Stellar structure and evolution of rotating stars

Reading material:

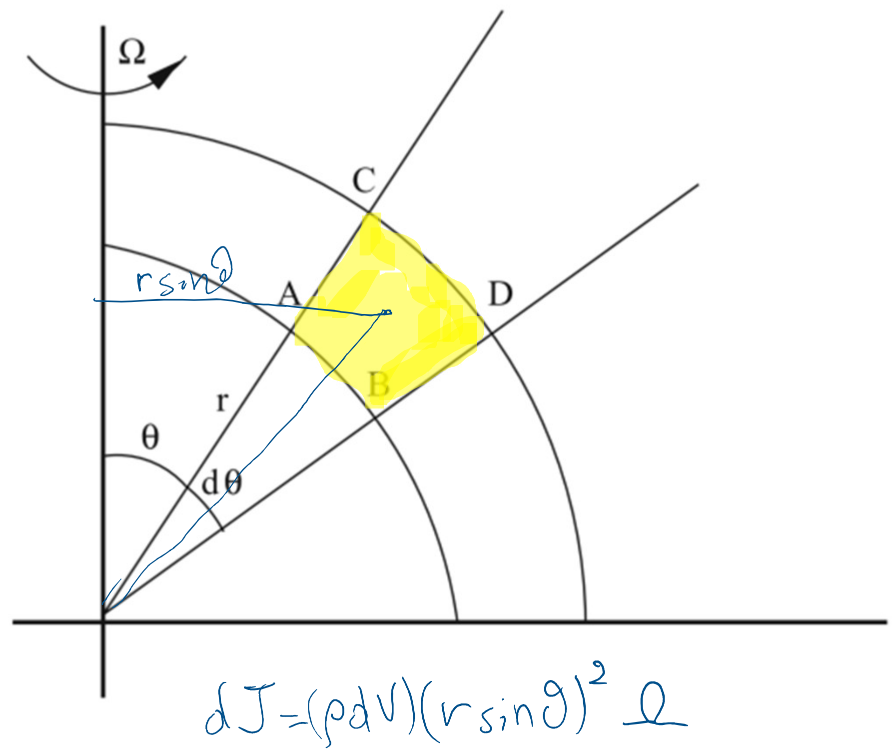
Chapters 2 & 4

Parts of chapters 6 & 10

from A. Maeder's book

ΜΕΡΟΣ Γ
Transport of Angular Momentum in Stars (Maeder ch. 10.5)

An element of volume limited by ABCD rotates around the vertical axis with an angular velocity $\Omega = \dot{\varphi}$



The mass of the volume element is

$\varrho r^2 \sin\vartheta d\vartheta d\varphi dr$

The component of the angular momentum $\boldsymbol{J}$ aligned with the rotational axis is

$dJ = \varrho r^4 \sin^3\vartheta d\vartheta d\varphi dr \Omega$

In Lagrangian coordinates, the derivative of the angular momentum of the moving fluid mass element is

$\varrho r^2 \sin\vartheta d\vartheta d\varphi dr \frac{d}{dt}\left(r^2 \sin^2\theta \Omega\right)_{M_r}$

$$\varrho \frac{d}{dt}\left(r^2\sin^2\vartheta\Omega\right)_{M_r} = \frac{d}{dt}\left(\varrho r^2\sin^2\vartheta\Omega\right)_{M_r} - r^2\sin^2\vartheta\Omega \left.\frac{d\varrho}{dt}\right|_{M_r}$$

Using relation between the Lagrangian and Eulerian derivatives:

$$\varrho \frac{d}{dt}\left(r^2\sin^2\vartheta\Omega\right)_{M_r} = \frac{\partial}{\partial t}\left(\varrho r^2\sin^2\vartheta\Omega\right)_r + \mathbf{U}\cdot\boldsymbol{\nabla}\left(\varrho r^2\sin^2\vartheta\Omega\right) - r^2\sin^2\vartheta\Omega \left.\frac{d\varrho}{dt}\right|_{M_r}$$

Also $\left.\frac{d\varrho}{dt}\right|_{M_r} = \left.\frac{\partial\varrho}{\partial t}\right|_r + \mathbf{U}\cdot\boldsymbol{\nabla}\varrho$

exercise

Continuity equation: $(\partial\varrho/\partial t)|_r = -\boldsymbol{\nabla}.(\varrho\mathbf{U})$

