Student Solutions Manual: *Modern General Relativity*

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This document gives the solutions for select problems (those marked by ***) at the ends of chapters for the first edition of *Modern General Relativity: Black Holes, Gravitational Waves, and Cosmology* by Mike Guidry (Cambridge University Press, 2019). Unless otherwise indicated, literature references, equation numbers, figure references, table references, and section numbers refer to the print version of that book.

1 Introduction

1.3 This question is ambiguous, since it does not specify whether the curvature is that of the surface itself (which is called *intrinsic curvature*) or whether it is the apparent curvature of the surface seen embedded in a higher-dimensional euclidean space (which is called the *extrinsic curvature*). In general relativity the curvature of interest is usually intrinsic curvature. Then the sheet of paper can be laid out flat and is not curved, the cylinder is *also flat,* with no intrinsic curvature, because one can imagine cutting it longitudinally and rolling it out into a flat surface, but the sphere has finite intrinsic curvature because it cannot be cut and rolled out flat without distortion. The reason that the cylinder seems to be curved is because the 2D surface is being viewed embedded in 3D space, which gives a non-zero *extrinsic curvature,* but if attention is confined only to the 2D surface it has no *intrinsic curvature*. This is a rather qualitative discussion but in later chapters methods will be developed to quantify the amount of intrinsic curvature for a surface.

2 Coordinate Systems and Transformations

2.1 Utilizing Eq. (2.31) to integrate around the circumference of the circle,

$$C = \oint ds = \oint (dx^2 + dy^2)^{1/2} = 2\int_{-R}^{+R} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

subject to the constraint $R^2 = x^2 + y^2$, where the factor of two and the limits are because x ranges from $-R$ to $+R$ over half a circle. The constraints yield $dy/dx = -(R^2 - x^2)^{-1/2}x$, which permits the integral to be written as

$$C = 2\int_{-R}^{R} dx \sqrt{\frac{R^2}{R^2 - x^2}}.$$

Introducing a new integration variable a through $a \equiv x/R$ then gives

$$C = 2R\int_{-1}^{+1} \frac{da}{\sqrt{1-a^2}} = 2\pi R,$$

since the integral is $\sin^{-1} a$. In plane polar coordinates the line element is given by Eq. (2.32) and proceeding as above the circumference is

$$\begin{aligned} C &= \oint ds = \oint (dr^2 + r^2 d\varphi^2)^{1/2} \\ &= \int_0^{2\pi} d\varphi \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2} = R\int_0^{2\pi} d\varphi = 2\pi R, \end{aligned}$$

where $r = R$ has been used, implying that $dr/d\varphi = 0$.

2.4 Using the spherical coordinates

$$u^1 = r \qquad u^2 = \theta \qquad u^3 = \varphi$$

defined through Eq. (2.2) and the results of Example 2.2,

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = 1 \qquad \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = r^2 \qquad \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = r^2 \sin^2\theta,$$

while all non-diagonal components vanish. Thus the metric tensor is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

The corresponding line element is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2,$$

where Eq. (2.29) has been used.

2.8 Taking the scalar products using Eqs. (2.8), (2.9), and (2.20) gives

$$\boldsymbol{e}^i \cdot \boldsymbol{V} = \boldsymbol{e}^i \cdot (V^j \boldsymbol{e}_j) = V^j \boldsymbol{e}^i \cdot \boldsymbol{e}_j = V^j \delta^i_j = V^i,$$
$$\boldsymbol{e}_i \cdot \boldsymbol{V} = \boldsymbol{e}_i \cdot (V_j \boldsymbol{e}^j) = V_j \boldsymbol{e}_i \cdot \boldsymbol{e}^j = V_j \delta^j_i = V_i,$$

which is Eq. (2.22).

2.9 Utilizing that the angle θ between the basis vectors is determined by $\cos\theta = \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 / |\boldsymbol{e}_1||\boldsymbol{e}_2|$, the area of the parallelogram is

$$\begin{aligned} dA &= |\boldsymbol{e}_1||\boldsymbol{e}_2| \sin\theta \, dx^1 dx^2 \\ &= |\boldsymbol{e}_1||\boldsymbol{e}_2| (1 - \cos^2\theta)^{1/2} dx^1 dx^2 \\ &= \left(|\boldsymbol{e}_1|^2 |\boldsymbol{e}_2|^2 - (\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)^2 \right)^{1/2} dx^1 dx^2. \end{aligned}$$

The components of the metric tensor g_{ij} are

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = g_{12} = g_{21} \qquad |\boldsymbol{e}_1||\boldsymbol{e}_1| = \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = g_{11} \qquad |\boldsymbol{e}_2||\boldsymbol{e}_2| = \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = g_{22},$$

so the area of the parallelogram may be expressed as

$$dA = \left(g_{11} g_{22} - g_{12}^2 \right)^{1/2} dx^1 dx^2 = \sqrt{\det g}\, dx^1 dx^2,$$

where $\det g$ is the determinant of the metric tensor. This is the 2D version of the invariant 4D volume element given in Eq. (3.48).

3 Tensors and Covariance

3.2 From Eqs. (3.50) and (3.51) with indices suitably relabeled

$$
\begin{aligned}
A'_{\mu,\nu} - \Gamma'^{\lambda}_{\mu\nu} A'_{\lambda} &= A_{\alpha,\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} + A_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\mu}} \\
&\quad - \left(\Gamma^{\kappa}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\kappa}} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\alpha}}\right)\frac{\partial x^{\gamma}}{\partial x'^{\lambda}} A_{\gamma} \\
&= A_{\alpha,\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} + A_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\mu}} \\
&\quad - \Gamma^{\kappa}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\kappa}}\frac{\partial x^{\gamma}}{\partial x'^{\lambda}} A_{\gamma} - \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\alpha}}\frac{\partial x^{\gamma}}{\partial x'^{\lambda}} A_{\gamma} \\
&= A_{\alpha,\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} + A_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\mu}} \\
&\quad - \Gamma^{\kappa}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} A_{\kappa} - A_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\mu}} \\
&= A_{\alpha,\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} - \Gamma^{\kappa}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} A_{\kappa} \\
&= \left(A_{\alpha,\beta} - \Gamma^{\kappa}_{\alpha\beta} A_{\kappa}\right)\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}},
\end{aligned}
$$

which is Eq. (3.52).

3.3 (a) Since δ^{ν}_{μ} is a rank-2 tensor with the same components in all coordinate systems (see Section 3.8), under a coordinate transformation $g_{\mu\alpha}g^{\alpha\nu} = \delta^{\nu}_{\mu}$ becomes $g'_{\mu\alpha}g'^{\alpha\nu} = \delta^{\nu}_{\mu}$. Since $g_{\mu\nu}$ is a tensor, if we assume $g^{\mu\nu}$ is also a tensor then

$$
g'_{\mu\alpha} = \frac{\partial x^{\kappa}}{\partial x'^{\mu}}\frac{\partial x^{\eta}}{\partial x'^{\alpha}} g_{\kappa\eta}. \qquad g'^{\alpha\nu} = \frac{\partial x'^{\alpha}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} g^{\rho\sigma}.
$$

Then evaluating $g'_{\mu\alpha}g'^{\alpha\nu}$,

$$
g'_{\mu\alpha}\frac{\partial x'^{\alpha}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} g^{\rho\sigma} = \frac{\partial x^{\kappa}}{\partial x'^{\mu}}\frac{\partial x^{\eta}}{\partial x'^{\alpha}} g_{\kappa\eta}\frac{\partial x'^{\alpha}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} g^{\rho\sigma} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \delta^{\nu}_{\mu},
$$

where we have used

$$
\frac{\partial x^{\eta}}{\partial x'^{\alpha}}\frac{\partial x'^{\alpha}}{\partial x^{\rho}} = \delta^{\eta}_{\rho} \qquad g_{\kappa\rho}g^{\rho\sigma} = \delta^{\sigma}_{\kappa}.
$$

Comparing the result

$$
g'_{\mu\alpha}\frac{\partial x'^{\alpha}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} g^{\rho\sigma} = \delta^{\nu}_{\mu}
$$

with $g'_{\mu\alpha}g'^{\alpha\nu} = \delta^\nu_\mu$ requires that

$$g'^{\alpha\nu} = \frac{\partial x'^\alpha}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}g^{\rho\sigma}.$$

which is the transformation law for a rank-2 contravariant tensor. Note that this result is an example of the quotient theorem described in Problem 3.13. Since $g_{\mu\alpha}g^{\alpha\nu} = \delta^\nu_\mu$ and $g_{\mu\nu}$ and δ^ν_μ are known to be tensors, $g^{\mu\nu}$ must also be a tensor.

(b) From Eq. (3.44) an arbitrary rank-2 tensor can be decomposed into a symmetric and antisymmetric part,

$$g_{\mu\nu} = \tfrac{1}{2}(g_{\mu\nu} + g_{\nu\mu}) + \tfrac{1}{2}(g_{\mu\nu} - g_{\nu\mu}).$$

Inserting this in the line element gives

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= \tfrac{1}{2}(g_{\mu\nu} + g_{\nu\mu})dx^\mu dx^\nu + \tfrac{1}{2}(g_{\mu\nu} - g_{\nu\mu})dx^\mu dx^\nu \\ &= \left[g_{\mu\nu} + \tfrac{1}{2}(g_{\nu\mu} - g_{\nu\mu})\right] dx^\mu dx^\nu \\ &= g_{\mu\nu}dx^\mu dx^\nu. \end{aligned}$$

Thus only the symmetric part of $g_{\mu\nu}$ contributes to the line element.

3.8 This problem is adapted from an example in Ref. [88]. From the transformation equations between spherical and cylindrical coordinates assuming $u = (r, \theta, \varphi)$ and $u' = (\rho, \varphi, z)$,

$$\begin{aligned} u'^1 &= \rho = r\sin\theta = u^1 \sin u^2 \\ u'^2 &= \varphi = u^3 \\ u'^3 &= z = r\cos\theta = u^1 \cos u^2 \end{aligned}$$

and the inverse transformations are

$$\begin{aligned} u^1 &= r = \sqrt{\rho^2 + z^2} = \sqrt{(u'^1)^2 + (u'^3)^2} \\ u^2 &= \theta = \tan^{-1}\left(\frac{\rho}{z}\right) = \tan^{-1}\left(\frac{u'^1}{u'^3}\right) \\ u^3 &= \varphi = u'^2. \end{aligned}$$

From these the partial derivative entries in the matrices U and $\hat{U}$ defined in Example 3.7 may be computed directly. For example,

$$\begin{aligned} U^1_2 &= \frac{\partial u'^1}{\partial u^2} = \frac{\partial}{\partial u^2}(u^1 \sin u^2) = u^1 \cos u^2 = r\cos\theta \\ \hat{U}^2_1 &= \frac{\partial u^2}{\partial u'^1} = \frac{\partial}{\partial u'^1}\left[\tan^{-1}\left(\frac{u'^1}{u'^3}\right)\right] = \frac{u'^3}{(u'^1)^2 + (u'^3)^2} = \frac{\cos\theta}{r}. \end{aligned}$$

Computing all the derivatives and assembling them gives

$$U = \begin{pmatrix} \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \qquad \hat{U} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \dfrac{\cos\theta}{r} & 0 & -\dfrac{\sin\theta}{r} \\ 0 & 1 & 0 \end{pmatrix},$$

and by explicit matrix multiplication, $\hat{U}U = I$.

3.12 Multiply both sides of $T_{\mu\nu} = U_{\mu\nu}$ by $\partial x^\mu / \partial x'^\alpha$ and $\partial x^\nu / \partial x'^\beta$ and take the implied sums to give

$$\frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} T_{\mu\nu} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} U_{\mu\nu}.$$

But from Eq. (3.36) this is just $T'_{\mu\nu} = U'_{\mu\nu}$.

3.16 This is a particular example of a scalar product, so it must transform as a scalar. Explicitly,

$$\begin{aligned} ds'^2 &= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\gamma} ds^\gamma \frac{\partial x'^\nu}{\partial x^\delta} ds^\delta \\ &= g_{\alpha\beta} ds^\gamma ds^\delta \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial x^\gamma} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\delta} \\ &= g_{\alpha\beta} ds^\gamma ds^\delta \frac{\partial x^\alpha}{\partial x^\gamma} \frac{\partial x^\beta}{\partial x^\delta} = g_{\alpha\beta} ds^\gamma ds^\delta \delta^\alpha_\gamma \delta^\beta_\delta \\ &= g_{\alpha\beta} ds^\alpha ds^\beta = ds^2 \end{aligned}$$

where Eq. (3.35) has been used. The squared line element (3.39) is clearly a scalar invariant and so it has the same value in all coordinate systems.

3.17 By the usual rank-2 tensor transformation law,

$$T'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}(x).$$

Upon differentiating Eq. (3.66),

$$\frac{\partial x'^\mu}{\partial x^\alpha} = \delta^\mu_\alpha + (\delta u)\, \partial_\alpha X^\mu(x),$$

which may be substituted into the first equation to give

$$\begin{aligned} T'^{\mu\nu}(x') &= \left(\delta^\mu_\alpha + (\delta u)\partial_\alpha X^\mu\right) \left(\delta^\nu_\beta + (\delta u)\partial_\beta X^\nu\right) T^{\alpha\beta}(x) \\ &= \left(\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\alpha (\delta u)\partial_\beta X^\nu + \delta^\nu_\beta (\delta u)\partial_\alpha X^\mu + \mathscr{O}\left(\delta u^2\right)\right) T^{\alpha\beta}(x) \\ &= T^{\mu\nu}(x) + \left[\partial_\beta X^\nu T^{\mu\beta} + \partial_\alpha X^\mu T^{\alpha\nu}\right] \delta u, \end{aligned}$$

where only terms first-order in δu have been retained.

3.21 (a) From Eqs. (3.15)–(3.17) and Example 3.4,

$$\begin{aligned} V(e^\mu) &= V^\nu e_\nu(e^\mu) = \delta^\mu_\nu V^\nu = V^\mu \\ \omega(e_\mu) &= \omega_\nu e^\nu(e_\mu) = \omega_\nu \delta^\nu_\mu = \omega_\mu, \end{aligned}$$

which is Eq. (3.19).

(b) For vectors $V = V^\alpha e_\alpha$, by the chain rule under a coordinate transformation $x^\mu \to x'^\mu$ the basis vectors transform as

$$e_\alpha \to e'_\alpha = \frac{\partial x^\nu}{\partial x'^\alpha} e_\nu .$$

Thus, to keep V invariant under $x^\mu \to x'^\mu$ its components must transform as

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu ,$$

which is equivalent to (3.31), since then

$$\begin{aligned} V \to V' = V'^\mu e'_\mu &= \frac{\partial x'^\mu}{\partial x^\nu} V^\nu \frac{\partial x^\alpha}{\partial x'^\mu} e_\alpha \\ &= \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\alpha}{\partial x'^\mu} V^\nu e_\alpha \\ &= \delta^\alpha_\nu V^\nu e_\alpha \\ &= V^\alpha e_\alpha = V. \end{aligned}$$

3.24 The infinitesimal displacement ds must be invariant under coordinate transformation: $ds = ds'$. Expand both sides in the basis e_μ to give

$$ds = dx^\mu e_\mu = dx'^\mu e'_\mu .$$

But $dx^\mu = \dfrac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$, so

$$\frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu e_\mu = dx'^\mu e'_\mu .$$

This is true generally only if $e'_\mu = \dfrac{\partial x^\nu}{\partial x'^\mu} e_\nu$. By a similar proof, $e'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} e^\nu$.

