Stellar structure and evolution of rotating stars

Reading material:

Chapters 2 & 4

Parts of chapters 6 & 10

from A. Maeder’s book

ΜΕΡΟΣ Α’

Introductory remarks

- Stellar rotation can cause the equatorial radius can be much larger than the polar radius, up to about 1.5 times.
- Stars do not rotate as solid bodies → they may have internal differential rotation

  e.g. the core rotating faster than the outer envelope.

- stellar rotation
  - produces a flattening of the equilibrium configuration of the star,
  - drives internal circulation motions and various instabilities which transport both the chemical elements and the angular momentum.
  - generates stellar magnetic fields within convective zones (often by a dynamo), converting rotational energy into magnetic field structure.

Equilibrium configurations

- Maclaurin spheroids : density ρ is assumed constant
- Roche model: assumes an infinite central condensation
- Real stars in between these extremes

Equation of Motion with Rotation

(cf undergraduate course in Mechanics)

- The accelerations measured in a rotating frame (quantities with a prime) are related to those in an inertial frame (without a prime)

$$\frac{d\boldsymbol{v}'}{dt} = \frac{d\boldsymbol{v}}{dt} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}') - \frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{r}' - 2\boldsymbol{\Omega} \times \boldsymbol{v}'$$

(primed quantities in the rotating frame)

Centrifugal terms: $- \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}') - \frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{r}'$

Coriolis term: $- 2\boldsymbol{\Omega} \times \boldsymbol{v}'$

- In a stationary rotating star $\frac{d\boldsymbol{\Omega}}{dt} = 0$
- In the Navier–Stokes equation $\frac{d\boldsymbol{v}}{dt}$ and $\frac{\partial \boldsymbol{v}}{\partial t}$ are in the intertial system.

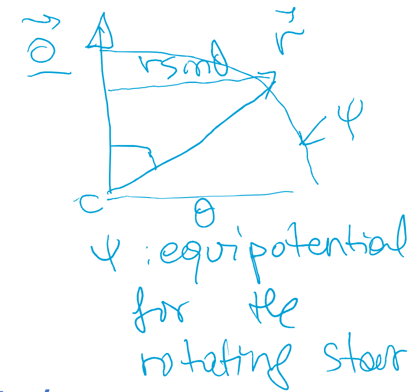
$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{a} - \frac{1}{\varrho}\nabla P + \nu\nabla^2\boldsymbol{v}$$

- To get the equation in the rotating frame $(d\boldsymbol{v}/dt) \rightarrow (d\boldsymbol{v}'/dt)$

$$\frac{d\boldsymbol{v}'}{dt} = \frac{\partial \boldsymbol{v}'}{\partial t} + (\boldsymbol{v}' \cdot \nabla)\boldsymbol{v}'$$

$$= \boldsymbol{a} - \frac{1}{\varrho}\nabla P + \nu\nabla^2\boldsymbol{v}' - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}') - \frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{r}' - 2\boldsymbol{\Omega} \times \boldsymbol{v}'$$

Hydrostatic Equilibrium for Solid Body Rotation



- assume solid body rotation $\Omega = \text{const}$
- assume viscosity negligible
- the Navier-Stokes equation gives for hydrostatic equilibrium:

$$0 = \boldsymbol{a} - \frac{1}{\varrho}\nabla P - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}')$$

- Gravitational force (per unit mass) $\boldsymbol{a} = \boldsymbol{g} = -\nabla\Phi$, where $\Phi = -GM_r/r$

  so $-\nabla\Phi = -\frac{GM_r}{r^2}\frac{\boldsymbol{r}}{r}$

Roche assumption: The inner layers are considered as spherical, i.e. not distorted by rotation, which gives the same external potential as if the whole mass is concentrated at the center.

- Centrifugal force (per unit mass) can be derived from the potential V: **(EXERCISE)**

  $V = -\frac{1}{2}\Omega^2\varpi^2$ where $\varpi = r\sin\vartheta$, and $-\nabla V = \Omega^2\boldsymbol{\varpi}$

- Total potential (per unit mass) $\Psi = \Phi + V$

  so the equation of hydrostatic equilibrium becomes:

  $\frac{1}{\varrho}\nabla P = -\nabla\psi = g_{eff}$

- the pressure is constant on an equipotential, i.e., one has $P \;=\; P(\Psi)$

  equipotentials and isobars coincide and the star is said to be barotropic

- $\nabla P = (dP/d\Psi)\nabla\Psi$ οπότε η $\frac{1}{\varrho}\nabla P = -\nabla\psi = g_{eff}$ γίνεται $\frac{1}{\varrho}\frac{dP}{d\Psi} = -1$

  → $\rho = \rho(\Psi)$ and via the eq. of state $T = T(\Psi)$

Stellar surface and gravity

- The stellar surface is an equipotential $\Psi = const.$

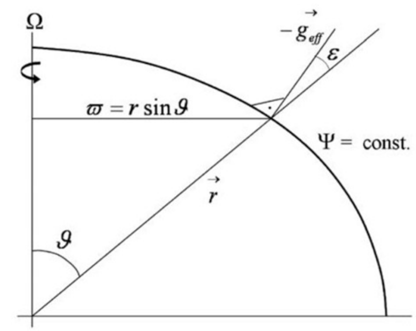
$$\Psi(r, \vartheta) = -\frac{GM_r}{r} - \frac{1}{2}\Omega^2 r^2 \sin^2\vartheta$$

unmodified by rotation in the Roche approximation

- Let us consider a star of total mass M and call R(ϑ) the stellar radius at colatitude ϑ.

