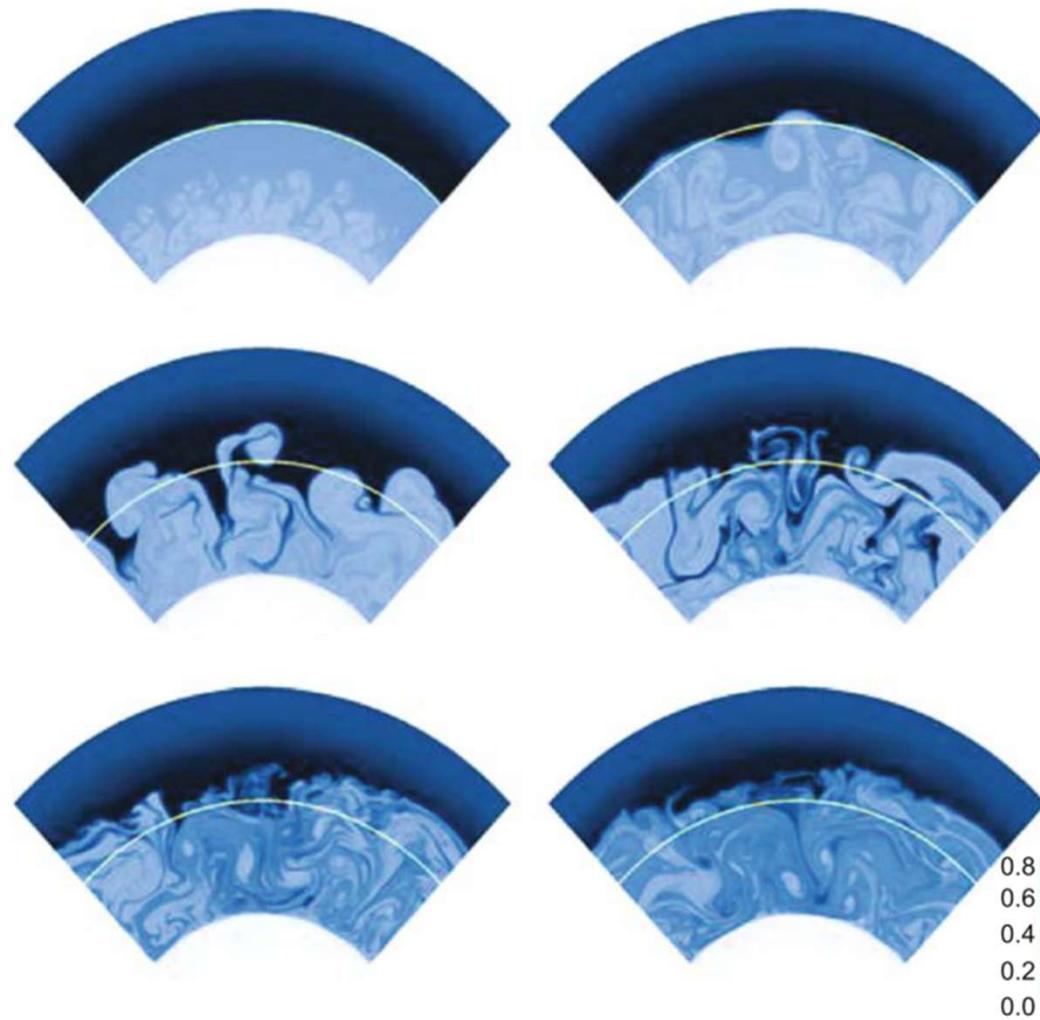
- Difficulties come near the boundaries of convective zones, with the problem of overshooting
- boundaries of convective regions from the Schwarzschild and Ledoux criteria → the **dynamical limit**  $r_{\Delta T}$  in the star, where the average temperature excess  $\Delta T$  and thus the acceleration of the convective cells are **zero**.
- But convective motions extend up to the **kinematical limit**  $r_v$ , where the **velocity** of the fluid elements is **zero**.
- **The difference** between the kinematical and the dynamical edges,  $r_v$  and  $r_{\Delta T}$  is the distance of overshooting  $d_{\text{over}} = |r_v - r_{\Delta T}|$
- **Overshooting is a non-local process**, namely the extent of overshooting critically depends on the properties of the adjacent convective layers → cannot be treated with a local theory

Study : chapter 6, section 6.1 from the book of Maeder

## Some comments

- $d_{\text{over}}$  is the average distance up to which convective mixing extends beyond the formal limit, defined by the Schwarzschild or Ledoux criteria.
- Overshooting may occur **above convective cores** or **below convective envelopes**.
- The overshooting from convective core determines the amount of nuclear fuel available for the star.
- The overshooting below the solar envelope is an “observable” parameter in helioseismology.





Notice that the convection cell motion is far from radial in these simulations

**Fig. 6.3** 2D numerical simulations of convection in the advanced phase of central O burning. Here, the mixed region (*cloudy*) overshoot by about 30% of the size of the convective region defined by Schwarzschild's criterion (*white curve*). Courtesy from Patrick Young [631]

# The Open Cluster NGC 7789. II. CCD VI Photometry

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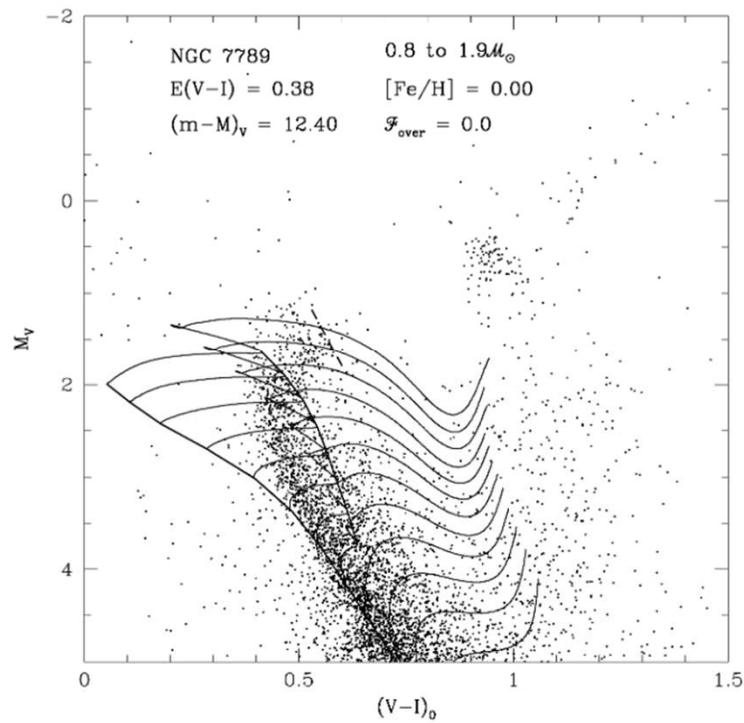


FIG. 10a

