Stellar structure and evolution of rotating stars

Reading material:

Chapters 2 & 4

Parts of chapters 6 & 10

from A. Maeder's book

ΜΕΡΟΣ Β’

Radiative Transfer in Rotating Stars

INTRODUCTION

Rotation modifies how radiation is transported inside stars.

Consequences of rotation

- Radiative flux is not constant on equipotential surfaces
- Effective gravity varies with latitude
- Surface temperature T_{eff} becomes latitude dependent

Observational implications

- Different T_{eff} at different latitudes
- Composite stellar spectrum
- Rotational Doppler broadening

Gravity darkening

In rotating stars, as we saw:

$g_{\mathrm{eff}} = g_{\mathrm{grav}} + g_{\mathrm{cent}}$

depends on (co-)latitude.

Therefore

$g_{\mathrm{eff}}(\vartheta)$

$T_{\mathrm{eff}}(\vartheta)$

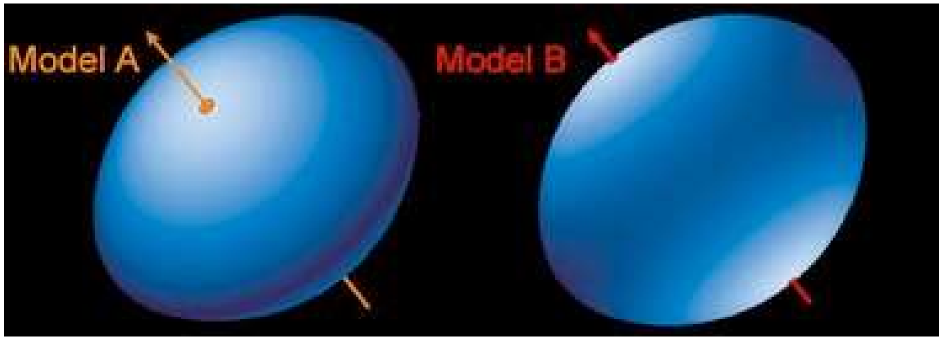
vary across the stellar surface.

Result

→poles hotter

→equator cooler

This leads to **gravity darkening**.



Model view of Achernar, based on the profile measured with the VLTI. Two different models are shown: in "A", the polar axis is inclined 50° to the line-of-sight; in "B", this angle is 90°.
ESO - https://www.eso.org/public/images/eso0316c/

Interior Consequences

Inside the star:

radiative equilibrium **breaks down on equipotential surfaces**

→ Drive of **large-scale flows**

→ **Meridional circulation**

These currents

✓transport angular momentum

✓mix chemical elements

✓affect stellar evolution.

Radiative Flux in Rotating Stars – breakdown of radiative equilibrium - the example of solid body rotation

$$\mathbf{F} = -\frac{4acT^3}{3\kappa\rho}\nabla T$$

For solid-body rotation:

$$T = T(\Psi)$$

where

$\Psi =$ *gravitational + centrifugal potential*

thus

$$\nabla T = \frac{dT}{d\Psi}\nabla\Psi$$

and the radiative flux at a level r will be

$$\mathbf{F} = -\frac{4acT^3}{3\kappa\rho}\nabla T = -\left(\frac{4acT^3}{3\kappa\rho}\frac{dT}{d\Psi}\right)\nabla\Psi$$

→ $\mathbf{F} \parallel \nabla\Psi$ cf $\left(\frac{4acT^3}{3\kappa\rho}\frac{dT}{d\Psi}\right)$ is constant on an equipotential

→Flux magnitude varies with position on equipotentials

→energy transport is **not uniform on level surfaces**

[Reminder] For hydrostatic equilibrium in a solid-body rotating star

$$-\nabla V = \Omega^2 \boldsymbol{\varpi} \rightarrow$$
$$V = -\frac{1}{2}\Omega^2\varpi^2$$
$$\Psi = \Phi + V$$
$$\nabla^2\Psi = \nabla^2\Phi + \nabla^2 V \qquad \nabla^2\Phi = 4\pi G\varrho$$
$$\left(\nabla^2 V\right)_\varpi = \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(-\varpi^2\Omega^2\right) = -2\Omega^2$$
$$\nabla^2\Psi = 4\pi G\varrho - 2\Omega^2$$

$$\nabla \cdot \boldsymbol{F} = \frac{d}{d\Psi}\left(-\frac{4acT^3}{3\kappa\varrho}\frac{dT}{d\Psi}\right)(\nabla\Psi)^2 + \left(-\frac{4acT^3}{3\kappa\varrho}\frac{dT}{d\Psi}\right)\nabla^2\Psi$$

$\nabla^2\Psi = 4\pi G\varrho - 2\Omega^2$ constant on an equipotential

So, the only non-constant term in the above expression for $\nabla \cdot \boldsymbol{F}$ is $\nabla\Psi$

(the equipotentials are differently spaced as a function of the colatitude ϑ.)

$\rightarrow$Local breakdown of radiative equilibrium on an equipotential

if $\nabla \cdot \boldsymbol{F} > 0$ $\rightarrow$ excess of energy in the medium $\rightarrow$ heating $\rightarrow$ locally upward motion

if $\nabla \cdot \boldsymbol{F} < 0$ $\rightarrow$ the opposite

$\rightarrow$If no local sources/sinks: $<\nabla \cdot \mathbf{F}> = 0$ But because F varies along equipotentials radiative equilibrium cannot be satisfied locally.