- The centrifugal force is zero at the poles (ϑ=0), so we can fix the constant value of the equipotential at the stellar surface as $\frac{GM}{R_\mathrm{p}}$

$$\frac{GM}{R} + \frac{1}{2}\Omega^2 R^2 \sin^2\vartheta = \frac{GM}{R_\mathrm{p}}$$



- Surface gravity

$$\boldsymbol{g}_{\mathrm{eff}} = \left[-\frac{GM}{R^2(\vartheta)} + \Omega^2 R(\vartheta)\sin^2\vartheta\right]\boldsymbol{e}_r + \left[\Omega^2 R(\vartheta)\sin\vartheta\cos\vartheta\right]\boldsymbol{e}_\vartheta$$

(EXERCISE)

$$g_{\mathrm{eff}} = |\boldsymbol{g}_{\mathrm{eff}}| = \left[\left(-\frac{GM}{R^2(\vartheta)} + \Omega^2 R(\vartheta)\sin^2\vartheta\right)^2 + \Omega^4 R^2(\vartheta)\sin^2\vartheta\cos^2\vartheta\right]^{\frac{1}{2}}$$

Critical velocities

- Critical velocity, or break-up velocity:

  When modulus of the centrifugal force = modulus of the gravitational attraction at the equator ($\vartheta = \pi/2$).

$$\frac{GM}{R^2(\vartheta)} = \Omega^2 R(\vartheta) \Rightarrow \Omega_{\mathrm{crit}}^2 = \frac{GM}{R_{\mathrm{e,crit}}^3}$$

- So, from the equation of the surface, we get:

$$\frac{GM}{R} + \frac{1}{2}\Omega^2 R^2 \sin^2\vartheta = \frac{GM}{R_p} \Rightarrow \frac{GM}{R_{e,crit}} + \frac{1}{2}\Omega_{crit}^2 R_{e,crit}^2 = \frac{GM}{R_{p,crit}} \Rightarrow \frac{R_{e,crit}}{R_{p,crit}} = \frac{3}{2}$$

- Equatorial break-up velocity

$$v_{\mathrm{crit},1}^2 = \Omega_{\mathrm{crit}}^2 R_{\mathrm{e,crit}}^2 = \frac{GM}{R_{\mathrm{e,crit}}} = \frac{2GM}{3R_{\mathrm{p,crit}}}$$

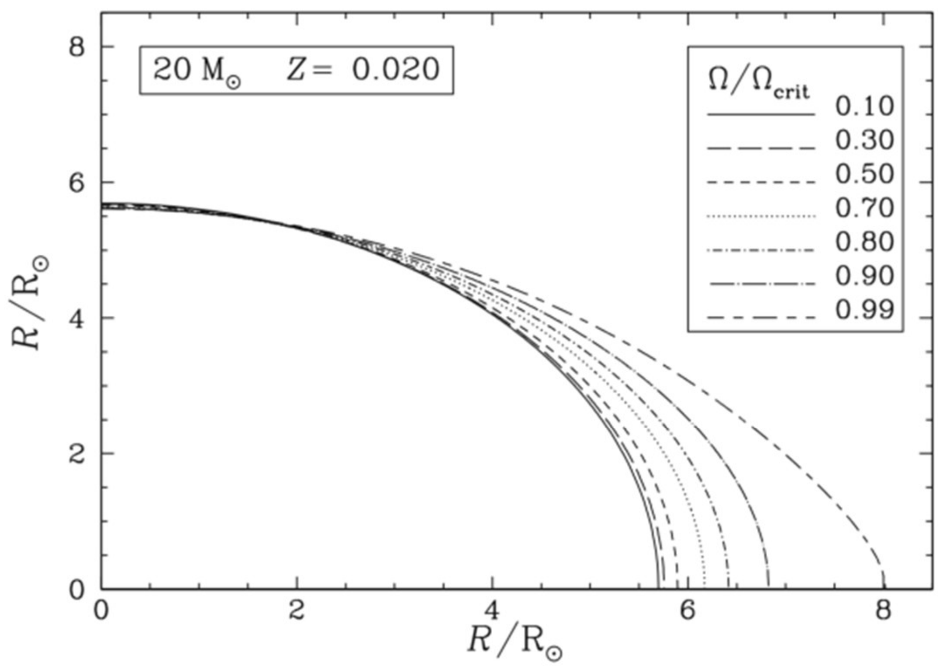


Fig. 2.2 The shape $R(\vartheta)$ of a rotating star in one quadrant. A $20\,M_\odot$ star with $Z = 0.02$ on the ZAMS is considered with various ratios $\omega = \Omega/\Omega_{crit}$ of the angular velocity to the critical value at the surface. One barely notices the small decrease of the polar radii for higher rotation velocities (cf. Fig. 2.7). Courtesy of S. Ekström