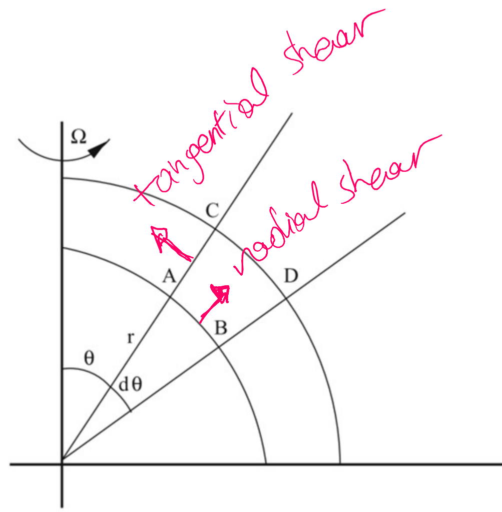
→ $\varrho \frac{d}{dt}\left(r^2\sin^2\vartheta\Omega\right)_{M_r} = \frac{\partial}{\partial t}\left(\varrho r^2\sin^2\vartheta\Omega\right)_r + \boldsymbol{\nabla}\cdot\left(\mathbf{U}\varrho r^2\sin^2\vartheta\Omega\right)$

→ $\frac{d\boldsymbol{J}}{dt} = \mathcal{M} = \{\frac{\partial}{\partial t}\left(\varrho r^2\sin^2\vartheta\Omega\right)_r + \boldsymbol{\nabla}\cdot\left(\mathbf{U}\varrho r^2\sin^2\vartheta\Omega\right)\}r^2\sin\vartheta d\vartheta d\varphi dr$

Transport of Angular Momentum by Shears

- ✓ Slab floating with velocity $\boldsymbol{v}$ (on the plane of the slab) on a viscous medium of density ρ and thickness d feels a viscous force parallel to the motion of the slab and in the opposite direction
- ✓ Newtonian law of viscosity

→ tension $\tau \equiv \frac{F}{S} \equiv \eta \frac{dv}{dz} = \frac{\eta v}{d}$, where η is the dynamic coefficient of viscosity

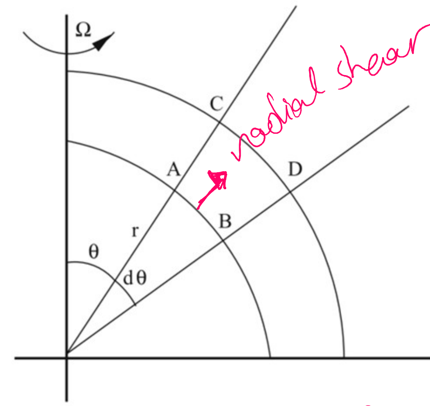


We will consider

- ➢ vertical shear
- ➢ Tangential shear

Vertical shear

✓ Applying this to a radial motion of a "slab" rotating around the axis, the tension will be:



$v_\varphi = \Omega r \sin\vartheta$

$$\eta_{\mathrm{v}} \frac{dv_\varphi}{dr} = \eta_{\mathrm{v}} r \sin\vartheta \frac{d\Omega}{dr}$$

→The viscous force applied on surface AB:

$$F_{\mathrm{AB}} = \eta_{\mathrm{v}} r^3 \sin^2\vartheta \frac{d\Omega}{dr} d\vartheta d\varphi$$

$(dS = r^2 \sin\vartheta \, d\vartheta \, d\varphi)$

Torque:

$\boldsymbol{\mathcal{M}} = \boldsymbol{r} \times \boldsymbol{F}_{\mathbf{AB}}$ → $\mathrm{d}\mathcal{M}_z = \eta_{\mathrm{v}} r^4 \sin^3\vartheta \frac{d\Omega}{dr} d\vartheta d\varphi$ **elementary viscous torque about the rotation axis exerted on the spherical surface element AB**

For the volume element with section ABCD the gain of torque over dr will be

$$\frac{\partial}{\partial r}\left(\eta_{\mathrm{v}} r^4 \sin^3\vartheta \, d\vartheta d\varphi \frac{\partial\Omega}{\partial r}\right) dr$$

Tangential shear

When $d\Omega/d\vartheta \neq 0$

The tension (force by unit surface) due to the tangential shear with a horizontal viscosity η_{h} is equal to $\eta_{\mathrm{h}} r \sin\vartheta\,[\partial\Omega/(r\partial\vartheta)]$