- radiative imbalance
- large-scale circulation

The Von Zeipel Theorem

For a solid-body rotating star, the flux is

$$F(\Omega, \vartheta) = -\chi \nabla T(\Omega, \vartheta) \qquad where \quad \chi = \frac{4acT^3}{3\kappa\varrho}$$

The equipotentials and isobars coincide so

$$\boldsymbol{F}(\Omega, \vartheta) = -\chi \frac{dT}{dP} \nabla P(\Omega, \vartheta) = -\varrho\chi \frac{dT}{dP} \boldsymbol{g}_{\mathrm{eff}}(\Omega, \vartheta)$$

Constant on an equipotential

(where we used the eq. of hydrostatic equilibrium for a barotropic star

$\frac{1}{\varrho} \nabla P = -\nabla\Psi = \boldsymbol{g}_{\mathrm{eff}}$)

→ The flux is proportional to the effective gravity on the equipotential

Calculation of the constant of proportionality between flux and effective gravity on the equipotential

The total luminosity on an equipotential of area Σ is:

$$L(\Omega) = \int_\Sigma \boldsymbol{F}(\Omega, \vartheta) \cdot \boldsymbol{d\sigma} = +\left(\varrho\chi\frac{dT}{dP}\right)\int_\Sigma \nabla\Psi(\Omega, \vartheta) \cdot \boldsymbol{d\sigma}$$

Using Gauss theorem

$$L(\Omega) = \left(\varrho\chi\frac{dT}{dP}\right)\int_V \nabla^2\Psi dV$$

but $\nabla^2\Psi = 4\pi G\varrho - 2\Omega^2$

$\bar{\varrho}_m$ the average density within the level surface considered

$$L(\Omega) = \left(\varrho\chi\frac{dT}{dP}\right)\int_V \left(4\pi G\varrho - 2\Omega^2\right)dV =$$

$$\left(\varrho\chi\frac{dT}{dP}\right)\left(4\pi GM_r - 2\Omega^2 V\right) =$$

$$\left(\varrho\chi\frac{dT}{dP}\right)(4\,\pi GM_r)\left(1 - \frac{\Omega^2 V}{2\pi GM_r}\right) \Rightarrow$$

$$\left(\varrho\chi\frac{dT}{dP}\right) = \frac{L(\Omega)}{4\pi GM_r\left(1 - \frac{\Omega^2}{2\pi G\bar{\varrho}_m}\right)}$$

$$\rightarrow \boldsymbol{F}(\Omega,\vartheta) = -\varrho\chi\frac{dT}{dP}\boldsymbol{g}_{\text{eff}}(\Omega,\vartheta) = -\frac{L(\Omega)}{4\pi G M_r\left(1-\frac{\Omega^2}{2\pi G\bar{\varrho}_m}\right)}\boldsymbol{g}_{\text{eff}}(\Omega,\vartheta)$$

This equation applies to any equipotential

At the surface of the star:

$$\Rightarrow \boldsymbol{F}(\Omega,\vartheta) = -\frac{L}{4\pi G M^*}\boldsymbol{g}_{\text{eff}}(\Omega,\vartheta)$$

$$where\ M^* = M\left(1-\frac{\Omega^2}{2\pi G\bar{\varrho}_M}\right)$$

→ von Zeipel theorem

the radiative flux at the surface of a rotating star is proportional to the local effective gravity at the considered colatitude

$$\rightarrow T_{\text{eff}}(\Omega,\vartheta) = \left(\frac{L}{4\pi\sigma G M^*}\right)^{\frac{1}{4}}[g_{\text{eff}}(\Omega,\vartheta)]^{\frac{1}{4}}$$

equatorial regions are fainter and cooler
polar ones, regions are brighter and hotter
→ gravity-darkening.

→Small differences found for shellular rotation – small increase of contrast between equator and poles

Interactions of Rotation and Radiation Effects

- The total luminosity L(Ω) of a rotating star changes with rotation as a result of structural changes due to the modifications of the eqs of stellar structure.

$$\frac{dL_P}{dM_P} = \varepsilon_{\mathrm{nucl}} - \varepsilon_v + \varepsilon_{\mathrm{grav}}, \checkmark$$

$$\frac{d\ln T}{dM_P} = -\frac{GM_P}{4\pi r_P^4} f_P \min\left[\nabla_{\mathrm{ad}}, \nabla_{\mathrm{rad}}\frac{f_T}{f_P}\right],$$

$$\text{where } f_P = \frac{4\pi r_P^4}{GM_P S_P}\frac{1}{\langle g_{\mathrm{eff}}^{-1}\rangle}$$

$$\text{and } f_T = \left(\frac{4\pi r_P^2}{S_P}\right)^2 \frac{1}{\langle g_{\mathrm{eff}}\rangle\langle g_{\mathrm{eff}}^{-1}\rangle}.$$

- Generally, the luminosity decreases with rotation

  the expansion produced by rotation causes the star to be a bit cooler and the opacity higher.

$$L(\omega) = L(0)\left(1 - b\omega^2\right)$$

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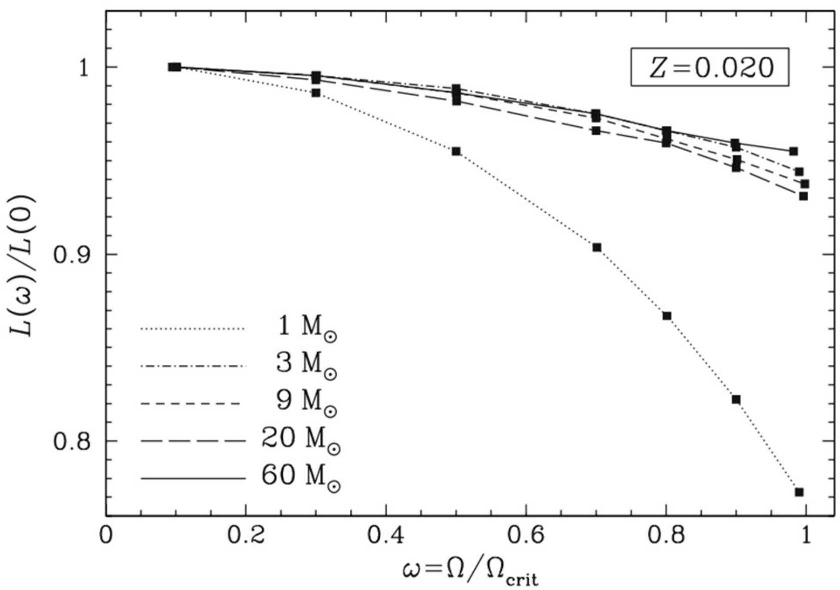


Fig. 4.2 Relative variations of the total stellar luminosity for models of various masses on the ZAMS at $Z = 0.02$ as a function of the rotation parameter $\omega = \Omega/\Omega_{\mathrm{crit}}$. From S. Ekström et al.