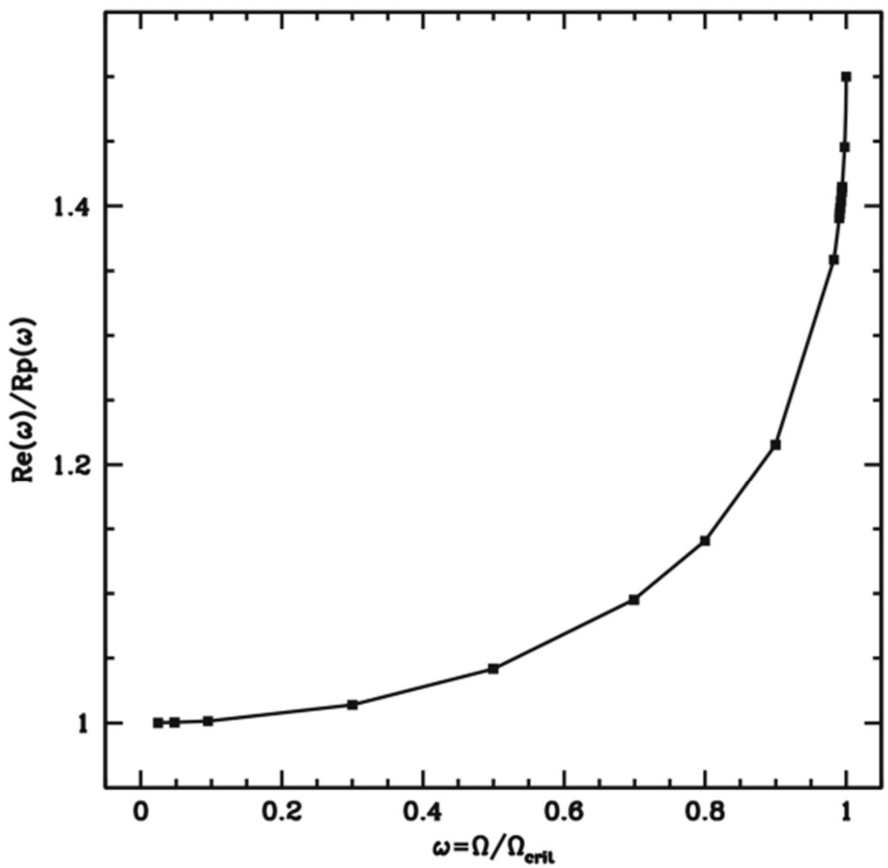


Fig. 2.3 The variation of the ratio R_e/R_p of the equatorial to the polar radius as a function of the rotation parameter ω in the Roche model

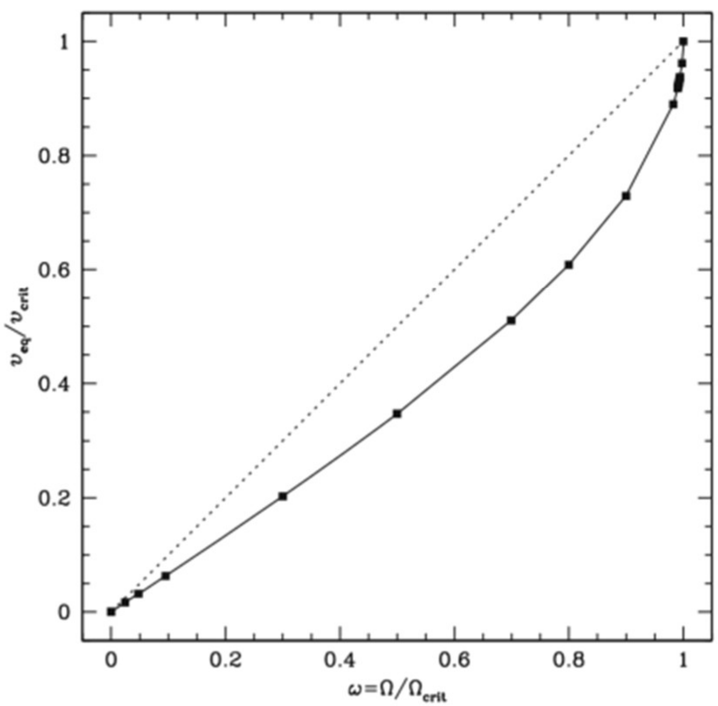


Fig. 2.5 The variation of the ratio $v/v_{\mathrm{crit},1}$ of the equatorial velocity to the critical velocity as a function of the rotation parameter ω in the Roche model. The polar radius is assumed not to change with the stellar mass here. The *dotted line* connects the origin to the maximum value

Set $x = \frac{R}{R_{\mathrm{p,crit}}}$ and

$$\omega = \frac{\Omega}{\Omega_{crit}} \Rightarrow \omega^2 = \frac{\Omega^2 R^3_{\mathrm{e,\ crit}}}{GM} \Rightarrow \Omega^2 = \frac{8}{27}\frac{GM\omega^2}{R^3_{\mathrm{p,crit}}}$$

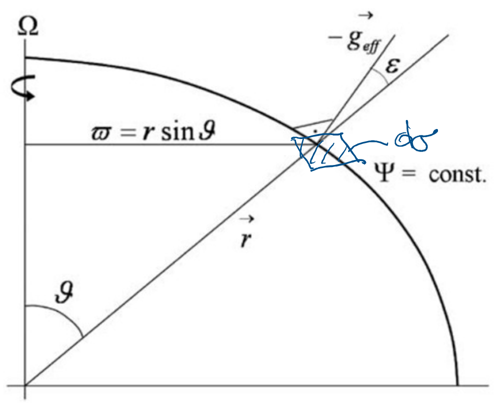
The surface equation becomes

$$\frac{GM}{R} + \frac{1}{2}\Omega^2 R^2 \sin^2\vartheta = \frac{GM}{R_{\mathrm{p}}} \Rightarrow \boxed{\frac{1}{x} + \frac{4}{27}\omega^2 x^2 \sin^2\vartheta = \frac{R_{\mathrm{p,}}}{R_{\mathrm{p}}(\omega)}}$$