The corresponding force on the surface described by AC will be

$\eta_{\mathrm{h}} r \sin^2\vartheta\,(\partial\Omega/\partial\vartheta) dr d\varphi$ $dS' = (r\sin\vartheta)d\varphi\,dr$

The corresponding torque

$\eta_{\mathrm{h}} r^2 \sin^3\vartheta\,(\partial\Omega/\partial\vartheta) dr d\varphi$

The gain on the torque when passing from AC to BD is

$$\frac{\partial}{r\partial\vartheta}\left(\eta_{\mathrm{h}} r^2 \sin^3\vartheta\, dr d\varphi \frac{\partial\Omega}{\partial\vartheta}\right) r d\vartheta$$

$$\frac{d\boldsymbol{J}}{dt} = \mathcal{M}$$

$$\varrho r^2 \sin\vartheta \frac{d}{dt}\left(r^2 \sin^2\vartheta\,\Omega\right)_{M_r}$$
$$= \frac{\partial}{\partial r}\left(\eta_{\mathrm{v}} r^4 \sin^3\vartheta \frac{\partial\Omega}{\partial r}\right) + \frac{\partial}{\partial\vartheta}\left(\eta_{\mathrm{h}} r^2 \sin^3\vartheta \frac{\partial\Omega}{\partial\vartheta}\right)$$

(we simplified by $dr d\vartheta d\varphi$)

$$\Rightarrow \varrho \frac{d}{dt}\left(r^2\sin^2\vartheta\Omega\right)_{M_r} = \frac{\partial}{\partial t}\left(\varrho r^2\sin^2\vartheta\Omega\right)_r + \nabla\cdot\left[\boldsymbol{U}\varrho r^2\sin^2\vartheta\Omega\right]$$
$$= \frac{\partial}{\partial t}\left(\varrho r^2\sin^2\vartheta\Omega\right)_r + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\varrho r^4\sin^2\vartheta U_r\Omega\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\varrho r^2\sin^3\vartheta U_\vartheta\Omega\right)$$
$$= \frac{\sin^2\vartheta}{r^2}\frac{\partial}{\partial r}\left(\varrho D_{\mathrm{v}} r^4 \frac{\partial\Omega}{\partial r}\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\varrho D_{\mathrm{h}}\sin^3\vartheta \frac{\partial\Omega}{\partial\vartheta}\right).$$

We have replaced the dynamic viscosity coefficients with the kinematic coefficients

$\nu_{\mathrm{v}} = \frac{\eta_{\mathrm{v}}}{\varrho}$ *and* $\nu_{\mathrm{h}} = \eta_{\mathrm{h}}/\varrho$, and used the fact that the kinematic viscosity coefficient is also the diffusion coefficients D_{v} *and* D_{h} for the chemical elements.

During contraction or expansion (radial) we replace $U_r \rightarrow U_r + \dot{r}$

For shellular rotation, internal rotation depends essentially on the distance to the stellar center and little on latitude the previous eq. is simplified

$$\Omega(r, \vartheta) = \overline{\Omega}(r) + \widehat{\Omega}(r, \vartheta)\ ,\ \ \widehat{\Omega} \ll \overline{\Omega}$$

with $\overline{\Omega}(r)$ being the angular velocity of a shell rotating like a solid body and having the same angular momentum as the considered actual shell.

$$\overline{\Omega}(r) = \frac{\int_0^{\pi} \Omega(r, \vartheta) \sin^3 \vartheta d\vartheta}{\int_0^{\pi} \sin^3 \vartheta d\vartheta}$$ (we will see why the mean is defined in this manner later)

In general, a vector field on the sphere can be represented by spherical harmonics. If the field is axially symmetric, it can be expressed in spherical functions.

$$\boldsymbol{U} = \underbrace{\sum_{l>0} U_l(r) P_l(\cos\vartheta)}_{U_r} \boldsymbol{e}_r + \underbrace{\sum_{l>0} V_l(r) \frac{dP_l(\cos\quad)}{d\vartheta}}_{U_\vartheta} \boldsymbol{e}_\vartheta$$

Limiting the development of the Legendre polynomials to second order:

$$\boldsymbol{U} = U_2(r) P_2(\cos\vartheta) \boldsymbol{e}_r + V_2(r) \frac{dP_2(\cos\vartheta)}{d\vartheta} \boldsymbol{e}_\vartheta$$