3.26 (a) The components may be evaluated by inserting basis dual vectors as arguments:

$$(U \otimes V)(e_\mu, e_\nu) = U(e_\mu) V(e_\nu) = U_\mu V_\nu ,$$

where Eq. (3.19) was used.

(b) Insert basis states $\{e^\mu, e^\nu, e_\lambda, e^\varepsilon\}$ as arguments, giving

$$(U \otimes V \otimes \Omega \otimes W)(e^\mu, e^\nu, e_\lambda e^\varepsilon) = U(e^\mu) V(e^\nu) \Omega(e_\lambda) W(e^\varepsilon) = U^\mu V^\nu \Omega_\lambda W^\varepsilon \equiv S^{\mu\nu}{}_\lambda{}^\varepsilon ,$$

where Eq. (3.19) was used.

(c) Generalizing part (a), the tensor product is defined through

$$(T \otimes V)(A, B, C)) = T(A, B) V(C).$$

Inserting basis states as the arguments gives for the mixed-tensor components

$$T(e^\mu, e_\nu) V(e_\gamma) = T^\mu{}_\nu V_\gamma \equiv S^\mu{}_{\nu\gamma} .$$

(d) The tensor product gives

$$(e_\mu \otimes e_\nu)(e^\alpha, e^\beta) = e_\mu(e^\alpha) e_\nu(e^\beta) = \delta_\mu^\alpha \, \delta_\nu^\beta,$$

where Eqs. (3.19) and (3.17) were used. Hence

$$T^{\mu\nu}(e_\mu \otimes e_\nu)(e^\alpha, e^\beta) = T^{\mu\nu}\delta_\mu^\alpha \delta_\nu^\beta = T^{\alpha\beta},$$

which are the contravariant components of T. Therefore we can expand T as

$$T = T^{\mu\nu}(e_\mu \otimes e_\nu),$$

and we see that $e_\mu \otimes e_\nu$ acts as a basis for $T = U \otimes V$. More generally, we can expand T in any of the forms

$$T = T^{\mu\nu}(e_\mu \otimes e_\nu) = T_{\mu\nu}(e^\mu \otimes e^\nu) = T_\mu{}^\nu(e^\mu \otimes e_\nu) = T^\mu{}_\nu(e_\mu \otimes e^\nu)$$

by inserting different combinations of basis vectors or basis dual vectors in the preceding derivation.

4 Lorentz Covariance and Special Relativity

4.1 After the transformation given by Eq. (4.20) the line element is

$$\begin{aligned}
ds'^2 &= -c^2(dt')^2 + (dx')^2 + (dy')^2 + (dz')^2 \\
&= -(c\cosh\xi\, dt + \sinh\xi\, dx)^2 + (c\sinh\xi\, dt + \cosh\xi\, dx)^2 + dy^2 + dz^2 \\
&= -c^2dt^2 + dx^2 + dy^2 + dz^2 = ds^2,
\end{aligned}$$

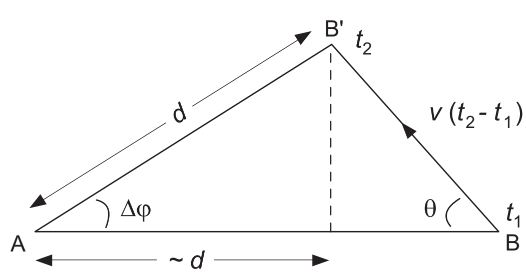
so it is invariant under the transformation.

4.6 The variable ξ is a relativistic velocity parameter so the inverse transformation corresponds to $\xi \to -\xi$. Since $\sinh(-x) = \sinh x$ and $\cosh(-x) = \cosh x$, one has for the inverse of the transformation defined by Eq. (4.20),

$$\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix}.$$

Because $i\sinh x = \sin(ix)$ and $\cosh x = \cos(ix)$, the Lorentz boost of Eq. (4.20) may be interpreted a a rotation through an imaginary angle.

4.8 From the diagram



(i) $\Delta\varphi \simeq v\delta t \sin\theta/d$. Observer A sees the light from B at t_1' and from B$'$ at t_2', with

$$t_1' = t_1 + \frac{d + v\delta t\cos\theta}{c} \qquad t_2' = t_2 + \frac{d}{c}.$$

The time measured at A for the source to move from B to B$'$ is

$$\begin{aligned}
\Delta t &= t_2' - t_1' = t_2 + \frac{d}{c} - t_1 - \frac{d + v\delta t\cos\theta}{c} \\
&= t_2 - t_1 - \frac{v\delta t\cos\theta}{c} = \delta t(1 - \beta\cos\theta),
\end{aligned}$$

where $\beta \equiv v/c$ and $\delta t \equiv t_2 - t_1$. Then the apparent transverse velocity for the motion B to B′ observed at A is

$$\beta_{\rm T} \equiv \frac{v_{\rm T}}{c} = \frac{d}{c}\frac{\Delta\varphi}{\Delta t} = \frac{d}{c}\frac{v\delta t \sin\theta/d}{\delta t(1-\beta\cos\theta)} = \frac{\beta\sin\theta}{1-\beta\cos\theta}.$$

(ii) The maximum for $\beta_{\rm T}$ is found from the above formula by setting $\partial\beta_{\rm T}/\partial\theta = 0$. Taking the derivative, setting it equal to zero, and using $\sin(\cos^{-1}\beta) = (1-\beta^2)^{1/2}$, yields that the maximum value of $\beta_{\rm T}$ is

$$\beta_{\rm T}^{\rm max} = \frac{\beta}{1-\beta^2},$$

where β is the actual velocity (in units of c) and $\beta_{\rm T}$ is the apparent velocity. Thus, as β approaches its physical maximum of unity, the apparent transverse velocity grows without bound and it is possible to observe any transverse velocity, even those appearing to exceed the speed of light.

(iii) Setting $\theta = 10^\circ$ and $\beta = 0.995$, gives from the preceding formula $\beta_{\rm T}^{\rm max} = 8.6$. Thus the apparent transverse velocity is observed to be 8.6 times that of light (superluminal), even though the actual transverse velocity is only $v = 0.995c$ (subluminal).

4.15 (a) From Eq. (4.50) the field tensor is defined by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Under the gauge transformation $A^\mu \to A^\mu - \partial^\mu\chi$ given by (4.46), this transforms as

$$\begin{aligned} F'^{\mu\nu} &= \partial^\mu(A^\nu - \partial^\nu\chi) - \partial^\nu(A^\mu - \partial^\mu\chi) \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}, \end{aligned}$$

where $\partial^\mu\partial^\nu = \partial^\nu\partial^\mu$ was used. Thus $F^{\mu\nu}$ is gauge invariant. It is also Lorentz invariant, since it is by explicit construction a rank-2 Lorentz tensor [Eq. (4.50) is a tensor equation].

(b) Appeal to Eqs. (4.33) to construct the components of the electric and magnetic fields in terms of the potentials. For example, writing some components of Eq. (4.33) out explicitly,

$$E^1 = \partial^1 A^0 - \partial^0 A^1 = F^{10} = -F^{01} \qquad B^2 = \partial^1 A^3 - \partial^3 A^1 = F^{13} = -F^{31}.$$

Proceeding in this manner, one finds that the six independent components of $\boldsymbol{E}$ and $\boldsymbol{B}$ are elements of the antisymmetric rank-2 *electromagnetic field tensor*

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

given by Eqs. (4.50) and (4.51).

(c) The equivalence may be established by multiplying the terms of Eqs. (4.53) and (4.54) out explicitly using Eqs. (4.51), (4.52), and (4.43). For example, setting $\nu = 0$ in Eq. (4.53) and using Eq. (4.51) gives

$$\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3 = j^0,$$

which is equivalent to Eq. (4.28).

5 Lorentz-invariant Dynamics

5.1 The variational principle $\delta \int ds = 0$ with line element $ds^2 = \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu d\tau^2$ (where $\dot{x}^\mu \equiv dx^\mu/d\tau$) implies the Euler–Lagrange equation (5.18),

$$-\frac{d}{d\tau}\left(\frac{\partial L}{\partial(dx^\mu/d\tau)}\right) + \frac{\partial L}{\partial x^\mu} = 0 \qquad L \equiv \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}.$$

But $\partial L/\partial x^\mu = 0$ and the preceding equation becomes

$$\frac{d}{d\tau}\left[\frac{\partial}{\partial \dot{x}^\mu}(-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{1/2}\right] = 0.$$

Using that $\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ is independent of time, this reduces to $\eta_{\mu\nu}\frac{d}{d\tau}\dot{x}^\nu = \eta_{\mu\nu}\ddot{x}^\nu = 0$, where $\ddot{x}^\nu \equiv d^2x^\nu/d\tau^2$. In matrix form this is

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{t} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = 0,$$

implying that trajectories obey

$$\frac{d^2t}{d\tau^2} = 0 \qquad \frac{d^2x}{d\tau^2} = 0 \qquad \frac{d^2y}{d\tau^2} = 0 \qquad \frac{d^2z}{d\tau^2} = 0.$$

This corresponds to straight lines in Minkowski space.

5.8 *By dimensional analysis:* For period P, total mass M, and reduced mass μ, the luminosity L in geometrized units is

$$L = \frac{128}{5}4^{2/3}M^{4/3}\mu^2\left(\frac{\pi}{P}\right)^{10/3},$$

Let the standard unit of length be $\mathscr{L}$, the standard unit of mass be $\mathscr{M}$, and the standard unit of time be $\mathscr{T}$. In standard units L has dimension $\mathscr{M}\mathscr{L}^2\mathscr{T}^{-3}$ (e.g., $\mathrm{erg\,s^{-1}}$). Since μ and M have units of $\mathscr{M}$ and P has units of $\mathscr{T}$, the right side of the equation for L must be multiplied by a factor of powers of G and c having the units of $\mathscr{M}^{-7/3}\mathscr{T}^{1/3}\mathscr{L}^2$ to be dimensionally correct. Because of the mass dependence of the equation G can enter only as the $7/3$ power so the required overall factor is $G^{7/3}/c^5$ and in standard units

$$L = \frac{128}{5}4^{2/3}\frac{G^{7/3}}{c^5}M^{4/3}\mu^2\left(\frac{\pi}{P}\right)^{10/3}.$$

Inserting G in units of ${M_\odot}^{-1}\mathrm{cm^3 s^{-2}}$ and c in units of $\mathrm{cm\,s^{-1}}$, the factor $G^{7/3}/c^5$ can be evaluated and units apportioned conveniently as

$$L = 2.3 \times 10^{45} \left(\frac{M}{M_\odot}\right)^{4/3} \left(\frac{\mu}{M_\odot}\right)^2 \left(\frac{1\ \mathrm{s}}{P}\right)^{10/3} \ \mathrm{erg\ s^{-1}}.$$

More automatically: Table B.1 indicates that conversion from geometrized units to standard units requires the replacements $M \to GM/c^2$, $\mu \to G\mu/c^2$, $P \to cP$, and $L \to (G/c^5)L$. Making these replacements and rearranging gives the same equations as above.

5.11 Substituting $x^\mu \to x'^\mu = x^\mu + \varepsilon K^\mu$ into

$$g_{\mu\nu}(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}\, g_{\alpha\beta}(x')$$

(in the following terms of order ε^2 are discarded) leads to

$$\begin{aligned} g_{\mu\nu}(x) &= \frac{\partial}{\partial x^\mu}\left(x^\alpha + \varepsilon K^\alpha\right)\frac{\partial}{\partial x^\nu}\left(x^\beta + \varepsilon K^\beta\right) g_{\alpha\beta}(x^\gamma + \varepsilon K^\gamma) \\ &= \left(\delta^\alpha_\mu + \varepsilon \partial_\mu K^\alpha\right)\left(\delta^\beta_\nu + \varepsilon \partial_\nu K^\beta\right) g_{\alpha\beta}(x'). \end{aligned}$$

Now expand $g_{\alpha\beta}(x')$ in a Taylor series around $g_{\alpha\beta}(x)$,

$$g_{\alpha\beta}(x') = g_{\alpha\beta}(x) + \left.\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right|_x \Delta x^\gamma = g_{\alpha\beta}(x) + \varepsilon K^\gamma \partial_\gamma g_{\alpha\beta}(x) + \mathscr{O}\left(\varepsilon^2\right),$$

where $\Delta x^\gamma = \varepsilon K^\gamma$. Combining the preceding two equations gives

$$\begin{aligned} g_{\mu\nu}(x) &= \left(\delta^\alpha_\mu + \varepsilon \partial_\mu K^\alpha\right)\left(\delta^\beta_\nu + \varepsilon \partial_\nu K^\beta\right)\left(g_{\alpha\beta}(x) + \varepsilon K^\gamma \partial_\gamma g_{\alpha\beta}(x)\right) \\ &= g_{\mu\nu}(x) + \varepsilon\left[K^\gamma \partial_\gamma g_{\mu\nu}(x) + \partial_\mu K^\beta g_{\beta\nu} + \partial_\nu K^\beta g_{\mu\beta} + \mathscr{O}\left(\varepsilon^2\right)\right] \end{aligned}$$

Subtracting $g_{\mu\nu}(x)$ from both sides and noting that ε is arbitrary, this can hold generally only if the quantity in square brackets vanishes. This leads to

$$g_{\mu\beta}\partial_\nu K^\beta + g_{\nu\beta}\partial_\mu K^\beta + K^\gamma \partial_\gamma g_{\mu\nu} = 0,$$

or equivalently,

$$\partial_\nu K_\mu + \partial_\mu K_\nu + K^\gamma \partial_\gamma g_{\mu\nu} = 0,$$

where contraction with the metric tensor was used to lower indices.

5.12 From Problem 5.11 or Box 5.3

$$\partial_\nu K_\mu + \partial_\mu K_\nu + K^\gamma \partial_\gamma g_{\mu\nu} = 0.$$

But from the formula for the Lie derivative given in Eq. (3.74)

$$\begin{aligned} \mathscr{L}_K g_{\mu\nu} &= K^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu K^\alpha + g_{\nu\alpha}\partial_\mu K^\alpha \\ &= \partial_\nu K_\mu + \partial_\mu K_\nu + K^\gamma \partial_\gamma g_{\mu\nu}. \end{aligned}$$

For the Lie derivative in a metric space, partial and covariant derivative operations are interchangeable (see Section 3.13.5), so

$$\begin{aligned}\mathscr{L}_K g_{\mu\nu} &= \partial_\nu K_\mu + \partial_\mu K_\nu + K^\gamma \partial_\gamma g_{\mu\nu} \\ &= \nabla_\nu K_\mu + \nabla_\mu K_\nu + K^\gamma \nabla_\gamma g_{\mu\nu} = 0,\end{aligned}$$

and since $\nabla_\gamma g_{\mu\nu} = 0$ from Eq. (3.63),

$$\nabla_\nu K_\mu + \nabla_\mu K_\nu = \partial_\nu K_\mu + \partial_\mu K_\nu = 0,$$

which is Killing's equation.