→solved numerically using the Newton method

$$g_{\mathrm{eff}} = \frac{GM}{R^2_{\mathrm{p,}}}\left[\left(-\frac{1}{x^2} + \frac{8}{27}\omega^2 x \sin^2\vartheta\right)^2 + \left(\frac{8}{27}\omega^2 x \sin\vartheta\cos\vartheta\right)^2\right]^{\frac{1}{2}}$$

$$\cos\varepsilon = -\frac{\boldsymbol{g_{eff}}\cdot\boldsymbol{r}}{|\boldsymbol{g_{eff}}|\cdot|\boldsymbol{r}|} = \frac{\frac{1}{x^2} - \frac{8}{27}\omega^2 x \sin^2\vartheta}{\left[\left(-\frac{1}{x^2} + \frac{8}{27}\omega^2 x \sin^2\vartheta\right)^2 + \left(\frac{8}{27}\omega^2 x \sin\vartheta\cos\ \right)^2\right]^{\frac{1}{2}}}$$



Surface element on equipotential $d\sigma = \frac{r^2 \sin\vartheta\, d\varphi\, d\vartheta}{\cos\varepsilon}$

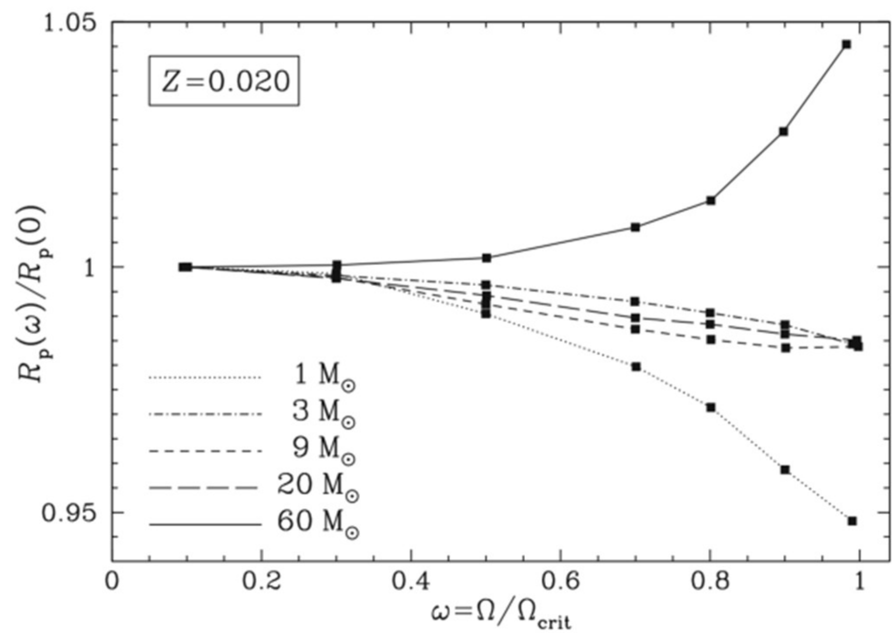
Polar radius as a function of rotation

- As a first approximation polar radii are independent of rotation
- In reality there is a small dependence $R_p(\omega)$ from the small changes (few %) of internal structure brought about by centrifugal force.
- This dependence can be described by

$$R_{\mathrm{p}}(\omega) = R_{\mathrm{p}}(0)\left(1 - a\omega^2\right)$$

where α is a constant for stars of the same mass

Metallicity also plays a role (see fig 2.8)



Variations of the polar radius as a function of the rotation parameter normalized to the value without rotation for stars of different initial masses at Z = 0.02. Fig. 2.7 from Maeder

Summary of solid-body rotation approximation

- $\Omega = constant$ throughout the star (solid-body rotation)
- every layer rotates with the **same angular velocity**
- there is **no internal shear ($\nabla\Omega = 0$)**
- centrifugal acceleration is simple and can be derived as the gradient of a potential
- effective potential $\Phi_{\text{eff}} = -\frac{GM}{r} - \frac{1}{2}\Omega^2 r^2 sin^2\vartheta$
- In hydrostatic equilibrium, surfaces of constant pressure, density, and temperature coincide with **equipotentials**
- Thus the structure can still be treated almost like a 1D stellar model.

Next approximation: Shellular rotation
Equations of Stellar structure

- Angular velocity is **constant on isobaric shells**, but can vary with isobaric radius.

  $\Omega = \Omega(P)$ (no dependence on ϑ) which is equivalent to $\Omega \approx \Omega(r)$

  where r the radial coordinate of the isobar (rotation ~constant on shells).

- Why is this a good approximation in radiative zones
  - ✓ In radiative zones, horizontal turbulence is strong (no buoyancy along isobars) because stable stratification (stable against convection → vertical motions suppressed) inhibits vertical mixing.
  - ✓ So turbulence transports angular momentum much more efficiently **along isobaric surfaces** than radially.
  - ✓ This leads to $\Omega(r, \vartheta) \approx \Omega(r)$ because horizontal shear gets erased quickly.
  - ✓ So the rotation becomes almost uniform on each shell.