$b = 0.23, 0.07, 0.065, 0.06, 0.05$

For masses

1, 3, 9, 20 and 60 $\mathrm{M}_{\odot}$

Reminder of the Eddington Luminosity for non-rotating stars

Let us consider a particle submitted at level r to only the inward acceleration of gravity and outward acceleration of radiative pressure:

$\boldsymbol{g} = -\frac{GM_r}{r^2}\frac{\boldsymbol{r}}{r}$ and $\boldsymbol{g}_{\rm rad} = -\frac{1}{\varrho}\frac{dP_{\rm rad}}{dr}\frac{\boldsymbol{r}}{r}$

$dP_{\rm rad}/dr < 0$

$\boldsymbol{g}_{\rm rad} = \frac{\kappa F}{c}\frac{\boldsymbol{r}}{r}$

$F = \frac{L_r}{4\pi r^2}$

At a level r of the star the sum of the two accelerations is:

$$\boldsymbol{g}_{\rm tot} = \boldsymbol{g} + \boldsymbol{g}_{\rm rad} = -\frac{GM_r}{r^2}\frac{\boldsymbol{r}}{r}\left[1 - \frac{\kappa L_r}{4\pi c G M_r}\right] = -\frac{GM_r}{r^2}\frac{\boldsymbol{r}}{r}[1-\Gamma_r]$$

where $\Gamma_r \equiv \frac{L_r}{4\pi c G \ {}^{r}/_{\kappa}} \equiv \frac{L_r}{L_{Edd}}$, with $\boxed{L_{\rm Edd} = \frac{4\pi c G M}{\kappa}}$

Eddington limit

$\boldsymbol{g}_{\rm tot} = 0 \ \Rightarrow \ \Gamma = 1$

Combined Eddington and Rotation Limits

Von Zeipel theorem for rotating stars →

✓Change of total luminosity

✓Local brightness changes over the surface

→ local changes in radiation pressure

→Eddington luminosity varies on the surface of a rotating star

For a rotating star the total acceleration of a mass particle at level r will be:

$$\boldsymbol{g}_{\mathrm{tot}} = \boldsymbol{g}_{eff} + \boldsymbol{g}_{\mathrm{rad}} = \boldsymbol{g}_{\mathrm{grav}} + \boldsymbol{g}_{\mathrm{rot}} + \boldsymbol{g}_{\mathrm{rad}}$$

The $\Omega\Gamma$ limit occurs when $\boldsymbol{g}_{\mathrm{tot}} = \boldsymbol{g}_{\mathrm{grav}} + \boldsymbol{g}_{\mathrm{rot}} + \boldsymbol{g}_{\mathrm{rad}} = 0$ (significant effects both due to rotation and due to radiation)

(compare with the Ω (break-up) limit when $\boldsymbol{g}_{eff} = \boldsymbol{g}_{\mathrm{grav}} + \boldsymbol{g}_{\mathrm{rot}} = 0$)

$$\boldsymbol{g}_{\text{tot}} = \boldsymbol{g}_{eff} + \boldsymbol{g}_{\text{rad}}$$

$$\boldsymbol{g}_{\text{rad}}(\Omega,\vartheta) = \frac{\kappa(\Omega,\vartheta)}{c}\boldsymbol{F}(\Omega,\vartheta)$$

$$\boldsymbol{F}(\Omega,\vartheta) = -\frac{L}{4\pi GM\left(1-\frac{\Omega^2}{2\pi G\bar{\varrho}_M}\right)}\boldsymbol{g}_{\text{eff}}(\Omega,\vartheta)$$

In all these, we have assumed solid body rotation

$$\boldsymbol{g}_{\text{tot}} = \boldsymbol{g}_{\text{eff}}\left[1-\frac{\kappa(\Omega,\vartheta)L(P)}{4\pi cGM\left(1-\frac{\Omega^2}{2\pi G\bar{\varrho}_M}\right)}\right] \Rightarrow \boldsymbol{g}_{\text{tot}} = \boldsymbol{g}_{\text{eff}}[1-\Gamma(\Omega,\vartheta)]$$

$$\boldsymbol{g}_{\text{tot}} = 0 \Rightarrow \boxed{L_{\Omega\Gamma} = \frac{4\pi cG}{\kappa(\Omega,\vartheta)}\left(1-\frac{\Omega^2}{2\pi G\bar{\varrho}_M}\right)} \qquad \boxed{\Gamma(\Omega,\vartheta) = L/L_{\Omega\Gamma}}$$

(here, generally $\boldsymbol{g}_{\text{eff}} \neq 0$)

So, for a certain angular velocity Ω on the stellar surface, the maximum permitted luminosity of a star is reduced by rotation with respect to the usual Eddington limit.

Discussion on break-up velocities for shellular rotation

$\rightarrow \Omega_{\text{crit}}$ for $g_{\text{eff}} = 0$ well defined for solid body rotation

But, for shellular rotation whether break-up occurs or not depends on timescales:

- Timescale for angular momentum redistribution (e.g. with meridional circulation → ~10^4y in massive ZAMS stars)
- Timescale or structural change (e.g. contraction causing spin-up) – short for massive stars
- Break-up velocity→ equatorial mass loss, formation of equatorial decretion disks (e.g. Be stars)

Effects of rotation on convection

Reminder: Brunt–Väisälä Frequency for non-rotating star

Let us consider a fluid element in a star at some level r_0 in equilibrium with the surrounding medium. If this cell is displaced vertically and the motion is adiabatic without viscous effects, the equation of motion is:

$$\varrho_{\mathrm{int}} \frac{d^2 r}{dt^2} + g(\varrho_{\mathrm{int}} - \varrho_{\mathrm{ext}}) = 0$$

For a small displacement ($r - r_0$), one may develop to the first order:

$$\varrho_{\mathrm{int}}(r) = \varrho_{\mathrm{int}}(r_0) + \left.\frac{d\varrho_{\mathrm{int}}}{dr}\right|_{r_0} (r - r_0)$$

$$\varrho_{\mathrm{ext}}(r) = \varrho_{\mathrm{ext}}(r_0) + \left.\frac{d\varrho_{\mathrm{ext}}}{dr}\right|_{r_0} (r - r_0)$$