5.13 Choose an inertial frame in which the star at rest and assume the emitted light to be monochromatic with frequency ω_0. The wavevector for the photon is $k^\mu = (\omega_0, \omega_0, 0, 0)$. For photons $E = \hbar\omega$, so from Eq. (5.25)

$$E = \hbar\omega = -p \cdot u = -\hbar k \cdot u,$$

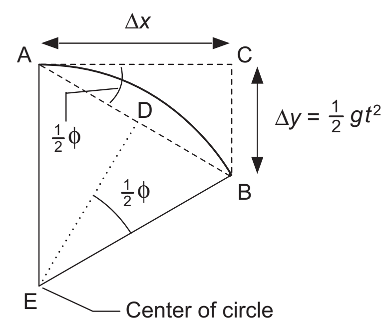
implying that $\omega = -k \cdot u$, where u is the velocity of the observer in the inertial frame of the star. Writing this scalar product out explicitly gives

$$\begin{aligned}\omega &= -k \cdot u = -\eta_{\mu\nu} k^\mu u^\nu = k^0 u^0 - k^1 u^1 \\ &= \omega_0 \left(\cosh(\tau/a) - \sinh(\tau/a)\right) = \omega_0 e^{-\tau/a},\end{aligned}$$

where τ is the proper time and $u = (\cosh(\tau/a), \sinh(\tau/a))$ from the solution of Problem 5.10 was used.

6 The Principle of Equivalence

6.1 Consider the following figure, where the arc approximates the path of the photon.



The flight time of the photon is $t \simeq \Delta x/c$ for a small arc. In the time t the elevator falls a distance $\Delta y = \frac{1}{2}gt^2$, where g is the gravitational acceleration. From the geometry, triangles EBD and ABC contain the same angles, so $\overline{BC}\,/\,\overline{AC} = \overline{DB}\,/\,\overline{DE}$. But from the figure one obtains

$$r_\mathrm{c} \simeq \overline{DE} \qquad \overline{AC} = \Delta x \qquad \overline{BC} = \Delta y = \tfrac{1}{2}gt^2 \qquad \overline{DB} = \frac{\Delta x}{2\cos(\varphi/2)},$$

where r_c is the curvature radius and the last relation follows because $\overline{AD} = \overline{DB}$, implying that $\frac{1}{2}\Delta x/\overline{DB} = \cos(\varphi/2)$. Therefore,

$$\frac{\frac{1}{2}gt^2}{\Delta x} = \frac{\Delta x/[2\cos(\varphi/2)]}{r_\mathrm{c}}.$$

Assuming a small deflection justifies the approximation $\cos(\varphi/2) \simeq 1$ and

$$r_\mathrm{c} = \frac{c^2}{g} = \frac{c^2R^2}{GM},$$

where in the rightmost expression g has been evaluated at the surface of a sphere of mass M and radius R. Quantities calculated for the Earth, a white dwarf, and a neutron star are displayed in the following table.

Object	R(km)	M(kg)	ρ(g cm^{-3})	g(m s^{-2})	r_c(km)	R/r_c
Earth	6378	6×10^{24}	5.6	9.8	9.2×10^{12}	6.9×10^{-10}
White dwarf	5500	2.1×10^{30}	$\sim 10^6$	4.6×10^6	1.9×10^7	2.8×10^{-4}
Neutron star	10	2×10^{30}	$\sim 10^{14}$	1.3×10^{12}	67.5	0.15

The corresponding vertical deflection of the light is $\Delta y = \frac{1}{2}gt^2$, where t is the flight time for the light. For an elevator width of 2 meters, $t = 6.67 \times 10^{-9}$ seconds and the vertical deflection Δy is 2.2×10^{-16} m for Earth, 1.0×10^{-10} m for the white dwarf, and 3×10^{-5} m for the neutron star.

6.2 The particle created at z_2 has mass $m = h\nu/c^2$, where h is Planck's constant and ν is the frequency of the photon. Upon dropping to z_1 in the gravitational field, the energy is $mc^2 + mg(z_2 - z_1)$. Thus, the system creates spontaneously an energy $mg(z_2 - z_1)$ in each cycle, unless the photon loses an energy $h\nu g(z_2 - z_1)/c^2$ in moving from z_1 to z_2.

6.3 Apply Kepler's laws to the approximately circular orbit of period 12 hours, giving $r \simeq 2.7 \times 10^7$ m and $v \simeq 3.9\,\mathrm{km\,s^{-1}}$. Defining $\beta = v/c$, the special relativistic time dilation factor for the satellite is $\gamma = (1-\beta^2)^{-1/2} \simeq 1 + \frac{1}{2}\beta^2$, where the small effect of Earth's rotation has been neglected. The fractional change in frequency is determined by the second term,

$$\frac{\nu_\mathrm{s} - \nu_0}{\nu_0} = -\tfrac{1}{2}\beta^2 = -8.5 \times 10^{-11},$$

where the negative sign is because the time is dilated ($\nu_\mathrm{s} < \nu_0$) for the satellite viewed from Earth. For the general relativistic time dilation, integrating Eq. (6.7) gives

$$\int_{\nu_0}^{\nu_\mathrm{s}} \frac{d\nu}{\nu} = \int_R^{r_\mathrm{s}} \frac{GM}{r^2c^2}dr.$$

Evaluating the integrals on both sides yields (see the solution of Problem 6.7)

$$\frac{\nu_\mathrm{s}}{\nu_0} = \exp\left[-\frac{GM}{c^2}\left(\frac{1}{r_\mathrm{s}} - \frac{1}{R}\right)\right] \simeq 1 - \frac{GM}{c^2}\left(\frac{1}{r_\mathrm{s}} - \frac{1}{R}\right).$$

Solving this for the fractional shift in frequency gives

$$\frac{\nu_\mathrm{s} - \nu_0}{\nu_0} = \frac{GM}{c^2}\left(\frac{1}{R} - \frac{1}{r_\mathrm{s}}\right) = 5.3 \times 10^{-10}.$$

This is opposite in sign relative to the special relativistic effect and about six times larger. Thus, for every second of elapsed time

1. Special relativistic time dilation slows the satellite clock relative to the ground clock by about $8.5 \times 10^{-11} \times 1$ second $= 0.085$ ns.
2. Gravitational time dilation (general relativity) slows the ground clock relative to the satellite clock by about $5.3 \times 10^{-10} \times 1$ second $= 0.53$ ns.

The net effect is that for every second the satellite clock gains about $0.53 - 0.085 = 0.445$ ns relative to the ground clock because of relativistic corrections. Suppose that an accuracy of two meters is desired from the GPS system for locations on the ground. Light takes 6.7 ns to travel two meters. Thus, without the above corrections for special and general relativistic time dilation an error in timing that begins to compromise two-meter resolution will have accumulated after about 15 seconds.

6.8 (a) From Eq. (6.7),

$$\frac{\Delta\nu}{\nu} = \frac{gh}{c^2} = 2.45 \times 10^{-15},$$

where a height of $h = 22.5\,\mathrm{m}$ was used.

(b) To compensate for the gravitational redshift a blue Doppler shift $\Delta\nu_{\mathrm{Doppler}} = v/c$ is required. Setting

$$\frac{gh}{c^2} = \frac{v}{c}$$

and solving for v gives $v = gh/c = 7.35 \times 10^{-5}\,\mathrm{cm\,s^{-1}}$. Notice that this is just the blueshift invoked in Section 6.5.1 to compensate for the gravitational redshift in the falling-elevator thought experiment.

6.10 From Eq. (6.12) the gravitational redshift at the surface of a spherical gravitating body of radius R and mass M is

$$z = 2.12 \times 10^{-6} \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right).$$

The gravitational shift in spectral lines for the Sun is then only $\Delta\lambda/\lambda \sim 2 \times 10^{-6}$. If the gravitational redshift z is parameterized as a velocity giving rise to an equivalent Doppler shift through $v = cz$,

$$v = 0.636 \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)\,\mathrm{km\,s^{-1}}.$$

For the Sun this gives only $\sim 0.6\,\mathrm{km\,s^{-1}}$, which must be disentangled from much larger kinematic Doppler shifts caused by motion of the Sun relative to the Earth and motion of gas in the solar surface. The situation is similar for other main sequence stars since the gravitational redshift is determined to lowest order by M/R, which is the same to within a factor of 2–3 for main sequence stars.

7 Curved Spacetime and General Covariance

7.1 This problem is patterned after an example in Cheng [64].

(a) If the space can be parameterized so that the metric is globally independent of the coordinates, all derivatives in Eq. (7.2) vanish and $K = 0$.

(b) Taking plane polar coordinates $(x^1, x^2) = (r, \theta)$ gives the position-dependent metric

$$g_{11} = 1 \qquad g_{22} = r^2,$$

but insertion into Eq. (7.2) again gives $K = 0$.

(c) Choosing the spherical coordinates $(x^1, x^2) = (S, \varphi)$ defined in Fig. 7.1 gives

$$g_{11} = 1 \qquad g_{22} = R^2 \sin^2\left(\frac{x^1}{R}\right).$$

Insertion into Eq. (7.2) gives $K = R^{-2}$, which is constant.

(d) Choosing the cylindrical coordinates $(x^1, x^2) = (r, \varphi)$ defined in Fig. 7.1 gives

$$g_{11} = \frac{R^2}{R^2 - r^2} \qquad g_{22} = r^2,$$

and insertion into Eq. (7.2) again yields $K = R^{-2}$.

7.2 Since $g_{\mu\nu}$ is a rank-2 covariant tensor its transformation law is given by Eq. (3.56) as

$$g_{\mu\nu;\lambda} = g_{\mu\nu,\lambda} - \Gamma^{\alpha}_{\mu\lambda} g_{\alpha\nu} - \Gamma^{\alpha}_{\nu\lambda} g_{\mu\alpha}.$$

In a local inertial frame space is locally flat and $g_{\mu\nu,\lambda}$ vanishes. Likewise, since from Eq. (7.30) the affine connection is proportional to the derivative of the metric tensor, it also vanishes in the local inertial frame. Thus $g_{\mu\nu;\lambda} = 0$ for a local inertial frame. But this is a tensor equation, so it is valid in all reference frames. Thus the covariant derivative of the metric tensor vanishes in any reference frame. This may be verified by direct computation. For example, it follows from Eqs. (7.30) and (3.56) and some algebra. More elegantly, substituting Eq. (7.29) into Eq. (3.56) gives

$$\begin{aligned}
g_{\mu\nu;\lambda} &= g_{\mu\nu,\lambda} - \Gamma^{\alpha}_{\mu\lambda} g_{\alpha\nu} - \Gamma^{\alpha}_{\nu\lambda} g_{\mu\alpha} \\
&= \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} + \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu} - \Gamma^{\alpha}_{\mu\lambda} g_{\alpha\nu} - \Gamma^{\alpha}_{\nu\lambda} g_{\mu\alpha} \\
&= \Gamma^{\alpha}_{\lambda\mu} g_{\alpha\nu} + \Gamma^{\alpha}_{\lambda\nu} g_{\alpha\mu} - \Gamma^{\alpha}_{\lambda\mu} g_{\alpha\nu} - \Gamma^{\alpha}_{\lambda\nu} g_{\alpha\mu} = 0
\end{aligned}$$

where the last step used that ρ and α are dummy indices and that $g_{\mu\nu}$ and $\Gamma^{\alpha}_{\lambda\mu}$ are symmetric in their lower indices.

7.4 For 2-dimensional polar coordinates (r,θ) the line element is $ds^2 = dr^2 + r^2 d\theta^2$, corresponding to a metric $g_{ij} = \mathrm{diag}(1, r^2)$. From Eq. (5.16) without the minus sign the Lagrangian is

$$L = \left[\left(\frac{dr}{d\sigma} \right)^2 + r^2 \left(\frac{d\theta}{d\sigma} \right)^2, \right]^{1/2}$$

and from Eq. (5.18) the equations of motion are

$$\frac{d^2 r}{d\tau^2} = r \left(\frac{d\theta}{d\tau} \right)^2 \qquad \frac{d}{d\tau} \left(r^2 \frac{d\theta}{d\tau} \right) = \frac{d^2\theta}{d\tau^2} + \frac{2}{r} \frac{dr}{d\tau} \frac{d\theta}{d\tau} = 0,$$

where $\tau = \int_0^1 d\sigma L$, with σ parameterizing a path in the space. For the current 2-dimensional case the geodesic equation (7.21) reduces to

$$\frac{d^2 r}{d\tau^2} = -\Gamma^0_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} \qquad \frac{d^2\theta}{d\tau^2} = -\Gamma^1_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau}$$

Thus, by comparing

$$\frac{d^2 r}{d\tau^2} = r \left(\frac{d\theta}{d\tau} \right)^2 \quad \longleftrightarrow \quad \frac{d^2 r}{d\tau^2} = -\Gamma^0_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau}$$

term by term one deduces that

$$\Gamma^0_{11} = -r \qquad \Gamma^0_{00} = \Gamma^0_{10} = \Gamma^0_{01} = 0,$$

and a corresponding comparison of

$$\frac{d}{d\tau} \left(r^2 \frac{d\theta}{d\tau} \right) = 0 \quad \longleftrightarrow \quad \frac{d^2\theta}{d\tau^2} = -\Gamma^1_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau}$$

yields

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{1}{r} \qquad \Gamma^1_{00} = \Gamma^1_{11} = 0.$$

It is easily checked that the same coefficients result from solution of Eq. (7.30). For example,

$$\Gamma^0_{11} = \frac{1}{2} g^{00} \left(\frac{\partial g_{10}}{\partial \theta} + \frac{\partial g_{10}}{\partial \theta} - \frac{\partial g_{11}}{\partial r} \right) = -r.$$

7.12 The inner product is $g_{\alpha\beta} A^\alpha B^\beta$ and its absolute derivative on a path parameterized by u is

$$\frac{D(g_{\alpha\beta} A^\alpha B^\beta)}{Du} = \frac{D(g_{\alpha\beta})}{Du} A^\alpha B^\beta + g_{\alpha\beta} \frac{DA^\alpha}{Du} B^\beta + g_{\alpha\beta} A^\alpha \frac{DB^\beta}{Du},$$

since the absolute derivative obeys the usual Leibniz rule for derivatives of products. But $DA^\alpha/Du = DB^\beta/Du = 0$ (definition of parallel transport of vectors) and $D(g_{\alpha\beta})/Du = 0$ (property of metric connection; see Section 7.8). Thus $D(g_{\alpha\beta} A^\alpha B^\beta)/Du = 0$ on the path and the inner product is unchanged by parallel transport if the connection is a metric connection.

7.18 If the fluid is at rest $u = (u^0, 0, 0, 0)$, and the normalization $u \cdot u = g_{\mu\nu} u^\mu u^\nu = -1$

gives $u_0 u^0 = -1$, since $u_\mu u^\nu = 0$ unless $\mu = \nu = 0$. Hence all non-diagonal elements of $T^\mu{}_\nu$ vanish and

$$T^0{}_0 = (\varepsilon + P)u_0 u^0 + P = -(\varepsilon + P) + P = -\varepsilon \qquad T^1{}_1 = T^2{}_2 = T^3{}_3 = P$$

in the rest frame of a perfect fluid.