- But there are always **small horizontal variations** that allow baroclinicity (when surfaces of constant pressure and constant density do not coincide), circulation currents and angular momentum redistribution, in real stars

- $\Omega(r,\vartheta) = \overline{\Omega}(r) + \widehat{\Omega}(r,\vartheta)$
  - ✓ $\widehat{\Omega} \ll \overline{\Omega}$
  - ✓ $\overline{\Omega}(r)$ the average Ω on an isobar with radius r:

    The horizontal average $\overline{\Omega}$ is defined as the angular velocity of a shell rotating like a solid body and having the same angular momentum as the considered actual shell.

$$\overline{\Omega}(r) = \frac{\int_0^{\pi} \Omega(r,\vartheta) \sin^3 \vartheta d\vartheta}{\int_0^{\pi} \sin^3 \vartheta d\vartheta}$$

- $\widehat{\Omega}(r,\vartheta)$ can be developed in terms of the Legendre polynomials.
  - ✓ To second order:

$$\widehat{\Omega}(r,\vartheta) = \Omega_2(r) P_2(\cos\vartheta)$$

angular momentum of a mass element on a shell of radius r and width Δr

- perp. distance of dm from rotation axis $\varpi = r\sin\theta$
- azimuthal velocity $v_\varphi = \Omega r \sin\theta$
- angular momentum per unit mass $j = \varpi v_\varphi = r^2 \sin^2\theta\, \Omega$
- angular momentum of element dm: $dJ = r^2 \sin^2\theta\, \Omega\, dm$
- $dm = \rho\, dV = \rho r^2 \sin\theta\, dr\, d\theta\, d\varphi$

$$dJ = \Omega \rho r^4 \sin^3\theta\, dr\, d\theta\, d\varphi \implies J_{shell} = r^4 \Delta r \left(\int_0^{2\pi} d\varphi\right) \int_0^{\pi} \Omega \sin^3\theta\, d\theta$$

- If the shell rotated as a solid body $J_{solid} = r^4 \Delta r\, 2\pi\, \bar{\Omega} \int_0^{\pi} \sin^3\theta\, d\theta$

$$J_{shell} = J_{solid} \implies \bar{\Omega} = \frac{\int_0^{\pi} \Omega(r,\theta) \sin^3\theta\, d\theta}{\int_0^{\pi} \sin^3\theta\, d\theta}$$

Properties of the isobars

- Let us consider the surface of constant Ψ

$$\Psi = \Phi - \frac{1}{2}\Omega^2 r^2 \sin^2\vartheta = \text{const} \;\; (\text{where } \Omega = \Omega(r,\vartheta) = \overline{\Omega}(r) + \widehat{\Omega}(r,\vartheta))$$

- The gravitational potential is defined (as in the case of solid rotation)

$-\nabla\Phi = -\frac{GM_r}{r^2}\frac{\mathbf{r}}{r}$, and $\Phi = -\frac{GM_r}{r}$ in the Roche approximation.

- The components of the gradient of Ψ

$$\frac{\partial\Psi}{\partial r} = \underbrace{\frac{\partial\Phi}{\partial r} - \Omega^2 r \sin^2\vartheta}_{-g_{eff,r}} - r^2\sin^2\vartheta\,\Omega\frac{\partial\Omega}{\partial r}$$

$$\frac{1}{r}\frac{\partial\Psi}{\partial\vartheta} = \underbrace{\frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}}_{0} \underbrace{- \Omega^2 r\sin\vartheta\cos\vartheta}_{-g_{eff,\theta}} - r^2\sin^2\vartheta\,\Omega\frac{1}{r}\frac{\partial\Omega}{\partial\vartheta}$$

$$\vec{g}_{eff} = -\vec{\nabla}\Psi - r^2\sin^2\theta\,\Omega\vec{\nabla}\Omega$$

$$\left(\vec{\nabla}\Omega = \frac{\partial\Omega}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\Omega}{\partial\theta}\hat{e}_\theta\right)$$

From the equation of hydrostatic equilibrium:

$$\boldsymbol{\nabla} P = \varrho \boldsymbol{g_{eff}} \Rightarrow \nabla P = -\varrho\left(\nabla\Psi + r^2 \sin^2\vartheta\, \Omega\, \nabla\Omega\right)$$

on an isobar, $\Omega = \Omega(P) \Rightarrow \nabla\Omega = \frac{d\Omega}{dP}\, \nabla P$

So ∇P parallel to $\nabla\Psi$

i.e. the surfaces defined by $\Psi = const$. are isobaric surfaces

but they are not equipotential and the star is said to be **baroclinic**.

(for solid body rotation, isobars and equipotentials coincide and the star is barotropic.)

Expressions for hydrostatic equilibrium and continuity for shellular rotation

- Characteristic radius r_{P} of an isobar, defined as

$V_P \equiv \frac{4\pi}{3} r_P^3$ where V_P is the volume inside the isobar

- For any quantity q, which is <u>not constant over an isobaric surface</u>, we define a mean value over the surface:

$$< q > \equiv \frac{1}{S_P} \oint q d\sigma$$

where S_P is the total surface of the isobar and $d\sigma = \frac{r^2 \sin\vartheta d\varphi d\vartheta}{\cos\varepsilon}$

- The effective gravity $\vec{g}_{eff} = -\vec{\nabla}\Psi - r^2 \sin^2\theta\Omega\vec{\nabla}\Omega$ can be written more simply using the fact that $\nabla\Omega$ parallel to $\nabla\Psi$

$\nabla\Omega = -\alpha\nabla\Psi$ with $\alpha = \left|\frac{d\Omega}{d\Psi}\right|$

$\rightarrow \vec{g}_{eff} = -\vec{\nabla}\Psi - \alpha r^2 \sin^2\theta\Omega\vec{\nabla}\Psi = -(1 - r^2 \sin^2\vartheta\,\Omega\alpha)\,\vec{\nabla}\Psi$