Exercise: show you why mathematically only the $l = 2$ term gives a non-zero angular momentum flux

Some comments

- $l = 0$ component not allowed because if $U_r \propto P_0 = \text{constant}$

  → the radial velocity would be the same at all latitudes

  → net mass flow through the spherical shell

  But meridional circulation must satisfy mass conservation on each shell $\int U_r \, d\sigma = 0$

- $l = 1,3,5, \ldots$ odd terms are antisymmetric about the equator, while the star is assumed to be equatorially symmetric (no preferred hemisphere) - so these components must also vanish

Continuity equation:

$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0$ (density fluctuations are small, flows are slow)

Inelastic approximation $\frac{\partial \rho}{\partial t} \cong 0 \rightarrow \nabla \cdot (\rho \mathbf{U}) = 0$

$$\nabla \cdot (\rho \mathbf{U}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho U_r\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}(\rho U_\vartheta \sin\vartheta)$$

$$U_r = U_2(r) P_2(\cos\vartheta)$$
$$U_\vartheta = V_2(r)\frac{dP_2(\cos\vartheta)}{d\vartheta}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho U_r\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\rho \sin\vartheta\, V_2 \frac{dP_2}{d\vartheta}\right) = 0\,,$$

where $P_2(\cos\vartheta) = \frac{1}{2}\left(3\cos^2\vartheta - 1\right) \Rightarrow \frac{dP_2(\cos\vartheta)}{d\vartheta} = -3\cos\vartheta\sin\vartheta$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \varrho U_r\right) = -\frac{\varrho V_2}{r \sin\vartheta}\left(3\sin^3\vartheta - 6\sin\vartheta\cos^2\vartheta\right)$$
$$= -\frac{\varrho V_2}{r}\left(3 - 9\cos^2\vartheta\right) = 6\frac{\varrho V_2}{r} P_2(\cos\vartheta)$$

and finally we get

$$\frac{1}{r}\frac{d}{dr}[\varrho r^2 U_2(r)] - 6\varrho V_2(r) = 0$$ exercise

→This expression provides $V_2(r)$ – if $U_2(r\ ) is\ known$ – and hence the horizontal component of the meridional velocity

→ $U_\vartheta = V_2(r)\frac{dP_2(\cos\vartheta)}{d\vartheta}$

In the course of stellar evolution, we account for circulation velocities as well as for expansion or contraction:

$\rightarrow U_r = U_2(r)P_2(\cos\vartheta) + \dot{r}$

The patterns of meridional circulation during stellar evolution are obtained from the components U_r and U_θ

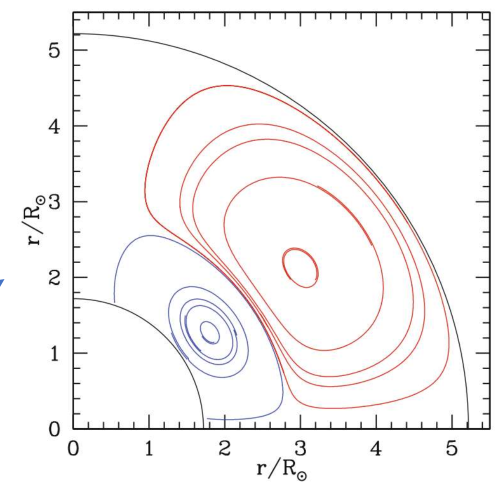


Fig. 11.2 Circulation currents in a 20 $M_\odot$ star in the middle of the H-burning phase. The initial rotation velocity is 300 km/s. The inner loop is raising along the polar axis, while the outer loop, the Gratton–Öpik circulation cell, is going up in the equatorial plane. Courtesy by G. Meynet

Transport in Shellular Rotation

We will now derive the equation of transport of the angular momentum in the vertical direction for the case of shellular rotation

✓We start with eq.