7.19 Suppose a vector field $V(\lambda)$ defined only along a curve $x^\mu(\lambda)$ parameterized by λ in a manifold. Assuming that $V(\lambda) = V^\mu(\lambda)e_\mu(\lambda)$, where $e_\mu(\lambda)$ is a coordinate basis vector evaluated at the point on the curve labeled by λ, then

$$\begin{aligned}\frac{dV}{d\lambda} &= \frac{dV^\mu}{d\lambda}e_\mu + V^\mu \frac{de_\mu}{d\lambda}\\ &= \frac{dV^\mu}{d\lambda}e_\mu + V^\mu \frac{\partial e_\mu}{\partial x^\nu}\frac{dx^\nu}{d\lambda},\end{aligned}$$

where the chain rule was used in the last step. As in Section 2.3, expand the partial derivative factor in the vector basis,

$$\frac{\partial e_\mu}{\partial x^\nu} = \Gamma^\alpha_{\mu\nu} e_\alpha,$$

which gives

$$\begin{aligned}\frac{dV}{d\lambda} &= \frac{dV^\mu}{d\lambda}e_\mu + \Gamma^\alpha_{\mu\nu}V^\mu \frac{dx^\nu}{d\lambda}e_\alpha\\ &= \frac{dV^\mu}{d\lambda}e_\mu + \Gamma^\mu_{\alpha\nu}V^\alpha \frac{dx^\nu}{d\lambda}e_\mu\\ &= \left(\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\nu}V^\alpha \frac{dx^\nu}{d\lambda}\right)e_\mu\\ &\equiv \frac{DV^\mu}{D\lambda}e_\mu,\end{aligned}$$

where in line two dummy summation indices in the second term were interchanged. Thus

$$\frac{DV^\mu}{D\lambda} = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\nu}V^\alpha \frac{dx^\nu}{d\lambda},$$

which is Eq. (3.65) for the absolute or intrinsic derivative.

7.20 (a) Differentiation of the 4-vector V gives

$$\begin{aligned}\frac{dV}{d\lambda} = \frac{d}{d\lambda}\left(V^\mu e_\mu\right) &= \frac{dV^\mu}{d\lambda}e_\mu + \frac{de_\mu}{d\lambda}V^\mu\\ &= \frac{\partial V^\mu}{\partial x^\nu}\frac{dx^\nu}{d\lambda}e_\mu + \frac{\partial e_\mu}{\partial x^\nu}\frac{dx^\nu}{d\lambda}V^\mu\\ &= \partial_\nu V^\mu u^\nu e_\mu + V^\mu u^\nu \partial_\nu e_\mu,\end{aligned}$$

where we have defined

$$u^\nu \equiv \frac{dx^\nu}{d\lambda} \qquad \partial_\nu \equiv \frac{\partial}{\partial x^\nu}.$$

Expanding $\partial_\nu e_\mu = \Gamma^\alpha_{\mu\nu} e_\alpha$ as in Section 2.3 gives

$$\begin{aligned}\frac{dV}{d\lambda} &= \partial_\nu V^\mu u^\nu e_\mu + \Gamma^\alpha_{\mu\nu} V^\mu u^\nu e_\alpha \\ &= \left(\partial_\nu V^\mu + \Gamma^\mu_{\alpha\nu} V^\alpha\right) u^\nu e_\mu,\end{aligned}$$

where dummy summation indices were switched in the second term. This is Eq. (7.14).

(b) From the first part

$$\frac{dV}{d\lambda} = \left(\frac{\partial V^\mu}{\partial x^\nu} + \Gamma^\mu_{\alpha\nu} V^\alpha\right) \frac{dx^\nu}{d\lambda} e_\mu.$$

Multiply both sides by $d\lambda/dx^\beta$ to give

$$\frac{dV}{dx^\beta} = \left(\frac{\partial V^\mu}{\partial x^\beta} + \Gamma^\mu_{\alpha\beta} V^\alpha\right) e_\mu.$$

Take the scalar product of both sides with e^η to give

$$\frac{dV^\mu}{dx^\nu} \equiv \nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\alpha} V^\alpha,$$

where some indices have been renamed. Comparison with Eq. (3.55) indicates that this defines the covariant derivative of the vector V.

8 The General Theory of Relativity

8.2 Take the line element in the form

$$ds^2 = -e^{\sigma} dt^2 + e^{\lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where $\sigma = \sigma(r,t)$ and $\lambda = \lambda(r,t)$. The corresponding metric is

$$g_{\mu\nu} = \mathrm{diag}\,(-e^{\sigma}, e^{\lambda}, r^2, r^2\sin^2\theta)$$
$$g^{\mu\nu} = g^{-1}_{\mu\nu} = \mathrm{diag}\left(-e^{-\sigma}, e^{-\lambda}. \frac{1}{r^2}, \frac{1}{r^2\sin^2\theta}\right)$$

From Eq. (7.30) the non-vanishing Christoffel symbols are

$$\Gamma^0_{00} = \tfrac{1}{2}\dot{\sigma} \qquad \Gamma^0_{01} = \tfrac{1}{2}\sigma' \qquad \Gamma^0_{11} = \tfrac{1}{2}e^{\lambda-\sigma}\dot{\lambda} \qquad \Gamma^1_{00} = \tfrac{1}{2}e^{\sigma-\lambda}\sigma' \qquad \Gamma^1_{01} = \tfrac{1}{2}\dot{\lambda}$$
$$\Gamma^1_{11} = \tfrac{1}{2}\lambda' \qquad \Gamma^1_{22} = -re^{-\lambda} \qquad \Gamma^1_{33} = -re^{-\lambda}\sin^2\theta \qquad \Gamma^2_{12} = 1/r$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta \qquad \Gamma^3_{13} = 1/r \qquad \Gamma^3_{23} = \cot\theta$$

where dots indicate time derivatives and primes indicate derivatives with respect to r.

The Riemann curvature tensor may then be calculated from Eq. (8.14); the non-zero components are

$$R_{0101} = \tfrac{1}{2}e^{\sigma}\sigma'' - \tfrac{1}{4}e^{\lambda}\dot{\lambda}^2 + \tfrac{1}{4}e^{\lambda}\dot{\sigma}\dot{\lambda} - \tfrac{1}{2}e^{\lambda}\ddot{\lambda} + \tfrac{1}{4}e^{\sigma}(\sigma')^2 - \tfrac{1}{4}e^{\sigma}\sigma'\lambda'$$
$$R_{0202} = \tfrac{1}{2}e^{\sigma-\lambda}\sigma' \qquad R_{0212} = \tfrac{1}{2}r\dot{\lambda} \qquad R_{0303} = \tfrac{1}{2}re^{\sigma-\lambda}\sigma'\sin^2\theta$$
$$R_{0313} = \tfrac{1}{2}r\dot{\lambda}\sin^2\theta \qquad R_{1212} = \tfrac{1}{2}r\lambda'$$
$$R_{1313} = \tfrac{1}{2}r\lambda'\sin^2\theta \qquad R_{2323} = r^2\sin^2\theta(1-e^{-\lambda}).$$

The Ricci tensor $R_{\mu\nu}$ then follows from Eq. (8.16), with non-vanishing components

$$R_{00} = -\tfrac{1}{2}e^{\sigma-\lambda}\sigma'' - \tfrac{1}{4}e^{\sigma-\lambda}(\sigma')^2 + \tfrac{1}{2}\ddot{\lambda} + \tfrac{1}{4}\dot{\lambda}^2 - \tfrac{1}{4}\dot{\sigma}\dot{\lambda} + \tfrac{1}{4}e^{\sigma-\lambda}\sigma'\lambda' - e^{\sigma-\lambda}\lambda'/r$$
$$R_{11} = \tfrac{1}{2}\sigma'' + \tfrac{1}{4}(\sigma')^2 - \tfrac{1}{2}e^{\lambda-\sigma}\ddot{\lambda} - \tfrac{1}{4}e^{\lambda-\sigma}\dot{\lambda}^2 + \tfrac{1}{4}e^{\lambda-\sigma}\dot{\sigma}\dot{\lambda} - \tfrac{1}{4}\sigma'\lambda' - \lambda'/r$$
$$R_{01} = -\dot{\lambda}/r \qquad R_{22} = \tfrac{1}{2}re^{-\lambda}\sigma' - \tfrac{1}{2}re^{-\lambda}\lambda' + e^{-\lambda} - 1 \qquad R_{33} = R_{22}\sin^2\theta.$$

The Ricci scalar R may then be constructed from Eq. (8.17),

$$R = 2\frac{e^{-\lambda}}{r^2} + 2\frac{e^{-\lambda}}{r}\sigma' - 2\frac{e^{-\lambda}}{r}\lambda' + e^{-\lambda}\sigma'' + \tfrac{1}{2}e^{-\lambda}(\sigma')^2$$
$$- e^{-\sigma}\ddot{\lambda} - \tfrac{1}{2}e^{-\sigma}\dot{\lambda}^2 - \frac{2}{r^2} + \tfrac{1}{2}e^{-\sigma}\dot{\sigma}\dot{\lambda} - \tfrac{1}{2}e^{-\lambda}\sigma'\lambda'.$$

and the Einstein tensor is

$$
\begin{aligned}
G_{00} &= \frac{e^{\sigma}}{r^2} - \frac{e^{\sigma-\lambda}}{r^2} + \frac{e^{\sigma-\lambda}}{r}\lambda' \qquad G_{01} = -\frac{\dot{\lambda}}{r} \qquad G_{11} = \frac{e^{\lambda}}{r^2} - \frac{1}{r^2} - \frac{\sigma'}{r} \\
G_{22} &= \tfrac{1}{2} r\lambda' e^{-\lambda} - \tfrac{1}{2} r\sigma' e^{-\lambda} - \tfrac{1}{2} r^2 e^{-\lambda}\sigma'' - \tfrac{1}{4} r^2 e^{-\lambda}(\sigma')^2 + \tfrac{1}{2} r^2 e^{-\sigma}\ddot{\lambda} \\
&\quad + \tfrac{1}{4} r^2 e^{-\sigma}(\dot{\lambda})^2 - \tfrac{1}{4} r^2 e^{-\sigma}\dot{\sigma}\dot{\lambda} + \tfrac{1}{4} r^2 e^{-\lambda}\sigma'\lambda' \\
G_{33} &= G_{22}\sin^2\theta,
\end{aligned}
$$

where Eq. (8.20) was used. These results for a general spherical metric are summarized in Appendix C.

8.5 For the $\mu = 0$ part of Eq. (8.4) $d^2x^0/d\tau^2 = 0$. For the $\mu = i$ components, from (8.4) and (8.8),

$$
\frac{d^2x^i}{d\tau^2} - \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}\left(\frac{dx^0}{d\tau}\right)^2 = 0.
$$

But for the first term

$$
\begin{aligned}
\frac{d^2x^i}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{dx^i}{dx^0}\frac{dx^0}{d\tau}\right) = \frac{dx^0}{d\tau}\frac{d}{d\tau}\left(\frac{dx^i}{dx^0}\right) \\
&= \frac{dx^0}{d\tau}\frac{d}{dx^0}\frac{dx^0}{d\tau}\left(\frac{dx^i}{dx^0}\right) = \left(\frac{dx^0}{d\tau}\right)^2\frac{d}{dx^0}\left(\frac{dx^i}{dx^0}\right) \\
&= \frac{1}{c^2}\left(\frac{dx^0}{d\tau}\right)^2\frac{d}{dt}\left(\frac{dx^i}{dt}\right) = \frac{1}{c^2}\left(\frac{dx^0}{d\tau}\right)^2\frac{d^2x^i}{dt^2}
\end{aligned}
$$

where $dx^0 = cdt$ and $dx^0/d\tau =$ constant has been used. Inserting this result in the first equation above yields the second equation of Eq. (8.9).

8.8 From the symmetries (8.15) it may be shown that all contractions of one upper index on the curvature tensor with one of its lower indices either gives $\pm R_{\mu\nu}$ defined in Eq. (8.16), or zero. For example

$$
g^{\lambda\sigma}R_{\mu\lambda\sigma\nu} = R_{\mu}{}^{\sigma}{}_{\sigma\nu} \equiv R'_{\mu\nu}.
$$

But also since $R_{\sigma\mu\nu\lambda} = -R_{\mu\sigma\nu\lambda}$,

$$
g^{\lambda\sigma}R_{\mu\lambda\sigma\nu} = -g^{\lambda\sigma}R_{\lambda\mu\sigma\nu} = -R^{\sigma}{}_{\mu\sigma\nu} = -R_{\mu\nu}.
$$

Thus, $R'_{\mu\nu} = -R_{\mu\nu}$. As another example, multiply $R_{\sigma\mu\nu\lambda} = -R_{\mu\sigma\nu\lambda}$, by $g^{\sigma\mu}$ to give

$$
g^{\sigma\mu}R_{\sigma\mu\nu\lambda} = -g^{\sigma\mu}R_{\mu\sigma\nu\lambda} \;\longrightarrow\; R^{\mu}{}_{\mu\nu\lambda} = -R^{\sigma}{}_{\sigma\nu\lambda} \;\longrightarrow\; R^{\mu}{}_{\mu\nu\lambda} = -R^{\mu}{}_{\mu\nu\lambda}
$$

where dummy (repeated) indices were relabeled. This can be true only if $R^{\mu}{}_{\mu\nu\lambda} = 0$.

8.12 Assume a general spherical metric with line element

$$
ds^2 = -e^{\sigma}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),
$$

where σ and λ are positive and independent of time. From Eqs. (7.10) and (3.57) the equation to be solved is

$$
T^{\mu}_{\nu;\mu} = T^{\mu}_{\nu,\mu} + \Gamma^{\mu}_{\alpha\mu}T^{\alpha}_{\nu} - \Gamma^{\alpha}_{\nu\mu}T^{\mu}_{\alpha} = 0.
$$

The non-zero connection coefficients required are given in the solution to Problem 8.2 and in Appendix C (but here all connection coefficients proportional to time derivatives are neglected, since the metric is static). The preceding equation corresponds to one equation for each of the four possible value of ν, with each equation involving implied sums over the repeated indices μ and α. When written out there are many terms, but a large number are identically zero because by inspection either the connection coefficient vanishes or the term is not diagonal in T^{μ}_{ν}. Collecting the terms that survive gives

$$\begin{aligned} P' + \Gamma^0_{10}T^1_\nu + \Gamma^1_{11}T^1_\nu + \Gamma^2_{12}T^1_\nu + \Gamma^3_{13}T^1_\nu + \Gamma^3_{23}T^2_\nu \\ - \Gamma^0_{\nu 0}T^0_0 - \Gamma^1_{\nu 1}T^1_1 - \Gamma^2_{\nu 2}T^2_2 - \Gamma^3_{\nu 3}T^3_3 = 0, \end{aligned}$$

where a prime denotes a partial derivative with respect to r. This represents four separate equations for the respective choices $\nu = 0,1,2,3$. The only non-trivial result corresponds to setting $\nu = 1$, which gives

$$P' + (P+\rho)\frac{\sigma'}{2} = 0,$$

upon substituting the expressions for the connection coefficients and using $T^0_0 = -\rho$ and $T^1_1 = T^2_2 = T^3_3 = P$.