Since at r_0 $\varrho_{\mathrm{int}}(0) = \varrho_{\mathrm{ext}}(0)$,

$$\rightarrow \varrho_{\mathrm{int}} \frac{d^2 r}{dt^2} + g\left(\frac{d\varrho_{\mathrm{int}}}{dr} - \frac{d\varrho_{\mathrm{ext}}}{dr}\right)(r - r_0) = 0$$

equation of harmonic motion without damping if $N^2 \equiv \frac{g}{\varrho}\left(\frac{d\varrho_{\mathrm{int}}}{dr} - \frac{d\varrho_{\mathrm{ext}}}{dr}\right) > 0$ i.e. $\frac{d\varrho_{\mathrm{int}}}{dr} > \frac{d\varrho_{\mathrm{ext}}}{dr}$

→N is the Brunt–Väisälä Frequency STABILITY

We may express the Brunt–Väisälä frequency in terms of temperature gradients instead of density gradients, using the eqution of state

$$\frac{d\ln \varrho_{\text{int}}}{dr} = \alpha \frac{d\ln P_{\text{int}}}{dr} - \delta \frac{d\ln T_{\text{int}}}{dr}$$

$$\frac{d\varrho}{\varrho} = \alpha \frac{dP}{P} - \delta \frac{dT}{T} + \varphi \frac{d\mu}{\mu},$$

$$\alpha = \left(\frac{\partial \ln \varrho}{\partial \ln P}\right)_{T,\mu}, \ \delta = -\left(\frac{\partial \ln \varrho}{\partial \ln T}\right)_{P,\mu}, \ \varphi = \left(\frac{\partial \ln \varrho}{\partial \ln \mu}\right)_{P,T}.$$

$$\frac{d\ln \varrho_{\text{ext}}}{dr} = \alpha \frac{d\ln P_{\text{ext}}}{dr} - \delta \frac{d\ln T_{\text{ext}}}{dr} + \varphi \frac{d\ln \mu_{\text{ext}}}{dr}$$

Subtract these equations:

$$\rightarrow \frac{1}{\varrho}\frac{d(\Delta\varrho)}{dr} = \delta\left(\frac{d\ln T_{\text{ext}}}{dr} - \frac{d\ln T_{\text{int}}}{dr}\right) - \varphi \frac{d\ln \mu_{\text{ext}}}{dr}$$

(assuming pressure equilibrium of fluid element with surroundings – OK for subsonic motions)

$$\frac{1}{\varrho}\frac{d(\Delta\varrho)}{dr} = \delta\left(\frac{d\ln T_{\text{ext}}}{dr} - \frac{d\ln T_{\text{int}}}{dr}\right) - \varphi\,\frac{d\ln \mu_{\text{ext}}}{dr}$$

$$N^2 \equiv \frac{g}{\varrho}\left(\frac{d\varrho_{\text{int}}}{dr} - \frac{d\varrho_{\text{ext}}}{dr}\right) = \frac{g}{\varrho}\frac{d(\Delta\varrho)}{dr} > 0$$

$$\Rightarrow \quad N^2 = \frac{g\delta}{H_P}\left(\nabla_{\text{int}} - \nabla + \frac{\varphi}{\delta}\nabla_\mu\right) > 0$$

where

$$\nabla \equiv \frac{d\ln T_{\text{ext}}}{d\ln P},\ \nabla_{\text{int}} \equiv \frac{d\ln \ {}_{\text{int}}}{d\ln P},\ \nabla_\mu \equiv \frac{d\ln \mu_{\text{ext}}}{d\ln P},\ H_P \equiv -\frac{dr}{dP}P$$

Ledoux criterion

$$\frac{g\delta}{H_P}\left(\nabla_{\text{int}} - \nabla + \frac{\varphi}{\delta}\nabla_\mu\right) < 0 \Rightarrow \nabla < \nabla_{\text{int}} + \frac{\varphi}{\delta}\nabla_\mu$$

→ Schwarzschild criterion

For chemically homogeneous medium $\nabla < \nabla_{\text{int}}$

Oscillation frequency in a rotating medium

- the displaced element also feels a change in the centrifugal force as it rises (or sinks). This will modify N^2
- We assume cylindrical symmetry
- The equation of motion of the displaced element (no viscous effects) can be written as:

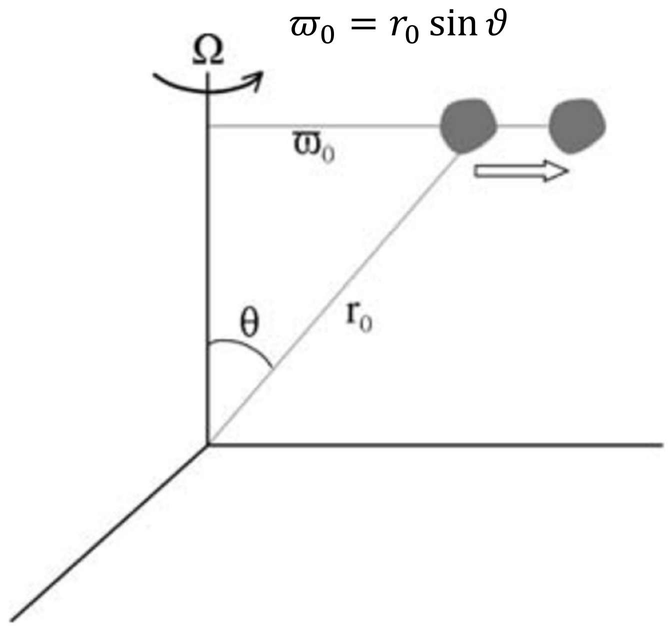
$$\frac{d^2r}{dt^2} + \frac{g}{\varrho_{\rm int}}(\varrho_{\rm int} - \varrho_{\rm ext}) - \varpi(\Omega_{\rm int}^2 - \Omega_{\rm ext}^2) = 0$$

- Specific angular momentum $j = \varpi^2\Omega$ →

$$\varpi(\Omega_{\rm int}^2 - \Omega_{\rm ext}^2) = \frac{1}{\varpi^3}(j_{\rm int}^2 - j_{\rm ext}^2)$$