Isobaric surface: $P = \text{const}$

Unit normal to this surface: $\hat{n} = \dfrac{\nabla P}{|\vec{\nabla} P|}$

The differential displacement along the normal to an isobaric surface:

$dn = \hat{n} \cdot d\vec{r}$ (i.e. dn is the infinitesimal displacement along the normal direction to the isobar)

$$\rightarrow \frac{d}{dn} = \hat{n} \cdot \vec{\nabla}$$

$$\underbrace{\hat{n} \cdot \vec{g}_{eff}}_{= g_{eff}} = -(1 - r^2 \sin^2 \vartheta \, \Omega \alpha)\, \hat{n} \cdot \vec{\nabla} \Psi \rightarrow \boxed{g_{eff} = -(1 - r^2 \sin^2\theta \, \Omega \alpha) \frac{d\Psi}{dn}}$$

→ the magnitude of the effective gravity along the normal to an isobaric surface

➢Using the above, the equation of hydrostatic equilibrium

$$\boldsymbol{\nabla} P = \varrho \boldsymbol{g_{eff}}$$

can be written as:

$$\frac{dP}{dn} = -\varrho\left(1 - r^2 \sin^2\vartheta\Omega\alpha\right)\frac{d\Psi}{dn}$$

constant on isobars.

g_{eff} $\Rightarrow \frac{dn}{d\Psi} = \frac{1-r^2\sin^2\vartheta\,\Omega a}{g_{eff}}$

➢To use the mass M_P inside an isobar as the independent variable (as we have done for the non-rotating stars):

$$dM_P = \int_{\Psi=con} \varrho\, dn\, d\sigma = d\Psi \int_{\Psi=const} \varrho \frac{dn}{d\Psi} d\sigma$$

$$= d\Psi \int_{\Psi=const} \varrho \frac{\left(1 - r^2 \sin^2 \vartheta\Omega\alpha\right)}{g_{\rm eff}} d\sigma$$

Using $< g_{\text{eff}}{}^{-1} > \equiv \frac{1}{S_P} \oint g_{\text{eff}}{}^{-1} d\sigma$ we get

$$\frac{d\Psi}{dM_P} = \frac{1}{\varrho(1-r^2 \sin^2 \vartheta \Omega \alpha) < g_{\text{eff}}^{-1} > S_P}$$

$$\Rightarrow \frac{dP}{dM_P} = \frac{-1}{\langle g_{\text{eff}}^{-1} \rangle S_P}$$

We define: $f_P \equiv \frac{4\pi r_P^4}{G M_P S_P} \frac{1}{\langle g_{\text{eff}}^{-1} \rangle}$

So the eq. of hydrostatic equilibrium is written as:

$$\boxed{\frac{dP}{dM_P} = -\frac{G M_P}{4\pi r_P^4} f_P}$$ ($f_P = 1$, for non-rotating stars)

For the continuity equation:

The volume of a shell can be written as:

$$dV_P = 4\pi r_P^2 dr_P$$

Also it can be written as:

$$dV_P = \int_{\Psi=\mathrm{const}} dn d\sigma = d\Psi \int_{\Psi=\mathrm{const}} \frac{dn}{d\Psi} d\sigma$$
$$= d\Psi \int_{\Psi=\mathrm{const}} \frac{\left(1-r^2\sin^2\vartheta\Omega\alpha\right)}{g_{\mathrm{eff}}} d\sigma$$

(where we used that $g_{\mathrm{eff}} = \left(1 - r^2 \sin^2\vartheta\Omega\alpha\right)\frac{d\Psi}{dn}$)

Using also $< \mathrm{q} > \equiv \frac{1}{S_P} \oint q d\sigma$

(careful here we do not have $\rho\left(1 - r^2\sin^2\vartheta\Omega\alpha\right)$ which is constant on the isobar, but just $\left(1 - r^2\sin^2\vartheta\Omega\alpha\right)$)

we get $dV_P = d\Psi S_P\left[< g_{\mathrm{eff}}^{-1} > - < g_{\mathrm{eff}}^{-1} r^2 \sin^2\vartheta > \Omega\alpha\right]$

- $dV_P = d\Psi S_P[< g_{\rm eff}^{-1} > - < g_{\rm eff}^{-1} r^2 \sin^2\vartheta > \Omega\alpha]$
- $dV_P = 4\pi r_P^2 dr_P$
- $\frac{d\Psi}{dM_P} = \frac{1}{\varrho(1-r^2\sin^2\vartheta\Omega\alpha)<g_{\rm eff}^{-1}>S_P}$

$$\frac{dr_p}{dM_p} = \frac{dr_p}{dV_p}\cdot\frac{dV_p}{dM_p} = \frac{dr_p}{dV_p}\cdot\frac{d\Psi\, S_p[<g_{\rm eff}^{-1}> - <g_{\rm eff}^{-1} r^2\sin^2\vartheta>\Omega\alpha]}{dM_p} \Rightarrow$$

$$\frac{dr_p}{dV_p} = \frac{1}{4\pi r_p^2}$$

$\Rightarrow \frac{dr_P}{dM_P} = \frac{1}{4\pi r_P^2 \bar{\varrho}}$, with $\bar{\varrho} = \frac{\varrho(1-r^2\sin^2\vartheta\Omega\alpha)<g_{\rm eff}^{-1}>}{\langle g_{\rm eff}^{-1}\rangle - <g_{\rm eff}^{-1} r^2\sin^2\vartheta>\Omega\alpha}$