8.15 From Eq. (8.18) the Bianchi identity is

$$\nabla_\lambda R_{\mu\nu\alpha\beta} + \nabla_\beta R_{\mu\nu\lambda\alpha} + \nabla_\alpha R_{\mu\nu\beta\lambda} = 0.$$

Contract this with $g^{\mu\alpha}$, remembering that since $\nabla_\mu g^{\mu\nu} = 0$, raising an index by contraction commutes with covariant differentiation,

$$\begin{aligned} \nabla_\lambda g^{\mu\alpha} R_{\mu\nu\alpha\beta} + \nabla_\beta g^{\mu\alpha} R_{\mu\nu\lambda\alpha} + \nabla_\alpha g^{\mu\alpha} R_{\mu\nu\beta\lambda} = 0 \\ \to \nabla_\lambda g^{\mu\alpha} R_{\mu\nu\alpha\beta} - \nabla_\beta g^{\mu\alpha} R_{\mu\nu\alpha\lambda} + \nabla_\alpha g^{\mu\alpha} R_{\mu\nu\beta\lambda} = 0 \\ \to \nabla_\lambda R_{\nu\beta} - \nabla_\beta R_{\nu\lambda} + \nabla_\alpha g^{\mu\alpha} R_{\mu\nu\beta\lambda} = 0, \end{aligned}$$

where in the second line the last two indices were switched in the second term using Eq. (8.15) so that the contraction is consistent with the definition (8.16) of the Ricci tensor [see the footnote following Eq. (8.16) and Problem 8.8], and in the last line Eq. (8.16) was used. Now contract with $g^{\nu\beta}$ to give

$$\begin{aligned} \nabla_\lambda g^{\nu\beta} R_{\nu\beta} - \nabla_\beta g^{\nu\beta} R_{\nu\lambda} + \nabla_\alpha g^{\mu\alpha} g^{\nu\beta} R_{\mu\nu\beta\lambda} = 0 \\ \to \nabla_\lambda R - \nabla_\beta R^{\beta}{}_{\lambda} - \nabla_\alpha g^{\mu\alpha} g^{\nu\beta} R_{\nu\mu\beta\lambda} = 0 \\ \to \nabla_\lambda R - \nabla_\beta R^{\beta}{}_{\lambda} - \nabla_\alpha g^{\mu\alpha} R_{\mu\lambda} = 0 \\ \to \nabla_\lambda R - \nabla_\beta R^{\beta}{}_{\lambda} - \nabla_\alpha R^{\alpha}{}_{\lambda} = 0 \\ \to \nabla_\lambda R - 2\nabla_\alpha R^{\alpha}{}_{\lambda} = 0, \end{aligned}$$

where in the second line the first two indices on R in the last term were switched to make the contraction compatible with (8.16) and the definition (8.17) was used, and in the last line the dummy summation indices were switched to the same variable so the last two terms could be added. Now contract with $g^{\mu\lambda}$ to raise the index on the last term,

$$\begin{aligned}\nabla_\lambda R g^{\mu\lambda} - 2\nabla_\alpha g^{\mu\lambda} R^\alpha{}_\lambda &= 0\\ \rightarrow \nabla_\lambda R g^{\mu\lambda} - 2\nabla_\alpha R^{\mu\alpha} &= 0\\ \rightarrow \nabla_\nu R g^{\mu\nu} - 2\nabla_\nu R^{\mu\nu} &= 0,\end{aligned}$$

where dummy summation indices have been switched. Finally, multiply both sides by $-\frac{1}{2}$ to give

$$\nabla_\nu (R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu}) = 0,$$

which is $\nabla_\mu G^{\mu\nu} = 0$ for the symmetric Einstein tensor $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}$ defined in Eq. (8.20).

9 The Schwarzschild Spacetime

9.1 It is convenient to introduce an exponential parameterization $B(r) \equiv e^{\nu(r)}$ and $A(r) \equiv e^{\lambda(r)}$, so that the metric is

$$g_{\mu\nu} = \text{diag}\left(-e^{\nu}, e^{\lambda}, r^2, r^2 \sin^2\theta\right) \qquad g^{\mu\nu} = \text{diag}\left(-e^{-\nu}, e^{-\lambda}, \frac{1}{r^2}, \frac{1}{r^2 \sin^2\theta}\right).$$

The unknown functions $\nu(r)$ and $\lambda(r)$ may then be determined by requiring that the metric be consistent with the vacuum (vanishing stress–energy tensor) Einstein equation. However, for a vacuum solution it is not necessary to construct the full Einstein tensor $G_{\mu\nu}$ because it may be shown (see Problem 22.1) that solution of the vacuum Einstein equation is equivalent to solving $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci tensor. The metric and Eq. (7.30) give the Christoffel symbols (affine connections) $\Gamma^{\gamma}_{\alpha\beta}$, and from these and Eqs. (8.14) and (8.16) the Ricci tensor may be constructed. Setting $R_{\mu\nu} = 0$ yields four equations, only three of which are independent. In particular,

$$\begin{aligned}
-R_{00} &= e^{\nu-\lambda}\left(\tfrac{1}{2}\nu'' + \tfrac{1}{4}(\nu')^2 - \tfrac{1}{4}\nu'\lambda' + \frac{\nu'}{r}\right) = 0 \\
R_{11} &= \tfrac{1}{2}\nu'' + \tfrac{1}{4}(\nu')^2 - \tfrac{1}{4}\nu'\lambda' - \frac{\lambda'}{r} = 0 \\
R_{22} &= e^{-\lambda}\left(\tfrac{1}{2}(\nu' - \lambda')r + 1\right) - 1 = 0,
\end{aligned}$$

where primes indicate derivatives with respect to r. The first two equations imply that $\nu' = -\lambda'$ and thus that $\nu = -\lambda + \text{constant}$. A time-independent solution is sought so the timescale may be shifted freely to make the constant zero and obtain $\nu = -\lambda$. Inserting this into the R_{22} equation gives $re^{\nu}\nu' + e^{\nu} = 1$, the left side of which is equivalent to $d(re^{\nu})/dr$, implying that $e^{\nu} = 1 - c/r$, where c is a constant. Choosing $c = 2M$, where M is another constant (that will be interpreted as the mass when compared with Newtonian gravity),

$$e^{\nu} = 1 - \frac{2M}{r} \qquad e^{\lambda} = e^{-\nu} = \left(1 - \frac{2M}{r}\right)^{-1},$$

which gives the Schwarzschild metric (9.5) when inserted into the equation for $g_{\mu\nu}$ above.

9.2 Use Eq. (9.30) to set $dV_{\text{eff}}/dr = 0$ and obtain

$$r_{\pm} = \frac{\ell^2}{2M} \pm \frac{1}{2}\sqrt{\frac{\ell^4}{M^2} - 12\,\ell^2},$$

from which the desired results follow.

9.5 From Eq. (9.23), $\ell = r^2\,(d\varphi/d\tau)$ if $\theta = \frac{\pi}{2}$ is assumed, and for classical orbital motion

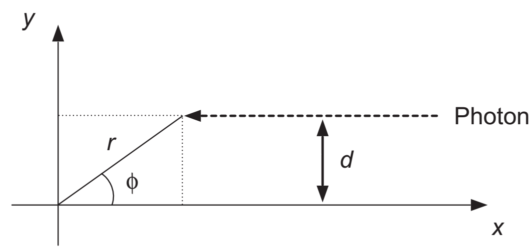
$$r^2\,\frac{d\varphi}{dt} = \sqrt{1-e^2}\left(\frac{2\pi}{P}\right)a^2.$$

Combining these equations assuming weak gravity gives

$$\ell^2 = \left(r^2\,\frac{d\varphi}{d\tau}\right)^2 \simeq \left(r^2\,\frac{d\varphi}{dt}\right)^2 = \frac{4\pi^2}{P^2}(1-e^2)a^4 = (1-e^2)GMa,$$

where in the last step $P^2 = (4\pi^2/GM)a^3$ (Kepler III) was used. Inserting this result in Eq. (9.42) then gives Eq. (9.43).

9.7 For the following diagram,



define polar coordinates in the $\theta = \pi/2$ plane,

$$x = r\cos\varphi \qquad y = r\sin\varphi.$$

Then the parameter b is

$$\begin{aligned} b = \left|\frac{\ell}{\varepsilon}\right| &= \frac{r^2\sin^2\theta\, d\varphi/d\lambda}{(1-2M/r)dt/d\lambda} \\ &= \frac{r^2}{(1-2M/r)}\,\frac{d\varphi}{dt} \\ &= \frac{r^2}{(1-2M/r)}\,\frac{d\varphi}{dr}\,\frac{dr}{dt}, \end{aligned}$$

where Eqs. (9.53)–(9.55) have been used. But for $r \gg 2M$ in the Schwarzschild metric,

$$1-\frac{2M}{r} \to 1 \qquad \varphi \to \frac{d}{r} \qquad \frac{dr}{dt} \to -1.$$

Therefore, $d\varphi/dr \simeq -d/r^2$ and

$$b = \frac{r^2}{(1-2M/r)}\,\frac{d\varphi}{dr}\,\frac{dr}{dt} = r^2\left(\frac{-d}{r^2}\right)(-1) = d,$$

so b may be interpreted as the impact parameter for the photon.

9.15 (a) Combine Eqs. (9.30) and (9.29) with the requirement that $V_{\text{eff}} = E$ to give Eq. (9.32).

(b) The angular velocity is given by

$$\Omega = \frac{d\varphi}{dt}.$$

Substitute Eqs. (9.22) and (9.23) to give Eq. (9.33).

(c) From Eqs. (9.25), (9.5), and (9.35),

$$-\left(1-\frac{2M}{r}\right)(u^t)^2 + r^2(u^t)^2\Omega^2 = -1,$$

This may be solved using Eq. (9.34) to give,

$$u^t = \left(1-\frac{3M}{r}\right)^{-1/2},$$

which is Eq. (9.36).

9.16 Since $u = (u^t, 0, 0, u^t\Omega)$ and $s \cdot u = 0$, utilizing the metric (9.5),

$$s \cdot u = g_{\mu\nu} s^\mu u^\nu = -\left(1-\frac{2M}{R}\right) s^t u^t + R^2 s^\varphi u^\varphi = 0,$$

from which

$$s^t = \frac{R^2 s^\varphi u^\varphi}{(1-2M/R)u^t} = \frac{R^2\Omega}{1-2M/R} s^\varphi,$$

where Eqs. (9.35) and (9.36) was used in the last step. This is Eq. (9.61).

9.17 (i) Our solution follows that of Hartle [110], Ch. 14. Utilizing

$$u = (u^t, 0, 0, u^t\Omega) \qquad s = (s^t, s^r, 0, s^\varphi),$$

the only non-vanishing combinations for the sums in Eq. (9.62) are $\Gamma^r_{tt} s^t u^t$ and $\Gamma^r_{\varphi\varphi} s^\varphi u^\varphi$. From Appendix C, for a general spherical metric with line element

$$ds^2 = -e^\sigma dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

the required connection coefficients are

$$\Gamma^r_{tt} = \Gamma^1_{00} = \tfrac{1}{2} e^{\sigma-\lambda}\sigma' \qquad \Gamma^r_{\varphi\varphi} = \Gamma^1_{33} = -re^{-\lambda}\sin^2\theta,$$

where $\sigma' \equiv d\sigma/dr$. Specializing to the Schwarzschild metric with $\theta = \frac{\pi}{2}$ and letting $r = R$,

$$e^\sigma = \left(1-\frac{2M}{R}\right) \qquad e^\lambda = \left(1-\frac{2M}{R}\right)^{-1} \qquad \sigma = \ln\left(1-\frac{2M}{R}\right)$$

$$\sigma' = \frac{d\sigma}{dr} = \frac{2M}{R^2}\left(1-\frac{2M}{R}\right)^{-1} \qquad \sin^2\theta = 1,$$

and the connection coefficients evaluate to

$$\Gamma^r_{tt} = \frac{M}{R^2}\left(1-\frac{2M}{R}\right) \qquad \Gamma^r_{\varphi\varphi} = -(R-2M)$$

Therefore Eq. (9.62) becomes

$$\frac{ds^r}{d\tau} + \frac{M}{R^2}\left(1-\frac{2M}{R}\right) s^t u^t - (R-2M) s^\varphi u^\varphi = 0.$$

Utilizing $u^t = dt/d\tau$ and Eq. (9.61), this may be written as

$$\frac{ds^r}{d\tau} - (R-3M)\Omega s^\varphi = 0.$$

(ii) Now consider Eq. (9.63). By similar considerations as above, the only nonvanishing contribution to the sum in the second term is $\Gamma^\varphi_{r\varphi} s^r u^\varphi$. From Appendix C the single required connection coefficient is $\Gamma^\varphi_{r\varphi} = r^{-1}$ and Eq. (9.63) becomes

$$\frac{ds^\varphi}{dt} + \frac{\Omega}{R} s^r = 0,$$

where $u^t = dt/d\tau$ and $u^\varphi = \Omega u^t$ from Eq. (9.35) were used (with $\Omega = d\varphi/dt$).

(iii) Thus the two equations

$$\frac{ds^r}{d\tau} - (R-3M)\Omega s^\varphi = 0 \qquad \frac{ds^\varphi}{dt} + \frac{\Omega}{R} s^r = 0$$

must be solved simultaneously. Take d/dt of the second equation and plug in ds^r/dt from the first equation to give

$$\frac{d^2 s^\varphi}{dt^2} + \omega^2 s^\varphi = 0,$$

where $\omega \equiv (1-3M/R)^{1/2}\Omega$. This is the equation of a harmonic oscillator and the solutions are given by Eq. (9.64).

9.18 From Eq. (9.68) evaluated assuming a satellite in orbit around the Earth, the geodetic precession per orbit is

$$\Delta\varphi \simeq \frac{3\pi GM}{c^2 R} = 8.6 \left(\frac{\text{km}}{R}\right) \text{arcsec orbit}^{-1}.$$

For GP-B, setting $R = a = 7027.4\,\text{km}$ gives 1.22×10^{-3} arcsec orbit^{-1}. By Kepler's 3rd law, the period for a circular satellite orbit around the Earth is

$$P = \sqrt{\frac{4\pi^2 a^3}{GM_\oplus}} = 9.95 \times 10^{-3} \left(\frac{a}{\text{km}}\right)^{3/2} \text{s}.$$

For GP-B with $a = 7027.4\,\text{km}$ this gives a period of 97.7 minutes, which translates to 5.38×10^3 orbits yr^{-1}. Therefore general relativity predicts that GP-B should exhibit a geodetic precession rate of

$$\frac{\Delta\varphi}{\Delta t} = (1.22 \times 10^{-3}\,\text{arcsec orbit}^{-1}) \times (5.38 \times 10^3\,\text{orbit yr}^{-1}) \simeq 6.6\,\text{arcsec yr}^{-1}.$$

As discussed in Box 9.3, GP-B measured a geodetic precession rate for its gyroscopes that was within 0.07% of this value.

9.19 The precession rate is given by Eq. (9.76). The angular momentum of the Earth can be expressed as $J_\oplus = \mathscr{I}_\oplus \Omega_\oplus$, where $\mathscr{I}_\oplus$ is the Earth's moment of inertia for rotation about the polar axis and $\Omega_\oplus$ is the angular velocity of the Earth. In planetary science the moment of inertia is parameterized as $\mathscr{I} = kMR^2$, where M is the mass, R is the radius, and k indicates how much the interior mass distribution differs from uniform (for example, a completely uniform sphere has $k = \frac{2}{5} = 0.4$, but Saturn with a centrally-concentrated mass distribution has $k = 0.21$). For the Earth (and other terrestrial planets) $k \simeq 0.33$, so Earth's moment of inertia is

$$\mathscr{I}_\oplus = 0.33 M_\oplus R_\oplus^2 = 8.06 \times 10^{44}\,\mathrm{g\,cm^2},$$

the Earth's angular velocity is

$$\Omega_\oplus = \frac{2\pi}{24\,\mathrm{h}r} = 0.262\,\mathrm{rad\,hr^{-1}} = 7.27 \times 10^{-5}\,\mathrm{rad\,s^{-1}},$$

and thus the Earth's angular momentum is $J = \mathscr{I}_\oplus \Omega_\oplus = 5.86 \times 10^{40}\,\mathrm{g\,cm^2\,s^{-1}}$. Hence the Lense–Thirring precession rate for a gyroscope in free fall on Earth's rotation axis is

$$\Omega_{\mathrm{LT}} = \frac{2GJ}{c^2 z^3} = 5.65 \times 10^{10} \left(\frac{1\,\mathrm{km}}{z}\right)^3 \mathrm{arcsec\,yr^{-1}}.$$

For Gravity Probe B illustrated in Box 9.3, the semimajor axis of the nearly circular polar orbit was 7027.4 km. As the satellite passes over the North Pole it is in free fall with $z =$ 7027.4 km. Inserting this in the above equation gives a precession rate of $0.16\,\mathrm{arcsec\,yr^{-1}}$. The Lense–Thirring precession rate depends on the latitude of a satellite in polar orbit, so the smaller general-relativistic prediction of $0.039\,\mathrm{arcsec\,yr^{-1}}$ per year shown in Box 9.3 represents an average over the satellite in polar orbit, which is less than our evaluation on the z axis. At any rate, this Lense–Thirring precession is a much smaller effect than the geodetic precession of more than 6 arcseconds per year.