- assuming conservation of the specific angular momentum of the displaced fluid element: $j_{\rm int}$ same at ϖ_0 and ϖ

$$\frac{1}{\varpi^3}(j_{\rm int}^2 - j_{\rm ext}^2) \approx -\frac{1}{\varpi^3}\frac{dj^2}{d\varpi}\bigg|_0 (\varpi - \varpi_0) = -\frac{1}{\varpi^3}\frac{d(\Omega^2\varpi^4)}{d\varpi}\bigg|_0 (\varpi - \varpi_0)$$



Developing the density as for the non-rotating case, we obtain for the eq. of motion:

$$\frac{d^2r}{dt^2} + \left[\frac{g}{\varrho_{\text{int}}}\left(\frac{d\varrho_{\text{int}}}{dr} - \frac{d\varrho_{\text{ext}}}{dr}\right) + \frac{1}{\varpi^3}\frac{d(\Omega^2\varpi^4)}{d\varpi}\sin\vartheta\right](r - r_0) = 0$$

with the derivatives taken at r_0

For stability, $\frac{g}{\varrho_{\text{int}}}\left(\frac{d\varrho_{\text{int}}}{dr} - \frac{d\varrho_{\text{ext}}}{dr}\right) + \frac{1}{\varpi^3}\frac{d(\Omega^2\varpi^4)}{d\varpi}\sin\vartheta > 0,$ or

$\boxed{N^2 = \frac{g\delta}{H_P}\left(\nabla_{\text{int}} - \nabla + \frac{\varphi}{\delta}\nabla_\mu\right) + \frac{1}{\varpi^3}\frac{d(\Omega^2\varpi^4)}{d\varpi}\sin\vartheta > 0}$ Solberg–Hoiland Criterion

(deep in stellar interiors $\nabla_{\text{int}} = \nabla_{\text{ad}}$)

The Rayleigh criterion is the criterion, which states

the condition for a distribution of angular velocity to be stable

$N_\Omega^2 = \frac{1}{\varpi^3}\frac{d(\Omega^2\varpi^4)}{d\varpi} > 0$ (since $\sin\vartheta$ is positive and equal to zero at the pole)

This condition (The Rayleigh criterion) means that the specific angular momentum

$j = \varpi^2\Omega$ must increase outward:

$\frac{dj}{d\varpi} = \varpi\left(\varpi\frac{d\Omega}{d\varpi} + 2\Omega\right) > 0$ or $\frac{d\ln}{d\ln\varpi} > -2$

✓If $\Omega \sim r^{-a}$ $a > 2$ → instability (violation of Rayleigh criterion)

✓On the stellar surface, the angular momentum must also increase from pole to equator.

Numerical simulations

Extreme theoretical cases

✓ **Case A**: In 1D stellar evolution models, it is generally assumed that convective regions are rotating like a solid body (as strong turbulent viscosity due to convective motions maintains a uniform distribution of angular velocity).

$$\Omega = \mathrm{const}$$

✓ **Case B**: an alternative possibility is that the distribution of the angular momentum is constant in convective regions.

$$j = \varpi^2 \Omega \Rightarrow \Omega \propto \varpi^{-2}$$

(large-scale motions dominate and conserve their angular momentum, thus establishing a uniform distribution of the specific angular momentum rather than angular velocity)

Reality is more complex than either extreme.

Results from 2D and 3D convection simulations applied to the solar convective envelope and convective cores

1. Solar convective envelope:

- Convection is time-dependent and highly turbulent
- Understanding convection is essential to explain the solar differential rotation, revealed by helioseismology
- Differential rotation plays a key role in the solar dynamo and magnetic activity cycles.

Mean rotation profile:

Equator: $\Omega(r)$increases outward (~10–15%)

Mid-latitudes: smaller radial increase

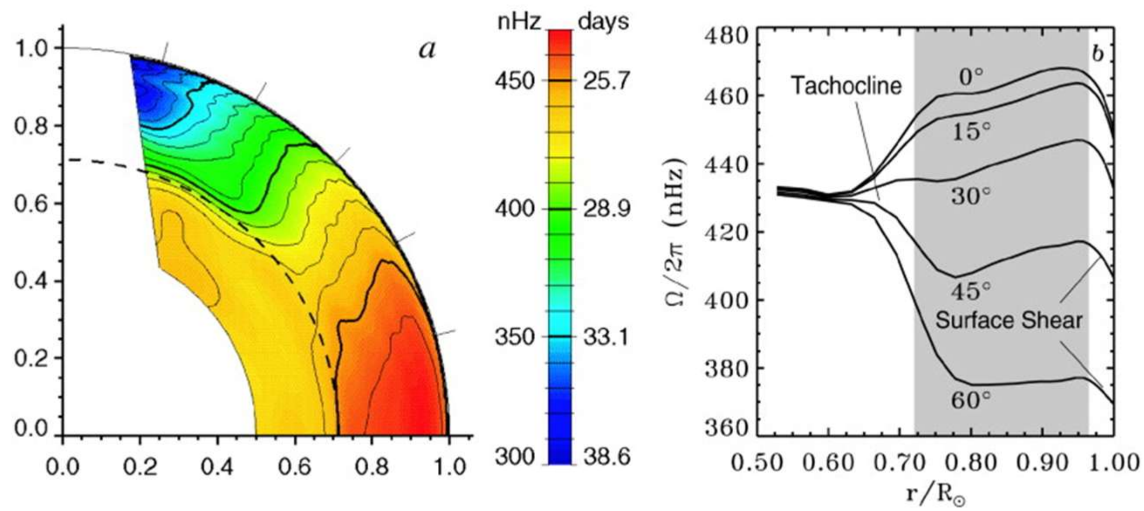
Poles: Ω $\approx$constant with depth
(~30% slower surface rotation than equator)

3D global simulations of turbulent convection in deep spherical shells.

Toomre and Brun 2002

Fig. 1

(*a*) Angular velocity profile $\Omega/2\pi$ with radius and latitude as deduced from helioseismology using SOI-MDI data, with red tones indicating fast rotation and blue-green the slowest rotation (adapted from Schou et al. 1998). (*b*) Time-averaged rotation rates from 5 years of GONG helioseismic data, plotted against radius at different latitudes. The surface shear layer and the tachocline at the base of the convective zone are indicated as well as the zone covered by our computational domain (*gray area*) (adapted from Howe et al. 2000b).