$\bar{\rho}$: average density in the element volume between two isobars

$<\rho>$: average density on an isobar

Generally $\bar{\rho} \neq <\rho>$

The Energy Conservation and Radiative Equilibrium in Rotating Stars

Equation of radiative transfer

exercise

$$F = -\frac{4acT^3}{3\kappa\varrho}\frac{dT}{dn}$$

$$dT/dn = (dT/dM_P)(dM_P/d\Psi)(d\Psi/dn)$$

$$g_{\text{eff}} = \left(1 - r^2\sin^2\vartheta\Omega\alpha\right)\frac{d\Psi}{dn}$$

$$\frac{d\Psi}{dM_P} = \frac{1}{\varrho(1-r^2\sin^2\vartheta\Omega\alpha)<g_{\text{eff}}^{-1}>S_P}$$

$$F = -\frac{4acT^3}{3\kappa\varrho}\frac{dT}{dM_P}\varrho < g_{\text{eff}}^{-1} > S_P g_{\text{eff}}$$

Integrating over the surface we obtain (using the definition of $< q >$)

exercise

$$L_P = -\frac{4a}{3} < g_{\rm eff}^{-1} > S_P^2 < \frac{T^3 g_{\rm eff}}{\kappa} \frac{dT}{dM_P} >$$

This expresses radiative transfer in rotating stars.

Conservation of energy in rotating stars

- The net energy outflow dL_P from a shell limited by the isobars Ψ and $\Psi + d\Psi$ is equal to

$$dL_P = \int_{\Psi=\mathrm{const}} \varepsilon\varrho dn d\sigma = d\Psi \int_{\Psi=\mathrm{con}} \varepsilon\varrho \frac{dn}{d\Psi} d\sigma$$

where ε is the net rate of energy production in the shell.

- Using $g_{\mathrm{eff}} = \left(1 - r^2 \sin^2 \vartheta\Omega\alpha\right)\frac{d\Psi}{dn}$

and the fact that $\varrho\left(1 - r^2 \sin^2 \vartheta\Omega\alpha\right)$ is constant on an isobar, we get

$$dL_P = d\Psi < \frac{\varepsilon}{g_{\mathrm{eff}}} > S_P\varrho\left(1 - r^2 \sin^2 \theta\, \Omega\alpha\right)$$

and using also

$$\frac{d\Psi}{dM_P} = \frac{1}{\varrho(1-r^2\sin^2\vartheta\Omega\alpha)<g_{\mathrm{eff}}^{-1}>S_P}$$

$$\frac{dL_P}{dM_P} = \frac{\langle \varepsilon g_{\mathrm{eff}}^{-1}\rangle}{\langle g_{\mathrm{eff}}^{-1}\rangle}$$

where generally $\varepsilon = \varepsilon_{\mathrm{nucl}} - \varepsilon_\nu + \varepsilon_{\mathrm{grav}}$

exercise

Stellar structure equations for rotating stars

✓ $\frac{dP}{dM_P} = -\frac{GM_P}{4\pi r_P^4} f_P$, where $f_P = \frac{4\pi r_P^4}{GM_P S_P} \frac{1}{\langle g_{\rm eff}^{-1} \rangle}$, $\quad \boldsymbol{g_{eff}} = \varrho\left(\boldsymbol{\nabla}\Psi + r^2 \sin^2\vartheta\ \Omega\ \boldsymbol{\nabla}\Omega\right)$

✓ $\frac{dr_P}{dM_P} = \frac{1}{4\pi r_P^2 \bar{\varrho}}$, where $\bar{\varrho} = \frac{\varrho\left(1 - r^2 \sin^2\vartheta \Omega \alpha\right) < g_{\rm eff}^{-1} >}{< g_{\rm eff}^{-1} > - < g_{\rm eff}^{-1} r^2 \sin^2\vartheta > \Omega\alpha}$

✓ $L_P = -\frac{4ac}{3} < g_{\rm eff}^{-1} > S_P^2 < \frac{T^3 g_{\rm eff}}{\kappa} \frac{dT}{dM_P} >$

✓ $\frac{dL_P}{dM_P} = \frac{<\left(\varepsilon_{\rm nucl} - \varepsilon_\nu + \varepsilon_{\rm grav}\right) g_{\rm eff}^{-1}>}{< g_{\rm eff}^{-1} >}$

$< q > \equiv \frac{1}{S_P} \oint q\, d\sigma$, S_P is the total surface of the isobar, $d\sigma = \frac{r^2 \sin\vartheta\, d\varphi\, d\vartheta}{\cos\varepsilon}$, $\cos\varepsilon = -\frac{\boldsymbol{g_{eff}} \cdot \boldsymbol{r}}{|\boldsymbol{g_{eff}}| \cdot |\boldsymbol{r}|}$

Note: density and temperature are not constant on isobars

“Simplified” Stellar structure equations for rotating stars

Independent variables:

- $\bar{\varrho} = \frac{\varrho(1-r^2\sin^2\vartheta\Omega\alpha)<g_{\text{eff}}^{-1}>}{<g_{\text{eff}}^{-1}>-<g_{\text{eff}}^{-1}r^2\sin^2\vartheta>\Omega\alpha}$
- $\bar{\mathrm{T}}$ derived from the EOS for the $\boldsymbol{P}$ of the isobar, and $\bar{\varrho}$

(The chemical composition is supposed to be homogeneous on an isobaric surface due to the strong horizontal turbulence.)