10 Neutron Stars and Pulsars

10.4 Solving the differential form of Eq. (10.15) for dr, substitution in Eq. (10.14), and a little algebra gives Eq. (10.16).

10.5 From Eqs. (10.2) and (10.11) the radial metric component for the Oppenheimer–Volkov solution is given by

$$g_{11}(r) = \left(1 - \frac{2M(r)}{r}\right)^{-1},$$

which is unity at the center where $M(r) = 0$. Thus the ratio of g_{11} at the surface to that at the center is

$$\frac{g_{11}(R)}{g_{11}(0)} = \left(1 - \frac{2M(R)}{R}\right)^{-1} \simeq 2.5,$$

where we've assumed a neutron star of radius 10 km and mass $M = 2M_\odot \simeq 3\,\text{km}$. Since the average spacing between neutrons is $\sim 10^{-13}$ cm, this means that the metric changes by only of order one part in 10^{19} over the internucleonic spacing. Thus, on that distance scale the metric is very flat. See Glendenning [98], Section 4.4 for further discussion.

10.8 From Eq. (6.5),

$$\varepsilon = \frac{GM}{Rc^2} = 7.416 \times 10^{-31} \left(\frac{M}{\text{kg}}\right)\left(\frac{\text{km}}{R}\right) = 1.475 \left(\frac{M}{M_\odot}\right)\left(\frac{\text{km}}{R}\right),$$

where M is the mass producing the gravitational field and R is the characteristic distance over which it acts. The innermost white dwarf has a mass of $\sim 0.2M_\odot$, for which a radius estimate is $\sim 15,000\,\text{km}$. For the neutron stars let's take a mass of $1.4M_\odot$ and radius of $\sim 10\,\text{km}$. Then

$$\varepsilon_{\text{WD}} \sim 2 \times 10^{-5} \qquad \varepsilon_{\text{NS}} \sim 0.2 \qquad \varepsilon_{\text{Earth}} \sim 7 \times 10^{-10} \qquad \varepsilon_{\text{Moon}} \sim 3 \times 10^{-11},$$

confirming the assertion that any deviations from the strong equivalence principle would be greatly amplified in PSR J0337+1715 relative to the Solar System. See also the related Problem 25.2.

11 Spherical Black Holes

11.1 Begin with Eq. (11.9). For the $r > 2M$ case, $\ln|r/2M - 1| = \ln(r/2M - 1)$ and differentiating (11.9) gives

$$dt = dv - \left(1 - \frac{2M}{r}\right)^{-1} dr.$$

For the $r < 2M$ case, $\ln|r/2M - 1| = \ln(1 - r/2M)$ and differentiating (11.9) gives the same result as above. Therefore,

$$dt^2 = dv^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv dr + \left(1 - \frac{2M}{r}\right)^{-2} dr^2.$$

Inserting this expression for dt^2 in the Schwarzschild line element (9.4) expressed in standard coordinates gives the Schwarzschild line element in Eddington–Finkelstein coordinates (11.10).

11.2 From Eq. (11.6), the proper time to fall from $r = 2M$ to $r = 0$ is $\tau = 4M/3$, in geometrized units. Restoring the G and c factors, $\tau \to c\tau$ and $M \to GM/c^2$, gives

$$\tau = \frac{4G}{3c^3} M = 6.6 \times 10^{-6} \left(\frac{M}{M_\odot}\right) \text{ seconds.}$$

Some times estimated from this formula to fall from the event horizon to the singularity of a spherical black hole are shown in the following table.

Type of black hole	Mass ($M_\odot$)	Time (s)
Typical stellar black hole	10	6.6×10^{-5}
GW150914 final black hole	62	4.1×10^{-4}
Milky Way central black hole	4.3×10^{6}	28.4
AGN central engine	10^{9}	6555

See also the related Problem 11.6.

11.9 From the discussion of invariant integration in Section 3.13.1, the area of the Schwarzschild horizon is

$$A = \int_0^{2\pi} d\varphi \int_0^{\pi} \sqrt{\det g}\, d\theta,$$

where the metric g for the horizon surface is 2-dimensional, corresponding to the Schwarzschild line element (9.4) evaluated at constant time and constant $r = r_\mathrm{s}$:

$$ds^2 = r_\mathrm{s}^2 d\theta^2 + r_\mathrm{s}^2 \sin^2\theta d\varphi^2.$$

Thus the metric is specified by the diagonal 2×2 matrix $g = \mathrm{diag}\,(r_\mathrm{s}^2, r_\mathrm{s}^2 \sin^2\theta)$, which has $\det g = r_\mathrm{s}^4 \sin^2\theta$. Substituting in the above expression for A then gives a horizon area

$$A = r_\mathrm{s}^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta = 4\pi r_\mathrm{s}^2 = 16\pi M^2,$$

where $r_\mathrm{s} = 2M$ was used.

12 Quantum Black Holes

12.7 The CMB has a temperature of about 2.7 K. From Eq. (12.4) a Hawking black hole of mass 4.6×10^{22} kg would have the same temperature and thus be in equilibrium with the CMB (this mass is a little less than that of the Moon, or about 0.008 that of Earth). Black holes with less mass than this would have higher temperature than the CMB and thus could radiate more energy than they absorb; black holes with more mass than this would absorb more energy from the CMB than they could radiate by Hawking radiation.

12.8 From the Stefan–Boltzmann law for a blackbody radiator the power is

$$P = A\sigma T^4,$$

where A is the surface area, the Stefan–Boltzmann constant is

$$\sigma = \frac{\pi^2 k^4}{60\hbar^3 c^2},$$

and T is the temperature. For a Schwarzschild black hole the area of the event horizon is

$$A = 16\pi M^2 = \frac{16\pi G^2 M^2}{c^4}$$

(see results of Problem 11.9 with factors of c and G reinstated using Table B.1). Inserting these and the temperature given by Eq. (12.4) in $P = A\sigma T^4$ gives Eq. (12.5). The power radiated by the black hole comes at the expense of its mass so

$$P = -c^2 \frac{dM}{dt},$$

which leads to Eq. (12.6) when Eq. (12.5) is substituted for P.

12.9 This problem is adapted from a discussion in Perkins [181]. Using Newtonian gravity, the magnitude of the tidal force acting over a distance Δr can be estimated as

$$dF = \frac{dF}{dr}\,dr = \frac{2mMG}{r^3}\,dr \quad \rightarrow \quad \Delta F \simeq \frac{2mMG}{r^3}\,\Delta r,$$

where M is the mass of the black hole. The energy required to create the particle–hole pair from the vacuum is $E \simeq mc^2$. By the uncertainty principle the virtual pair can live for a time $\Delta t \sim \hbar/E$, and thus could separate a maximum distance

$$\Delta r \sim c\Delta t \sim \frac{\hbar c}{E}$$

in the time Δt. Requiring that the work done $\Delta F \cdot \Delta r$ be comparable to the rest mass energy,

$$\Delta F \cdot \Delta r = \frac{2mMG}{r^3}(\Delta r)^2 \sim E,$$

substituting $m \sim E/c^2$ and $\Delta r \sim \hbar c/E$, and solving for E gives

$$E \sim \sqrt{\frac{2\hbar^2 GM}{r^3}}.$$

Evaluating this at the Schwarzschild radius $r = 2MG/c^2$ gives

$$E \sim \frac{\hbar c^3}{GM},$$

where a factor of $\frac{1}{2}$ has been dropped since our approximations are crude. But up to numerical factors this is the result of Eq. (12.4) for the average energy $k_{\mathrm{B}}T$ associated with the Hawking radiation.

13 Rotating Black Holes

13.4 The distances are gotten by integrating the line element (13.8) around the corresponding curves. For the equator, $\theta = \frac{\pi}{2}$ so $d\theta = 0$ and only the second term contributes. The distance around the equator is

$$L = \oint \sqrt{ds^2} = 2M \int_0^{2\pi} d\varphi = 4\pi M,$$

where $\sin\theta = 1$ has been used. For a meridian through the poles take $\varphi = 0$ and $d\varphi = 0$, so only the first term contributes. The corresponding distance is

$$L' = \oint \sqrt{\rho_+}\, d\theta = 2 \int_0^{\pi} \sqrt{r_+^2 + a^2 \cos^2\theta}\, d\theta.$$

For the special case of an extremal black hole $a = M$ and $r_+ = M$, so that $\rho_+ = M(1 + \cos^2\theta)^{1/2}$ and

$$L' = 2M \int_0^{\pi} \sqrt{1 + \cos^2\theta}\, d\theta \simeq 7.6M,$$

where the definite integral has been evaluated numerically using Maple. Thus, the ratio of the equatorial to polar circumferences for the horizon of an extremal Kerr black hole is $L/L' = 4\pi M/7.6M = 1.65$. Although the horizon corresponds to a constant Boyer–Lindquist coordinate r_+, it does not have a spherical geometry.

13.8 Assume $r \gg M$ and $r \gg a$, and drop terms quadratic in a. Then

$$\rho^2 = r^2 + a^2\cos\theta \simeq r^2 \qquad \Delta = r^2 - 2Mr + a^2 \simeq r^2 - 2Mr.$$

Substituting in the Kerr metric (13.1) gives

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 - \frac{4Ma}{r}\sin^2\theta d\varphi dt,$$

where $(1 - 2M/r)^{-1} \sim (1 + 2M/r)$ was used. Taking the limit $r \to \infty$ at constant M and a gives

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

which is the flat-space Minkowski metric in spherical coordinates. Hence the Kerr spacetime is asymptotically flat.

14 Observational Evidence for Black Holes

14.4 Evaluating the constants in Eq. (14.8) gives

$$M = 3.77 \times 10^{-11} \left(\frac{R}{\text{km}}\right) \left(\frac{\sigma}{\text{km/s}}\right)^2 M_\odot$$

(a) For M31 with $R = 0.8\,\text{pc} = 2.47 \times 10^{13}\,\text{km}$ and $\sigma = 240\,\text{km s}^{-1}$, the virial mass is

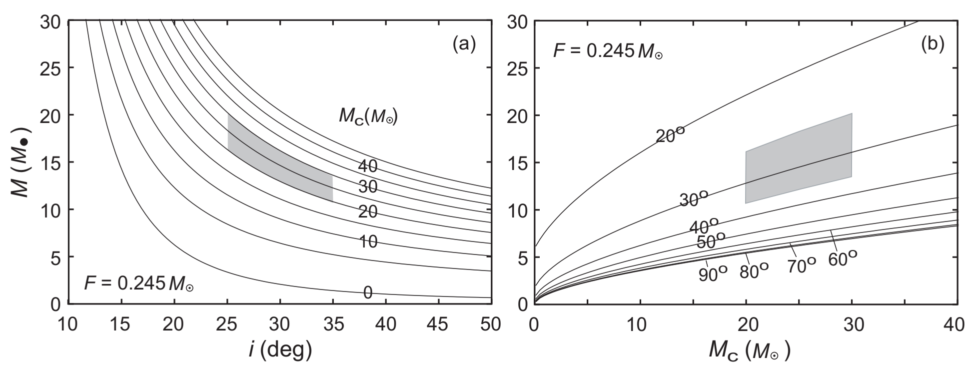
$$M = 3.77 \times 10^{-11}(2.47 \times 10^{13})(240)^2 M_\odot = 5.36 \times 10^7 M_\odot.$$

(b) For a typical Seyfert galaxy,

$$M = 3.77 \times 10^{-11}(3.09 \times 10^{15})(1000)^2 M_\odot = 1.16 \times 10^{11} M_\odot,$$

where $R = 100\,\text{pc} = 3.09 \times 10^{15}\,\text{km}$ and $\sigma = 1000\,\text{km s}^{-1}$ were assumed. (This suggests that 10% of the mass of the Seyfert galaxy is contained within a central region only 100 pc across.)

14.6 From Fig. 2 of Ref. [171], $P = 5.6$ days and $K \sim 75\,\text{km s}^{-1}$, which gives from Eq. (14.3) $F = PK^3/2\pi G = 0.245$. Inserting this in Eq. (14.6) and plotting M versus i and M versus M_c gives the following diagrams.



The gray areas indicate the ranges $i = 25$–35° and $M_c = 20$–$30 M_\odot$, which constrain the mass M of the unseen object to lie in the range ~ 10–$20 M_\odot$. A more diligent use of observational constraints described in Ref. [171] and Box 14.2 gives a value of $M = 14.8 \pm 1.0 M_\odot$. Since these mass estimates are far above the maximum mass expected for a neutron star, the unseen massive and compact object in Cygnus X-1 is almost certainly a black hole.

14.7 From Eq. (6.5), the intrinsic strength of gravity is governed by

$$\varepsilon = \frac{GM}{Rc^2} = 7.416 \times 10^{-31} \left(\frac{M}{\text{kg}}\right)\left(\frac{\text{km}}{R}\right) = 1.475 \left(\frac{M}{M_\odot}\right)\left(\frac{\text{km}}{R}\right),$$

where M is the mass producing the gravitational field and R is the characteristic distance over which it acts. For the Sun, taking the mass of the Sun as M and its radius as R gives $\varepsilon_\odot \sim 2.1 \times 10^{-6}$. For the Binary Pulsar, taking the separation at closest approach ($\sim 1.1 R_\odot$) as R and the mass of about $1.4 M_\odot$ for the other neutron star as M gives $\varepsilon_{\text{BP}} \sim 2.7 \times 10^{-6}$. For the star S0-2, taking the distance at closest approach of 17 light–hours as R and the mass $4.3 \times 10^6 M_\odot$ of the black hole as M gives $\varepsilon_{\text{S0-2}} = 3.45 \times 10^{-4}$ (which is comparable to the strength of gravity at the surface of a white dwarf). The star S0-102 comes even closer to the black hole so ε for it is larger but the same order of magnitude. Thus ε is about two orders of magnitude larger for stars in orbit around the black hole at Sgr A* than for gravity at the surface of the Sun or in the Binary Pulsar, and the orbits of stars like S0-2 and S0-102 can provide a test of general relativity in stronger gravity than for either Solar System measurements or binary pulsars.

15 Black holes as Central Engines

15.1 (a) From the masses before and after in nuclear reactions burning hydrogen to helium one finds a typical efficiency for mass to energy conversion of $\eta \sim 0.007$. Requiring that

$$\eta \dot{m} c^2 = 10^{47}\,\text{erg s}^{-1} = 3.2 \times 10^{54}\,\text{erg yr}^{-1}$$

yields that $\dot{m} \simeq 255 M_\odot\,\text{yr}^{-1}$ if this luminosity is supplied by hydrogen fusion. It is difficult to conjecture a mechanism consistent with observations that could account for this.