Main goals

- ✓ Reproduce the observed decrease of angular velocity toward the poles.
- ✓ Determine whether a strong equator–pole angular velocity contrast **(ΔΩ)** can be maintained in highly turbulent regimes.

Main results

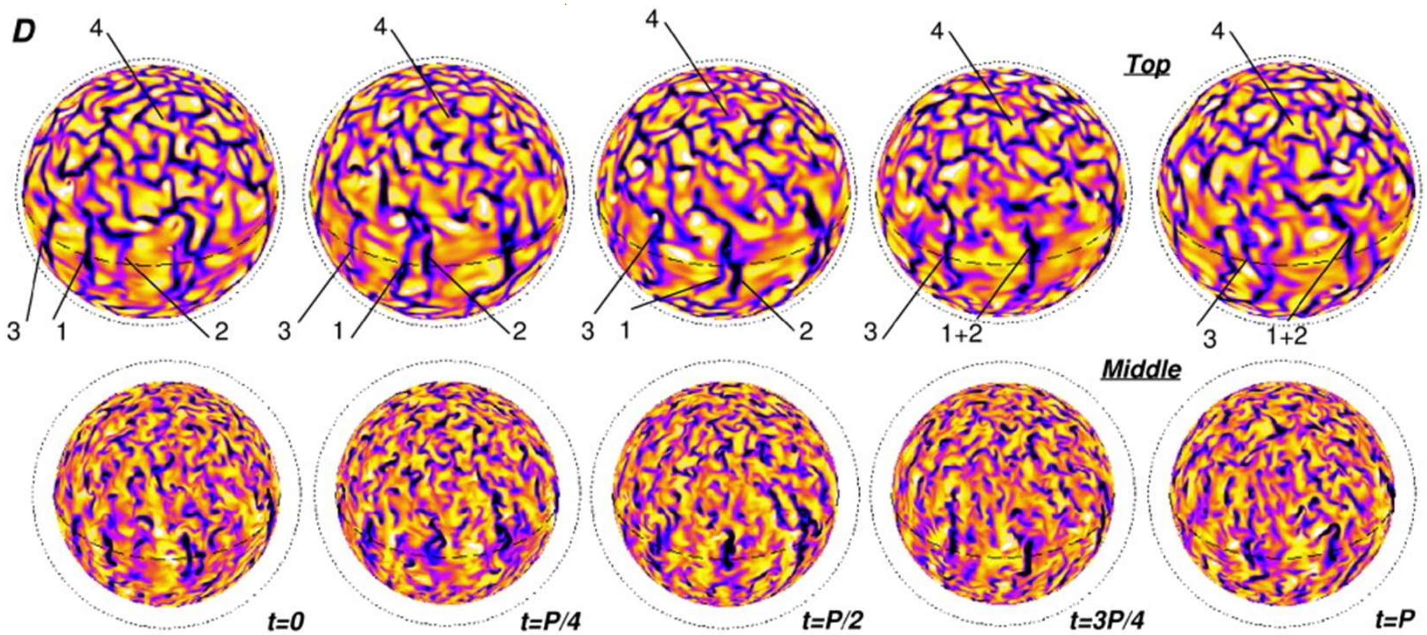
Simulations can reproduce solar-like differential rotation:

- fast rotation at the equator
- ~30% slower poles
- angular velocity roughly constant along radial lines at mid-latitudes.

→ A strong **latitudinal contrast in rotation can persist even in very turbulent convection**.

Evolution of the convection throughout one solar rotation, showing the radial velocity of case D near the top and at the middle of the domain. The time interval between each successive image is about 7 days. Features 1 and 2 exhibit the merging of the downflow lanes, feature 3 the shearing action of the differential rotation present in the shell, and feature 4 the appearance and deformation of a convective cell at higher latitudes.

Yellow: upflows
Purple: downflows



near the top of the convection zone

deeper in the convection zone

$t \longrightarrow$

Toomre and Brun 2002

2. Convective cores:

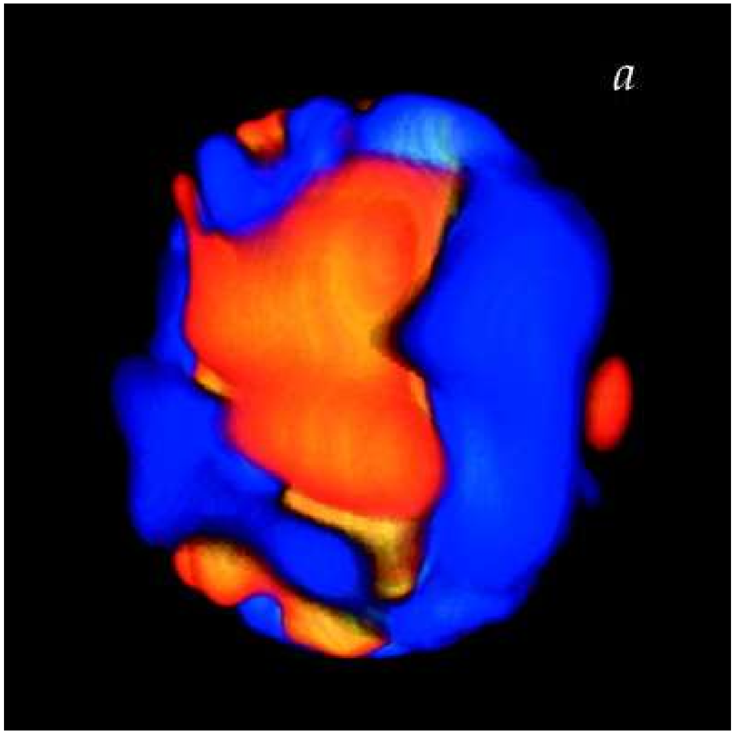
Browning, Brun, & Toomre 2004

The convective core is not spherical
The bright convective region is **elongated along the rotation axis.**
This is the **prolate core geometry** predicted by the Solberg–Høiland stability criterion (rotational modification of buoyancy).