Approximations:

- $\frac{<(\varepsilon_{\text{nucl}}-\varepsilon_{\nu}+\varepsilon_{\text{grav}})g_{\text{eff}}^{-1}>}{<g_{\text{eff}}^{-1}>} \approx \varepsilon_{\text{nucl}}(\bar{\varrho},\bar{T}) - \varepsilon_{\nu}(\bar{\varrho},\bar{T}) + \varepsilon_{\text{grav}}(\bar{\varrho},\bar{T})$
- $< \frac{T^3 g_{\text{eff}}}{\kappa}\frac{dT}{dM_P} > \approx \frac{\bar{T}^3<g_{\text{eff}}>}{\kappa(\bar{\varrho},\bar{T})}\frac{d\bar{T}}{dM_P}$
- $< \frac{d\ln T}{d\ln P} > = < \nabla_{\text{ad}} > \approx \frac{d\ln\bar{T}}{d\ln P}$ (for convective regions, where the temperature gradient is the adiabatic gradient)

→The equations of stellar structure become:

$$\frac{dP}{dM_P} = -\frac{GM_P}{4\pi r_P^4} f_P, \qquad \frac{dr_P}{dM_P} = \frac{1}{4\pi r_P^2 \overline{\varrho}},$$

$$\frac{dL_P}{dM_P} = \varepsilon_{\mathrm{nucl}} - \varepsilon_\nu + \varepsilon_{\mathrm{grav}},$$

$$\frac{d\ln T}{dM_P} = -\frac{GM_P}{4\pi r_P^4} f_P \min\left[\nabla_{\mathrm{ad}}, \nabla_{\mathrm{rad}} \frac{f_T}{f_P}\right],$$

where $f_P = \frac{4\pi r_P^4}{GM_P S_P} \frac{1}{\langle g_{\mathrm{eff}}^{-1} \rangle}$

and $f_T = \left(\frac{4\pi r_P^2}{S_P}\right)^2 \frac{1}{\langle g_{\mathrm{eff}} \rangle \langle g_{\mathrm{eff}}^{-1} \rangle}$.

$$\frac{d\ln T}{dM_P} = -\frac{GM_P}{4\pi r_P^4} f_P \min\left[\nabla_{\mathrm{ad}}, \nabla_{\mathrm{rad}}\frac{f_T}{f_P}\right]$$

$$L_P = -\frac{4ac}{3} < g_{\mathrm{eff}}^{-1} > S_P^2 < \frac{T^3 g_{\mathrm{eff}}}{\kappa}\frac{dT}{dM_P} > \cong -\frac{4a}{3} < g_{\mathrm{eff}}^{-1} > S_P^2 \frac{\bar{T}^3 <g_{\mathrm{eff}}>}{\kappa(\bar{\varrho},\bar{T})}\frac{d\bar{T}}{dM_P} \Rightarrow$$

$$\frac{d\bar{T}}{dM_P} = -\frac{3\kappa}{4a}\frac{L_P}{S_P^2\bar{T}^3}\frac{1}{<g_{\mathrm{eff}}> <g_{\mathrm{eff}}^{-1}>} \Rightarrow \frac{d\ln^-}{dM_P} = -\frac{3\kappa}{4ac}\frac{L_P}{S_P^2\bar{T}^4}\frac{1}{<g_{\mathrm{eff}}> <g_{\mathrm{eff}}^{-1}>}$$

$$\frac{dT}{dM_P} = \frac{dT}{dP}\frac{dP}{dM_P} = \frac{T}{P}\frac{dT/T}{dP/P}\frac{dP}{dM_P} = \frac{T}{P}\frac{d\ln T}{d\ln P}\frac{dP}{dM_P} \Rightarrow \frac{dT/T}{dM_P} = \frac{d\ln}{dM_P} = \nabla\frac{1}{P}\frac{dP}{dM_P}, \text{where } \nabla = \frac{d\ln}{d\ln}$$

$$\frac{dP}{dM_P} = -\frac{GM_P}{4\pi r_P^4} f_P$$

$$\frac{d\ln T}{dM_P} = -\frac{GM_P}{4\pi r_P^4} f_P \nabla \frac{1}{P}$$

$$\Rightarrow \nabla = -P\frac{d\ln T}{dM_P}\frac{4\pi r_P^4}{G f_P M_P}$$

for radiative transfer we will have $\nabla = P\frac{3\kappa}{4ac}\frac{L_P}{S_P^2\bar{T}^4}\frac{1}{<g_{\mathrm{eff}}> <g_{\mathrm{eff}}^{-1}>}\frac{4\pi r_P^4}{G f_P M_P} \Rightarrow \nabla = \frac{3\kappa L_P P}{16\pi ac G M_P \bar{T}^4}\frac{f_T}{f_P} = \nabla_{\mathrm{rad}}\frac{f_T}{f_P}$

with $f_T = \left(\frac{4\pi r_P^2}{S_P}\right)^2 \frac{1}{\langle g_{\mathrm{eff}}\rangle\langle g_{\mathrm{eff}}^{-1}\rangle}$

Comments

- ✓ the equations describing the hydrostatic equilibrium and the conservation of mass are strictly valid (in the Roche model approximation) in the case of a “shellular rotation law”, provided that $\bar{\varrho}$ is considered as the dependent variable for density.
- ✓ Moreover the strong horizontal turbulence homogenizes the chemical composition and reduces the ϱ and T contrasts on isobars.
- ✓ The above equations are used in models of rotating stars in hydrostatic equilibrium instead of the basic equations for non-rotating single stars
- ✓ small consequences on the evolution with respect to the larger effects of mixing and mass loss induced by rotation
- ✓ at the surface, rotation produces large distortions and enhances convection