(b) On the other hand, for black hole accretion the mass to energy conversion efficiency could lie in the range $\eta \simeq 0.1 - 0.4$. Then the luminosity could be sustained by accretion of $5 M_\odot - 20 M_\odot\,\text{yr}^{-1}$, for which there are plausible mechanisms. The Eddington luminosity (maximum luminosity for which radiation pressure would not reverse accretion infall) is given by

$$L_{\text{Edd}} = 1.3 \times 10^{38} \left(\frac{M}{M_\odot} \right) \text{erg s}^{-1}.$$

Equating this to the observed $\sim 10^{47}$ erg s^{-1} and solving for the mass of the central object gives $M \simeq 7.7 \times 10^8 M_\odot$.

(c) Observed light variation on timescales of days argues by causality that the source has a maximum diameter of 1 light-day or about 170 AU (twice the diameter of the Solar System). Thus, it may be inferred on rather general grounds that the AGN central engine has of order $10^9 M_\odot$ concentrated in a region not much larger than the Solar System. The most plausible explanation is a supermassive black hole. The Schwarzschild radius of a $10^9 M_\odot$ black hole would be

$$R \simeq 2.95 \left(\frac{M}{M_\odot} \right) \text{km} \simeq 3 \times 10^9\,\text{km} \simeq 20\,\text{AU},$$

which is approximately the radius of the orbit of Uranus.

15.3 From the geometry of the figure, the leading edge of the jet appears to move about 14 lightyears in two elapsed observing years. Thus the apparent transverse velocity is $v \sim 14/2 \sim 7c$.

15.4 From the inverse Compton boost factor given in Box 15.4, a visible photon of frequency 5×10^{14} Hz is boosted to a frequency

$$\nu = \gamma^2 \nu_0 = \frac{\nu_0}{1 - v^2/c^2} = \frac{5 \times 10^{14}\,\text{Hz}}{1 - (0.9999995)^2} = 5 \times 10^{20}\,\text{Hz},$$

which lies in the γ-ray region of the spectrum.

15.5 Neglecting the spectral energy distribution of the emitted flux,

$$\frac{S_{\text{observed}}}{S_{\text{emitted}}} = \frac{1}{[\gamma(1-\beta\cos\theta)]^3}.$$

Thus, if one assumes the same γ and β for both jets and that the approaching jet makes an angle θ with the line of sight, the ratio of observed flux densities for light emitted from the approaching and receding jets will be

$$\frac{S_{\text{approach}}}{S_{\text{recede}}} = \frac{[\gamma(1-\beta\cos\theta)]^{-3}}{[\gamma(1-\beta\cos(\theta+\pi))]^{-3}} = \left(\frac{1+\beta\cos\theta}{1-\beta\cos\theta}\right)^3.$$

If, for example, $\beta = 0.98$ and $\theta = 10^\circ$, this ratio is $\sim 10^5$, implying that the counterjet will appear to be much fainter than the jet. If one takes into account the spectral energy distribution of the emitted flux, the above formula remains valid except that the exponent of 3 is replaced by a somewhat smaller exponent ~ 2.7 (see Section 8.3.3 of Ref. [204]).

15.6 By the Hubble law (Ch. 16), the distance is

$$d = \frac{v}{H_0} \sim \frac{cz}{H_0},$$

since $z \sim v/c$ for redshifts that are not too large. Using $H_0 = 72\,\text{km s}^{-1}\ \text{Mpc}^{-1}$,

$$d = \frac{cz}{H_0} = \frac{(3\times10^5\,\text{km s}^{-1})(0.158)}{72\,\text{km s}^{-1}\ \text{Mpc}^{-1}} \simeq 660\,\text{Mpc}.$$

The apparent magnitude m and the absolute magnitude M for a distant object are related by the distance modulus formula

$$M = m - 5\log\frac{d\,(\text{pc})}{10},$$

where $d\,(\text{pc})$ is the distance to the object in parsecs. Using the apparent visual magnitude $m = +12.9$ and $d = 660\times10^6$ pc gives an absolute visual magnitude of $M = -26.2$ for 3C 273. Absolute magnitudes M and luminosities L for objects 1 and 2 are related by

$$\frac{L_1}{L_2} = 10^{0.4(M_2-M_1)}.$$

For the Sun the absolute visual magnitude is $+4.8$ and the above formula indicates that 3C 273 is 2.5×10^{12} times more luminous than the Sun at visual wavelengths. For M31, with absolute visual magnitude of about -21.5, the corresponding ratio is about 76, and for M87, with absolute visual magnitude of about -22, one obtains that 3C 273 is 48 times more luminous at visual wavelengths. Thus, 3C 273 is roughly 100 times more luminous at visual wavelengths than large galaxies. However, the quasar emits most of its light at nonvisual wavelengths. When the luminosities are integrated over all wavelengths 3C 273 is found to be about 1000 times more luminous than large normal galaxies.

15.7 For the thin disk radiating as a blackbody the radiation rate per unit area is σT^4, where σ is the Stephan–Boltzmann constant. Thus, if the disk has a radius R the luminosity L is

$$L = 2\pi R^2\sigma T^4,$$

where the factor of two comes from the disk having two sides. Assume the observed luminosity of the disk to be a fraction η of the Eddington luminosity (15.3). Then solving the preceding equation for T and using the expression (15.4) to approximate the Eddington luminosity gives for the temperature of the disk

$$T = \left(\frac{L}{2\pi R^2 \sigma}\right)^{1/4} = \left(\frac{\eta L_{\text{edd}}}{2\pi R^2 \sigma}\right)^{1/4} = 7.72 \times 10^7 \left(\eta \frac{M/M_\odot}{(R/\,\text{km})^2}\right)^{1/4} \text{K},$$

where M is the gravitational mass responsible for the accretion (which may be approximated by the mass of the compact object, since this is much larger than the mass in the accretion disk). For a neutron star, assuming

$$R \sim 10\,\text{km} \qquad M \sim 1M_\odot \qquad \eta \sim 1,$$

this formula yields $T \simeq 2.4 \times 10^7$ K. By the Wien law, the corresponding blackbody spectrum peaks at a wavelength

$$\lambda_{\text{peak}} = \frac{2.9 \times 10^{-3}\,\text{m K}}{T} \simeq 0.12\,\text{nm},$$

which is in the X-ray portion of the spectrum. For a Schwarzschild (spherical) black hole, approximate R by the radius of the innermost stable circular orbit, which from Eq. (9.34) is given by

$$R = \frac{6GM}{c^2},$$

with factors of G and c restored. In convenient units, $G/c^2 = 1.475\,\text{km}\,{M_\odot}^{-1}$ and the preceding equation for T may be written

$$T = 2.6 \times 10^7 \eta^{1/4} \left(\frac{M_\odot}{M}\right)^{1/4} \text{K}.$$

Assuming radiation near the Eddington limit so that $\eta \sim 1$, for a 10 $M_\odot$ black hole this formula and the Wien law give

$$T = 1.5 \times 10^7\,\text{K} \qquad \lambda_{\text{peak}} = 0.20\,\text{nm},$$

which is dominantly in the X-ray region of the spectrum. For a $10^8 M_\odot$ black hole, we find likewise that

$$T = 2.6 \times 10^5\,\text{K} \qquad \lambda_{\text{peak}} = 11\,\text{nm},$$

which is dominantly in the UV portion of the spectrum. This has been a rather crude approximation to the physics of accretion disks (for more realistic descriptions, see Refs. [89, 183]), but it indicates correctly that accretion disks around neutron stars or stellar-size black holes are expected to radiate in the X-ray region, but the corresponding accretion disks around supermassive black holes should have lower temperatures and radiate at longer wavelengths, largely in the UV portion of the spectrum.

15.8 Evaluation of constants allows Eq. (15.15) to be written as

$$
\begin{aligned}
\tau \simeq \frac{f\sigma_{\mathrm{T}} F D^2}{\delta t^2 m_{\mathrm{e}} c^4} &= 9.01\times 10^{-40} f \left(\frac{D}{\mathrm{cm}}\right)^2 \left(\frac{F}{\mathrm{erg\,cm^{-2}}}\right) \left(\frac{\mathrm{s}}{\delta t}\right)^2 \\
&= 8.58\times 10^{15} f \left(\frac{D}{\mathrm{Mpc}}\right)^2 \left(\frac{F}{\mathrm{erg\,cm^{-2}}}\right) \left(\frac{\mathrm{ms}}{\delta t}\right)^2 \\
&= 7.7\times 10^{13} f \left(\frac{D}{3000\,\mathrm{Mpc}}\right)^2 \left(\frac{F}{10^{-7}\,\mathrm{erg\,cm^{-2}}}\right) \left(\frac{10\,\mathrm{ms}}{\delta t}\right)^2 .
\end{aligned}
$$

Since f and the product of quantities in parentheses are of order one in the last expression for a typical gamma-ray burst, the resulting optical depth is huge ($\tau \sim 10^{14}$). This is inconsistent with the observed nonthermal spectrum for gamma-ray bursts, since a nonthermal spectrum typically requires a medium that is optically thin. The fallacy is that Eq. (15.15) is invalid for a gamma-ray burst because it must be modified to account for the ultrarelativistic kinematics of the burst. When that is done, as in Eq. (15.16), the above expression is multiplied by a factor approximately equal to $1/\gamma^{4+2\alpha}$, as discussed in Section 15.7.4. For a typical value $\alpha \sim 2$ this will yield optical depths smaller than one for γ of order 100 or larger.

16 The Hubble Expansion

16.3 Integration of the Hubble law $dr/dt = H_0 r$, assuming that H_0 is constant with time, gives $r(t) \sim e^{H_0 t}$. The volume of a spherical region is

$$V(t) = \tfrac{4}{3}\pi r^3 \simeq e^{3H_0 t}.$$

Since the volume expands but the density is assumed constant, matter must be created continuously to maintain the constant density. The total mass within a volume is $M = \rho V$, where ρ is the constant density. Then $\dot{M} = \rho \dot{V} = \rho \times 3H_0 V$ and the creation rate per unit volume is

$$\frac{\dot{M}}{V} = 3H_0\rho \simeq 7 \times 10^{-48}\,\mathrm{g\,s^{-1}cm^{-3}},$$

where a matter density of $\rho \sim 10^{-30}\,\mathrm{g\,cm^{-3}}$ and a Hubble parameter $H_0 \simeq 72\ \mathrm{km\,s^{-1}Mpc^{-1}}$ were assumed. This is equivalent to the creation of about one hydrogen atom per cubic meter every 10 billion years.

16.5 From Eq. (6.5), a general relativistic description is required if $GM/Rc^2 \simeq 1$. As a crude estimate take the Universe to be static and euclidean, with a radius given by the Hubble distance and a density comparable to the critical density. Then, treating the Universe as a spherical gravitating mass,

$$R \sim \frac{c}{H_0} \qquad M \sim \tfrac{4}{3}\pi R^3 \rho_{\mathrm{crit}} \qquad \rho_{\mathrm{crit}} \simeq \frac{3H_0^2}{8\pi G},$$

which implies that

$$\frac{GM}{Rc^2} = \frac{4}{3}\pi G \frac{R^2 \rho_{\mathrm{crit}}}{c^2} \simeq \frac{1}{2}.$$

Therefore, a correct cosmological description is expected to involve general relativity.

17 Energy and Matter in the Universe

17.2 From Eqs. (17.15)–(17.16),

$$\dot{a}^2 = a_0^2 H_0^2 \left(1 + \frac{a_0}{a}\Omega_0 - \Omega_0\right).$$

Choose $\Omega_0 = 1$ (flat Universe), take the square root of both sides, and then integrate both sides to give

$$\int_0^a a^{1/2} da = a_0^{3/2} H_0 \int_0^t dt.$$

Performing the integrals and solving for a/a_0 gives

$$\frac{a(t)}{a_0} = \left(\frac{2}{3}\right)^{3/2} \left(\frac{t}{t_{\mathrm{H}}}\right)^{2/3},$$

where $t_{\mathrm{H}} = H_0^{-1}$ has been used. The redshift z is given by $a_0/a(t) = 1 + z$. Choosing by convention $a_0 = 1$ gives

$$\frac{t(z)}{t_{\mathrm{H}}} = \tfrac{2}{3}(1+z)^{-3/2}.$$

Since $z = 0$ today, the age of a flat, dust-filled Universe is $t_0 \equiv t(z=0) = \frac{2}{3} t_{\mathrm{H}}$.

17.3 From Eq. (17.18), for a closed universe

$$\frac{a}{a_0} = \frac{\Omega}{2(\Omega-1)}(1 - \cos\psi),$$

and since $1 + z = a_0/a$,

$$\cos\psi = 1 - \frac{2(\Omega-1)}{\Omega(1+z)}.$$

Also, from Eq. (17.19),

$$\psi - \sin\psi = \frac{2(\Omega-1)^{3/2}}{\Omega}\left(\frac{t}{t_{\mathrm{H}}}\right),$$

where $t_{\mathrm{H}} = 1/H_0$ was used. Combining these relations and a substantial amount of algebra then gives

$$\frac{t(z)}{t_{\mathrm{H}}} = \frac{\Omega}{2(\Omega-1)^{3/2}} \left[\cos^{-1}\left(\frac{\Omega z - \Omega + 2}{\Omega + \Omega z}\right) - \frac{2(\Omega^2 z + \Omega - \Omega z - 1)^{1/2}}{\Omega + \Omega z}\right],$$

which reduces to Eq. (17.25) upon setting $z = 0$. For an open universe, start from Eqs.

(17.22) and (17.23) and proceed in a way analogous to above. After substantial algebra, one obtains

$$\frac{t(z)}{t_{\mathrm{H}}} = \frac{\Omega}{2(1-\Omega)^{3/2}}\left[\frac{2(-\Omega^2 z - \Omega + \Omega z + 1)^{1/2}}{\Omega + \Omega z} - \cosh^{-1}\left(\frac{\Omega z - \Omega + 2}{\Omega + \Omega z}\right)\right],$$

which reduces to Eq. (17.24) upon setting $z = 0$.

17.8 From the Hubble law $v = H_0 r$ and for small v/c the redshift is $z \simeq v/c$. Thus, $z \simeq H_0 r/c$. Expand the scale factor to first order in time,

$$a(t) \simeq a_0 - \left.\frac{da}{dt}\right|_{t_0}(t_0 - t) \equiv a_0 - \dot{a}_0 \Delta t.$$

The redshift can also be written as

$$\begin{aligned} z &= \frac{\lambda_0}{\lambda} - 1 = \frac{a_0}{a} - 1 = \frac{a_0 - a}{a} \\ &\simeq \frac{a_0 - a_0 + \dot{a}_0 \Delta t}{a_0 - \dot{a}_0 \Delta t} \simeq \frac{\dot{a}_0}{a_0}\Delta t = \frac{\dot{a}_0}{a_0}\frac{r}{c}, \end{aligned}$$

where $r = c\Delta t$ was used in the last step. Comparing with the earlier expression $z \simeq H_0 r/c$ gives $\dot{a}_0/a_0 = H_0$.