Overshooting makes the mixed region more spherical
→convective plumes penetrate into the radiative envelope
stronger near the equatorial regions

So, although the *formal convective boundary* is prolate,
the total region affected by mixing becomes closer to spherical**.**

3D Simulations of Core Convection in Rotating A-Type Stars: Differential Rotation and Overshooting



Red / orange regions → strong convective motions inside the core
Blue region → surrounding radiative zone
The irregular boundary shows convective penetration / overshooting

Transport of Angular Momentum in Stars (Maeder ch. 10.5)

An element of volume limited by ABCD rotates around the vertical axis with an angular velocity $\Omega = \dot{\varphi}$

The mass of the volume element is

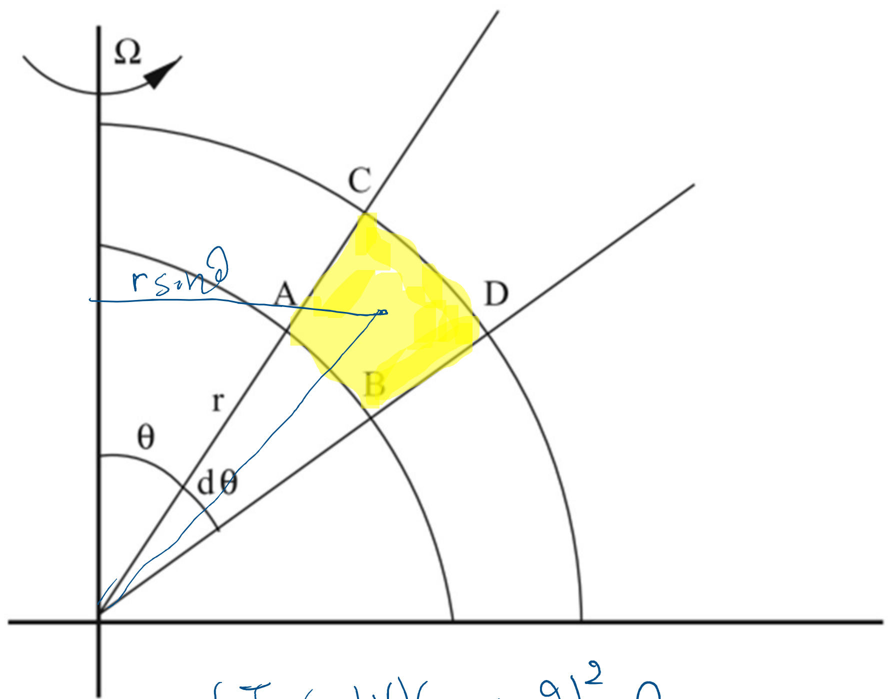
$\varrho r^2 \sin\vartheta d\vartheta d\varphi dr$

The component of the angular momentum $\boldsymbol{J}$ aligned with the rotational axis is

$dJ = \varrho r^4 \sin^3\vartheta d\vartheta d\varphi dr \Omega$

In Lagrangian coordinates, the derivative of the angular momentum of the moving fluid mass element is

$\varrho r^2 \sin\vartheta d\vartheta d\varphi dr \frac{d}{dt}\left(r^2 \sin^2\theta \Omega\right)_{M_r}$



$dJ = (\rho dV)(r \sin\theta)^2 \,\Omega$

$$\varrho \frac{d}{dt}\left(r^2 \sin^2 \vartheta \Omega\right)_{M_r} = \frac{d}{dt}\left(\varrho r^2 \sin^2 \vartheta \Omega\right)_{M_r} - r^2 \sin^2 \vartheta \Omega \left.\frac{d\varrho}{dt}\right|_{M_r}$$

Using relation between the Lagrangian and Eulerian derivatives:

$$\varrho \frac{d}{dt}\left(r^2 \sin^2 \vartheta \Omega\right)_{M_r} = \frac{\partial}{\partial t}\left(\varrho r^2 \sin^2 \vartheta \Omega\right)_r + \mathbf{U} \cdot \boldsymbol{\nabla}\left(\varrho r^2 \sin^2 \vartheta \Omega\right) - r^2 \sin^2 \vartheta \Omega \left.\frac{d\varrho}{dt}\right|_{M_r}$$

Also $\left.\frac{d\varrho}{dt}\right|_{M_r} = \left.\frac{\partial \varrho}{\partial t}\right|_r + \mathbf{U} \cdot \boldsymbol{\nabla} \varrho$

exercise

Continuity equation: $(\partial \varrho / \partial t)|_r = -\boldsymbol{\nabla}.(\varrho \mathbf{U})$

$\rightarrow \varrho \frac{d}{dt}\left(r^2 \sin^2 \vartheta \Omega\right)_{M_r} = \frac{\partial}{\partial t}\left(\varrho r^2 \sin^2 \vartheta \Omega\right)_r + \boldsymbol{\nabla} \cdot \left(\mathbf{U} \varrho r^2 \sin^2 \vartheta \Omega\right)$

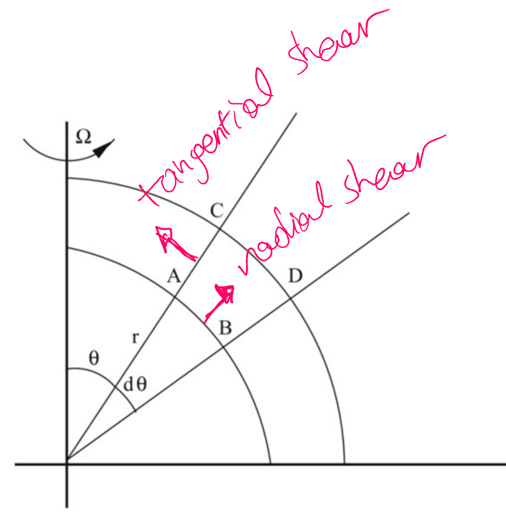
$\rightarrow \frac{d\boldsymbol{J}}{dt} = \mathcal{M} = \{\frac{\partial}{\partial t}\left(\varrho r^2 \sin^2 \vartheta \Omega\right)_r + \boldsymbol{\nabla} \cdot \left(\mathbf{U} \varrho r^2 \sin^2 \vartheta \Omega\right)\} r^2 \sin \vartheta \, d\vartheta \, d\varphi \, dr$