$$\varrho \frac{d}{dt}\left(r^2 \sin^2 \vartheta \Omega\right)_{M_r} = \frac{\partial}{\partial t}\left(\varrho r^2 \sin^2 \vartheta \Omega\right)_r + \nabla \cdot [\boldsymbol{U} \varrho r^2 \sin^2 \vartheta \Omega]$$

$$= \frac{\partial}{\partial t}\left(\varrho r^2 \sin^2 \vartheta \Omega\right)_r + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\varrho r^4 \sin^2 \vartheta U_r \Omega\right) + \frac{1}{r \sin \vartheta}\frac{\partial}{\partial \vartheta}\left(\varrho r^2 \sin^3 \vartheta U_\vartheta \Omega\right)$$

$$= \frac{\sin^2 \vartheta}{r^2}\frac{\partial}{\partial r}\left(\varrho D_{\mathrm{v}} r^4 \frac{\partial \Omega}{\partial r}\right) + \frac{1}{\sin \vartheta}\frac{\partial}{\partial \vartheta}\left(\varrho D_{\mathrm{h}} \sin^3 \vartheta \frac{\partial \Omega}{\partial \vartheta}\right).$$

✓Multiply by $\sin \vartheta \, d\vartheta$

✓Integrate from 0 to π

First term

$$\int_0^{\pi} \frac{\partial}{\partial t}\left(\varrho r^2 \sin^2\vartheta\Omega\right)_r \sin\vartheta d\vartheta$$

$$= \frac{\partial}{\partial t}\varrho r^2 \int_0^{\pi} \Omega(r,\vartheta)\sin^3\vartheta d\vartheta = \frac{\partial}{\partial t}\left(\varrho r^2\overline{\Omega}\right)_r \int_0^{\pi} \sin^3\vartheta d\vartheta$$

$$\overline{\Omega}(r) = \frac{\int_0^{\pi} \Omega(r,\vartheta)\sin^3\vartheta\, d\vartheta}{\int_0^{\pi} \sin^3\vartheta\, d\vartheta}$$

(horizontal averaging)

Second term

$$\int_0^{\pi} \frac{1}{r^2}\frac{\partial}{\partial r}\left(\varrho r^4 \sin^2\vartheta(U_r + \dot{r})\Omega(r,\vartheta)\right)\sin\vartheta d\vartheta$$

- For the term with $U_r = U_2(r)P_2(\cos\vartheta)$, and with $\Omega(r,\vartheta)\sim\overline{\Omega}(r)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\varrho r^4 U_2(r)\overline{\Omega}(r)\right)\int_0^{\pi} \sin^3\vartheta \frac{1}{2}\left(3\cos^2\vartheta - 1\right)d\vartheta$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\varrho r^4 U_2(r)\overline{\Omega}(r)\right)\left[\int_0^{\pi} \sin^3\vartheta d\vartheta - \frac{3}{2}\int_0^{\pi} \sin^5\vartheta d\vartheta\right]$$

$$= -\frac{1}{5}\frac{1}{r^2}\frac{\partial}{\partial r}\left(\varrho r^4 U_2(r)\overline{\Omega}(r)\right)\int_0^{\pi} \sin^3\vartheta d\vartheta$$

where we used $\int_0^{\pi} \sin^5\vartheta d\vartheta = (4/5)\int_0^{\pi} \sin^3\vartheta d\vartheta$

- For the term with $\dot{r}$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\varrho r^4\overline{\Omega}(r)\right)\dot{r}\int_0^{\pi} \sin^3\vartheta d\vartheta$$

Third term

$$\int_0^\pi \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\varrho r^2\sin^3\vartheta U_\vartheta \Omega(r,\vartheta)\right)\sin\vartheta d\vartheta$$

$$= \varrho r\overline{\Omega}(r)\int_0^\pi \frac{\partial}{\partial\vartheta}\left(U_\vartheta \sin^3\vartheta\right)d\vartheta$$

$$= -3\varrho r\overline{\Omega}(r)V_2(r)\int_0^\pi \frac{\partial}{\partial\vartheta}(\sin^4\vartheta\cos\vartheta)d\vartheta = 0$$

$$U_\vartheta = V_2(r)\frac{dP_2(\cos\vartheta)}{d\vartheta}$$

Fourth term

$$\int_0^\pi \frac{\sin^2\vartheta}{r^2}\frac{\partial}{\partial r}\left(\varrho D_\mathrm{h} r^4 \frac{\partial\Omega(r,\vartheta)}{\partial r}\right)\sin\vartheta\, d\vartheta$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\varrho D_\mathrm{h} r^4 \frac{\partial\overline{\Omega}}{\partial r}\right)\int_0^\pi \sin^3\vartheta\, d\vartheta$$

where one assumes that the coefficient D_h does not depend on ϑ.

Fifth term

$$\frac{\partial\Omega(r,\vartheta)}{\partial\vartheta} = \frac{\partial\overline{\Omega}(r)}{\partial\vartheta} = 0 \quad \text{(for shellular rotation)}$$

Putting together all terms and simplifying by $\int_0^\pi \sin^3\vartheta\, d\vartheta$ we get

$$\frac{\partial}{\partial t}\left(\varrho r^2\overline{\Omega}\right)_r = \frac{1}{5r^2}\frac{\partial}{\partial r}\left(\varrho r^4\overline{\Omega}[U_2(r) - 5\dot{r}]\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\varrho D_\mathrm{v} r^4 \frac{\partial\overline{\Omega}}{\partial r}\right)$$

Comments

Eulerian angular-momentum transport equation averaged on isobars.

$$\frac{\partial}{\partial t}\left(\varrho r^2\overline{\Omega}\right)_r = \frac{1}{5r^2}\frac{\partial}{\partial r}\left(\varrho r^4\overline{\Omega}[U_2(r) - 5\dot{r}]\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\varrho D_{\rm v} r^4\frac{\partial\overline{\Omega}}{\partial r}\right)$$

- The term involving radial expansion/contraction $\dot{r}$ is important mainly:
  - during pre-main-sequence phases
  - and inter-nuclear phases.

  → During rapid structural evolution (on t_{KH} ,(the $\dot{r}$ term dominates angular-momentum transport

- Where $U_2(r) > 0$ (typically inner radiative envelopes of massive stars):
  - Circulation rises along the polar axis. (*)
  - Expansion ($\dot{r} > 0$) opposes the circulation effect.
  - Meridional circulation transports angular momentum inward
- Where $U_2(r) < 0$ (often outer layers):
  - Circulation rises at the equator and sinks at the poles.
  - Expansion ($\dot{r} > 0$) reinforces the circulation effect.
  - Meridional circulation transports angular momentum outward

(*) $U_r(r,\vartheta) = U_2(r)P_2(\cos\vartheta)$, $P_2(\cos\vartheta) = \frac{1}{2}\left(3\cos^2\vartheta - 1\right)$

At the poles $\vartheta = 0 \Rightarrow P_2 = 1$ → $U_r(r,\vartheta) = U_2(r) > 0$ rising flow

At the equator $\vartheta = \frac{\pi}{2} \Rightarrow P_2 = -1/2$ → $U_r(r,\vartheta) = -\frac{1}{2}U_2(r) < 0$ sinking flow

Vertical shear diffusion term

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho D_v r^4 \frac{\partial \bar{\Omega}}{\partial r}\right)$$

Represents **diffusive transport of angular momentum due to vertical shear turbulence**.

Acts to **reduce radial gradients of angular velocity** (tends toward solid-body rotation).

If $\partial \bar{\Omega}/\partial r < 0$)usual case: surface rotates slower),
$\rightarrow$ diffusion transports **angular momentum outward**.

If $\partial \bar{\Omega}/\partial r > 0$,
$\rightarrow$diffusion transports **angular momentum inward**.

Unlike meridional circulation (advective, large-scale), this is a **local turbulent process**.

Boundary conditions

The transport equation for Ω is 4th order in radius → 4 boundary conditions are required.

They are applied at the bottom r_b and top r_t of the radiative zone (e.g. convective core edge and stellar surface in massive stars).

1. **No viscous torque at the boundaries**

- If no viscous angular-momentum flux crosses the boundaries:
- $\frac{\partial \Omega}{\partial r} = 0 \; at \; r = r_b, \; r = r_t$
- Meaning:
  - rotation profile has zero radial shear at the interfaces
  - ensures continuity of the first derivative of Ω.
- Also applies:
  - at the stellar center (solar-type stars with inner radiative zone).

2. Global angular-momentum conservation at boundaries

Derived by applying the **transport equation at the interfaces** and integrating interior/exterior zones.

✔ **At the bottom** r_b

$$\frac{1}{5}\rho r^4 \overline{\Omega} U_2 = \frac{d}{dt}\left[\overline{\Omega} \int_0^{r_b} r^4 \rho dr\right]$$

→Meridional circulation flux matches **time change of angular momentum of the inner region**.

✓**At the top** r_t

$$-\frac{1}{5}\rho r^4 \overline{\Omega} U_2 = \frac{d}{dt}\left[\overline{\Omega} \int_{r_t}^{R} r^4 \rho dr\right] + \mathcal{M}_\Omega$$

→ **external torque** (e.g. magnetic braking, tidal torque).