Transport of Angular Momentum by Shears

✓Slab floating with velocity $\boldsymbol{v}$ (on the plane of the slab) on a viscous medium of density ρ and thickness d feels a viscous force parallel to the motion of the slab and in the opposite direction

✓Newtonian law of viscosity

→ tension $\tau \equiv \frac{F}{S} \equiv \eta \frac{dv}{dz} = \frac{\eta v}{d}$, where η is the dynamic coefficient of viscosity

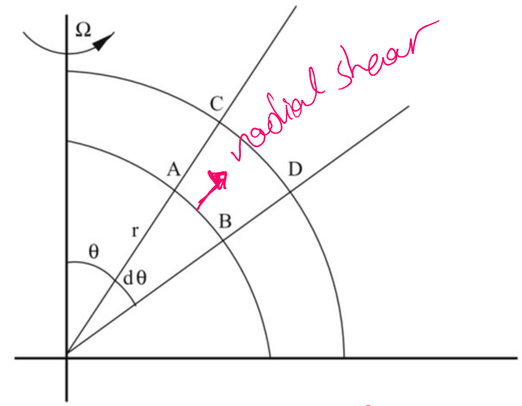


We will consider

- vertical shear
- Tangential shear

Vertical shear

✓ Applying this to a radial motion of a "slab" rotating around the axis, the tension will be:



$$\eta_v \frac{dv_\varphi}{dr} = \eta_v r \sin\vartheta \frac{d\Omega}{dr}$$

→The viscous force applied on surface AB:

$$F_{AB} = \eta_v r^3 \sin^2\vartheta \frac{d\Omega}{dr} d\vartheta d\varphi$$

$(dS = r^2 \sin\vartheta \, d\theta \, d\varphi)$

$v_\varphi = \Omega r \sin\vartheta$

Torque:

$\boldsymbol{\mathcal{M}} = \boldsymbol{r} \times \boldsymbol{F}_{\mathbf{AB}}$ → $\mathrm{d}\mathcal{M}_z = \eta_v r^4 \sin^3\vartheta \frac{d\Omega}{dr} d\vartheta d\varphi$ **elementary viscous torque about the rotation axis exerted on the spherical surface element AB**

For the volume element with section ABCD the gain of torque over dr will be

$$\frac{\partial}{\partial r}\left(\eta_v r^4 \sin^3\vartheta \, d\vartheta d\varphi \frac{\partial\Omega}{\partial r}\right) dr$$

Tangential shear

When $d\Omega/d\vartheta \neq 0$

The tension (force by unit surface) due to the tangential shear with a horizontal viscosity η_{h} is equal to $\eta_{\mathrm{h}} r \sin\vartheta\, [\partial\Omega/(r\partial\vartheta)]$

The corresponding force on the surface described by AC will be

$\eta_{\mathrm{h}} r \sin^2\vartheta\, (\partial\Omega/\partial\vartheta) dr d\varphi$ $\quad dS' = (r\sin\vartheta) d\varphi\, dr$

The corresponding torque

$\eta_{\mathrm{h}} r^2 \sin^3\vartheta\, (\partial\Omega/\partial\vartheta) dr d\varphi$

The gain on the torque when passing from AC to BD is

$$\frac{\partial}{r\partial\vartheta}\left(\eta_{\mathrm{h}} r^2 \sin^3\vartheta\, dr d\varphi \frac{\partial\Omega}{\partial\vartheta}\right) r d\vartheta$$

$$\frac{d\boldsymbol{J}}{dt} = \mathcal{M}$$

$$\varrho r^2 \sin\vartheta \frac{d}{dt}\left(r^2 \sin^2\vartheta\, \Omega\right)_{M_r}$$
$$= \frac{\partial}{\partial r}\left(\eta_{\mathrm{v}} r^4 \sin^3\vartheta \frac{\partial\Omega}{\partial r}\right) + \frac{\partial}{\partial\vartheta}\left(\eta_{\mathrm{h}} r^2 \sin^3\vartheta \frac{\partial\Omega}{\partial\vartheta}\right)$$

(we simplified by $drd\vartheta d\varphi$)

$$\Rightarrow \varrho \frac{d}{dt}\left(r^2 \sin^2\vartheta\Omega\right)_{M_r} = \frac{\partial}{\partial t}\left(\varrho r^2 \sin^2\vartheta\Omega\right)_r + \nabla\cdot\left[\boldsymbol{U}\varrho r^2 \sin^2\vartheta\Omega\right]$$
$$= \frac{\partial}{\partial t}\left(\varrho r^2 \sin^2\vartheta\Omega\right)_r + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\varrho r^4 \sin^2\vartheta U_r\Omega\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\varrho r^2 \sin^3\vartheta U_\vartheta\Omega\right)$$
$$= \frac{\sin^2\vartheta}{r^2}\frac{\partial}{\partial r}\left(\varrho D_{\mathrm{v}} r^4 \frac{\partial\Omega}{\partial r}\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\varrho D_{\mathrm{h}} \sin^3\vartheta \frac{\partial\Omega}{\partial\vartheta}\right).$$

We have replaced the dynamic viscosity coefficients with the kinematic coefficients $\nu_{\mathrm{v}} = \frac{\eta_{\mathrm{v}}}{\varrho}$ $\quad and \quad$ $\nu_{\mathrm{h}} = \eta_{\mathrm{h}}/\varrho,$ and used the fact that the kinematic viscosity coefficient is also the diffusion coefficient D for the chemical elements.

For shellular rotation, this eq. is simplified as $\Omega = \Omega(\mathrm{r})$

$$\Omega(r,\vartheta) = \overline{\Omega}(r) + \widehat{\Omega}(r,\vartheta)\ ,\ \ \widehat{\Omega} \ll \overline{\Omega}$$

with

$$\overline{\Omega}(r) = \frac{\int_0^\pi \Omega(r,\vartheta)\sin^3\vartheta d\vartheta}{\int_0^\pi \sin^3\vartheta d}$$

- We integrate the previous eq. after multiplying by $\sin\vartheta\ d\vartheta$