17.9 The coordinate distance is

$$d = c\int_t^{t_0} \frac{dt'}{a(t')} \simeq \frac{c}{a_0}\int_t^{t_0} \big(1 - H_0(t' - t_0)\big)\, dt',$$

where in the second step the expansion (17.36) was inserted, terms quadratic and higher in $t - t_0$ were dropped, and the integrand was expanded in a binomial series. Performing the integration gives

$$d = \frac{c}{a_0}\left((t_0 - t) + \tfrac{1}{2}H_0(t_0 - t)^2\right),$$

which is Eq. (17.39) with $a_0 = 1$.

17.10 Equation (17.36) follows directly from inserting Eqs. (17.34)–(17.35) into Eq. (17.33). From $z \equiv a_0 a^{-1} - 1$, and (17.36) for a,

$$z = \left(1 + H_0(t - t_0) - \tfrac{1}{2}H_0^2 q_0 (t - t_0)^2\right)^{-1} - 1.$$

Then Eq. (17.37) follows from a binomial expansion $(1+x)^{-1} \simeq 1 - x + x^2$, with terms higher than second order in $(t - t_0)$ discarded. Equation (17.37) is a quadratic equation in $(t_0 - t)$, which gives Eq. (17.38) when solved by the usual quadratic formula with the positive solution, and with the square root expanded according to $(1+x)^{1/2} \simeq 1 + \frac{1}{2}x - \frac{1}{8}x^2$.

17.12 From the identity $\ddot{a} = \frac{1}{2}d(\dot{a}^2)/da$ and Eq. (17.14),

$$\ddot{a} = \frac{1}{2}\frac{d}{da}\dot{a}^2 = \frac{-\frac{1}{2}H_0^2 a_0^3 \Omega}{a^2}.$$

Solving this for $d\dot{a}^2$ and integrating from the present time t_0 to a time t,

$$\int_{a_0}^{a} d\dot{a}^2 = -H_0^2 a_0^3 \Omega \int_{a_0}^{a} \frac{da}{a^2}.$$

Evaluating the integrals gives

$$\dot{a}^2 = \dot{a}_0^2 + H_0^2 a_0^3 \Omega \left(\frac{1}{a} - \frac{1}{a_0} \right),$$

and since $\dot{a}_0 = a_0 H_0$ (see Problem 17.8),

$$\dot{a}^2 = a_0^2 H_0^2 \left(1 + \Omega \frac{a_0}{a(t)} - \Omega, \right),$$

which is Eq. (17.15).

18 Friedmann Cosmologies

18.2 The condition $T^{\mu\nu}{}_{;\nu} = 0$ with $T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + Pg^{\mu\nu}$ implies that

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\alpha\nu} T^{\alpha\nu} + \Gamma^\nu_{\alpha\nu} T^{\mu\alpha} = 0.$$

The non-zero components of $T_{\mu\nu}$ are obtained from Eq. (18.31) and the corresponding $T^{\mu\nu}$ can be obtained from these by contraction with the R–W metric tensor (18.16):

$$\begin{aligned} T_{00} &= \varepsilon & T^{00} &= \varepsilon \\ T_{11} &= Pa^2/(1-kr^2) & T^{11} &= P(1-kr^2)/a^2 \\ T_{22} &= Pr^2a^2 & T^{22} &= P/(ra)^2 \\ T_{33} &= Pr^2a^2\sin^2\theta & T^{33} &= P/(ra\sin\theta)^2, \end{aligned}$$

where a is the scale parameter. Consider the $\mu = 0$ component:

$$\frac{\partial T^{0\nu}}{\partial x^\nu} + \Gamma^0_{\alpha\nu} T^{\alpha\nu} + \Gamma^\nu_{0\nu} T^{00} = 0,$$

which is explicitly

$$\begin{aligned} \Gamma^0_{00}T^{00} + \Gamma^0_{11}T^{11} + \Gamma^0_{22}T^{22} + \Gamma^0_{33}T^{33} + \Gamma^0_{00}T^{00} \\ + \Gamma^1_{01}T^{00} + \Gamma^2_{02}T^{00} + \Gamma^3_{03}T^{00} = 0. \end{aligned}$$

The required Christoffel symbols are given in Table 18.1,

$$\Gamma^0_{00} = 0 \qquad \Gamma^0_{11} = \frac{a\dot{a}}{1-kr^2} \qquad \Gamma^0_{22} = r^2 a\dot{a} \qquad \Gamma^0_{33} = r^2\sin^2\theta\, a\dot{a}$$

$$\Gamma^1_{01} = \frac{\dot{a}}{a} \qquad \Gamma^2_{02} = \frac{\dot{a}}{a} \qquad \Gamma^3_{03} = \frac{\dot{a}}{a}$$

and from the preceding expressions for the stress–energy tensor components

$$\frac{\partial T^{0\nu}}{\partial x^\nu} = \frac{\partial T^{00}}{\partial x^0} = \dot{\varepsilon}.$$

Inserting these results into the previous expression for the $\mu = 0$ component and collecting terms gives

$$\dot{\varepsilon} + 3(\varepsilon + P)\frac{\dot{a}}{a} = 0,$$

which expresses conservation of mass–energy.

18.3 Let's use the Robertson–Walker metric in the form (18.19). For positive curvature the spatial line element is

$$d\ell^2 = a^2(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\varphi^2)),$$

so the metric is diagonal with

$$g_{11} = a^2 \qquad g_{22} = a^2 \sin^2\chi \qquad g_{33} = a^2 \sin^2\chi \sin^2\theta$$

and

$$\sqrt{\det g} = a^3 \sin^2\chi \sin\theta.$$

Then the volume is (see Section 3.13.1; here a positive sign is used under the square root because $\det g$ is positive for the spatial part of the metric)

$$\begin{aligned} V &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\pi} \sqrt{\det g}\, d\chi \\ &= a^3 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_0^{\pi} \sin^2\chi d\chi \\ &= 2\pi^2 a^3. \end{aligned}$$

For negative curvature the line element is

$$d\ell^2 = a^2(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)),$$

so the metric is diagonal with

$$g_{11} = a^2 \qquad g_{22} = a^2 \sinh^2\chi \qquad g_{33} = a^2 \sinh^2\chi \sin^2\theta$$

and

$$\sqrt{\det g} = a^3 \sinh^2\chi \sin\theta.$$

Therefore, for negative curvature the volume is

$$V = 4\pi a^3 \int_0^{\infty} \sinh^2\chi d\chi = \infty.$$

For a flat metric the line element is

$$d\ell^2 = a^2(d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\varphi^2)),$$

and proceeding as above the volume for flat space is

$$V = 4\pi a^3 \int_0^{\infty} \chi d\chi = \infty.$$

Thus, the volume of a spatial slice described by a Robertson–Walker metric with positive curvature is finite but the volume of a spatial slice described by a R–W metric with negative curvature or no curvature is infinite.

18.10 Introduce the conformal time η through $dt = a(t)d\eta$. The flat ($k = 0$) Robertson–Walker metric (18.14) is then given by

$$\begin{aligned} ds^2 &= -dt^2 + a^2(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) \\ &= -a^2 d\eta^2 + a^2(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) \\ &= a^2(-d\eta^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2), \end{aligned}$$

which is the same form as the metric for a uniformly-expanding Minkowski space. For radial light rays $d\theta = d\varphi = ds = 0$, so the line element implies that $a^2(-d\eta^2 + dr^2) = 0$ and hence $d\eta = \pm dr$. Thus, in the η–r plane light rays move at 45-degree angles at all times.

19 Evolution of the Universe

19.1 From the Friedmann equations,

$$\frac{\dot{\varepsilon}}{\varepsilon} + 3\left(1 + \frac{P}{\varepsilon}\right)\frac{\dot{a}}{a} = 0.$$

If an equation of state $P = w\varepsilon$ is assumed, this may be written as

$$\frac{d\varepsilon}{\varepsilon} = -(3+3w)\frac{da}{a}.$$

Assume w to be constant and integrate both sides from now ($a = a_0$ and $\varepsilon = \varepsilon_0$) until some scale factor a to give

$$\ln\frac{\varepsilon(a)}{\varepsilon_0} = \ln\left(\frac{a}{a_0}\right)^{-3(1+w)},$$

which implies that

$$\varepsilon(a) = \varepsilon_0\left(\frac{a}{a_0}\right)^{-3(1+w)}.$$

Thus matter ($w \simeq 0$), radiation ($w = \frac{1}{3}$), and vacuum energy ($w < -\frac{1}{3}$) have very different histories: for matter, $\varepsilon_{\mathrm{m}} \propto a^{-3}$; for radiation, $\varepsilon_{\mathrm{r}} \propto a^{-4}$; for vacuum energy, ε_Λ is constant, if it is assumed that $w = -1$ (as implied by a cosmological constant).

19.3 (a) Assume air to be an ideal gas of nitrogen molecules N_2 with mass $\mu \sim 2 \times 14 = 28$ amu at $T = 300$ K. From Eq. (19.10), $w = kT/\mu c^2$. Thus w is basically the ratio of the thermal energy to the rest mass energy, which will be small for a nonrelativistic gas. Inserting the numbers gives $w = 9.9 \times 10^{-13}$.

(b) For an ideal gas $\langle v^2 \rangle = 3kT/\mu$, implying that $w = kT/\mu c^2 = \langle v^2 \rangle / 3c^2$. Thus w may also be interpreted as the ratio of the average of the velocity squared to the speed of light squared, which is a very small number for nonrelativistic gases.

19.10 For the angular part of the metric (18.19) displayed in Box 19.3, we have $\sqrt{\det g} = S_k^2 \sin\theta$ (where $a_0 \equiv 1$) and the spherical surface has a proper area $A_{\mathrm{p}}(t_0)$ that differs from $4\pi r^2$ in curved space:

$$A_{\mathrm{p}}(t_0) = \int_0^{2\pi} d\varphi \int_0^{\pi} \sqrt{\det g}\, d\theta = 4\pi S_k(r)^2.$$

This is a consequence of curvature, independent of whether the Universe is expanding. If the Universe is flat, $S_k = r$ and an area of $4\pi r^2$ is recovered.

20 The Big Bang

20.11 The scaling of the energy densities with expansion is

$$\varepsilon_{\rm dm} \sim a^{-3} = (1+z)^3 \qquad \varepsilon_\gamma \sim a^{-4} = (1+z)^4 \qquad \varepsilon_{\rm b} \sim a^{-3} = (1+z)^3$$

From Table 20.2 and Table 17.1, take the density parameters today to be

$$\Omega_{\rm m} = 0.308 \qquad \Omega_{\rm r} = 8\times 10^{-5} \qquad \Omega_{\rm b} = 0.048 \qquad \Omega_{\rm dm} = \Omega_{\rm m} - \Omega_{\rm b} = 0.26$$

and from Eq. (17.7) the critical energy density is

$$\varepsilon_{\rm c} = 1.05\times 10^{-2} h^2\,{\rm MeV\,cm^{-3}} = 4.83\times 10^{-3}\,{\rm MeV\,cm^{-3}},$$

where $h = 0.678$ inferred from Table 20.2 was used. Thus for dark matter at a redshift for last scattering $z_{\rm ls}$,

$$\varepsilon_{\rm dm}(z_{\rm ls}) = \Omega_{\rm dm}\varepsilon_{\rm c}(1+z_{\rm ls})^3 = 1.6\times 10^6\,{\rm MeV\,cm^{-3}},$$

where $z_{\rm ls} = 1080$ was used. Likewise, for photons

$$\varepsilon_\gamma(z_{\rm ls}) = \Omega_\gamma\varepsilon_{\rm c}(1+z_{\rm ls})^4 = 5.3\times 10^5\,{\rm MeV\,cm^{-3}},$$

and for baryons

$$\varepsilon_{\rm b}(z_{\rm ls}) = \Omega_{\rm b}\varepsilon_{\rm c}(1+z_{\rm ls})^3 = 2.9\times 10^5\,{\rm MeV\,cm^{-3}}.$$

The ratio of energy densities at decoupling was then $\varepsilon_{\rm dm} : \varepsilon_\gamma : \varepsilon_{\rm b} \sim 5.5 : 1.8 : 1$, and the Universe was dominated by dark matter at decoupling. See also related Problem 20.5.

20.12 Putting the parameters from Table 20.2 into the cosmological calculator at Ref. [5] gives an angular size distance $d_{\rm A} = 12.9\,{\rm Mpc}$ for $z = 1080$. From $D = d_{\rm A}\Delta\theta$ in Box 20.2, on the last scattering surface assuming this cosmology

$$D_{\rm lss} = d_{\rm A}\Delta\theta = 12.9\,{\rm Mpc}\left(\frac{\Delta\theta}{\rm rad}\right) = 3.75\left(\frac{\Delta\theta}{\rm arcmin}\right)\,{\rm kpc}.$$

The smallest angular size resolved in the Planck data is about 5 arcmin, which gives ~ 19 kpc when inserted in the above formula for the size on the LSS. This would have scaled up to

$$D(t_0) = D_{\rm lss}(1+z_{\rm lss}) \sim 20.5\,{\rm Mpc},$$

as observed today because of the Hubble expansion, assuming $z_{\rm lss} = 1080$. The current baryon mass density of the Universe is $\rho_{\rm b} = \Omega_{\rm b}\rho_{\rm c}$, which gives $\rho_{\rm b} \sim 4.15\times 10^{-31}\,{\rm g\,cm^{-3}}$ upon using $\Omega_{\rm b} = 0.048$ and $h = 0.678$ from Table 20.2 in Eq. (17.6). Using this density a sphere of diameter $D = 20.5\,{\rm Mpc}$ contains a total baryonic mass of $2.8\times 10^{13} M_\odot$. This is comparable (given our crude estimates) to the baryonic mass of a cluster of galaxies, which typically have total masses of $10^{14} - 10^{15}\,M_\odot$. For example, the total mass of the (fairly rich) Virgo Cluster is estimated to be about $\sim 10^{15}\,M_\odot$, of which probably 5-10% is baryonic.

20.13 Before decoupling the barons and photons were strongly coupled, with the photons greatly outnumbering the baryons in the fluid. Thus the speed of sound was essentially that of a photon gas, which is $v_s = c/\sqrt{3}$. The proper distance that sound could travel from the big bang to the time of last scattering t_{ls} was then

$$\begin{aligned}\ell_s(t_{ls}) &= a(t_{ls})\int_0^{t_{ls}} \frac{v_s(t)dt}{a(t)} \simeq \frac{1}{\sqrt{3}}a(t_{ls})\int_0^{t_{ls}} \frac{cdt}{a(t)}\\ &= \frac{1}{\sqrt{3}}\ell_h = \frac{1}{\sqrt{3}}(0.25\,\text{Mpc}) = 0.144\,\text{Mpc}\end{aligned}$$

where we've used that at the time of last scattering for the CMB the distance to the horizon ℓ_h was approximately 0.25 Mpc. From Box 20.2 the corresponding angular size on the CMB as viewed today is

$$\theta_s = \frac{\ell_s(t_{ls})}{d_A} = \frac{0.144\,\text{Mpc}}{12.9\,\text{Mpc}} = 0.011\,\text{rad} = 0.6^\circ,$$

where an angular size distance $d_A = 12.9\,\text{Mpc}$ was computed for the parameters in Table 20.2 at a redshift $z = 1080$ [5]. The corresponding length scale as viewed today is stretched by a factor $1 + z_{ls}$:

$$\ell_s(t_0) = (1 + z_{ls})\ell_s(t_{ls}) = (1 + 1080)(0.144\,\text{Mpc}) = 156\,\text{Mpc}.$$

The present total mass density is given by ρ_c from Eq. (17.6) since the Universe is flat. The total matter density is $\rho_m = \Omega_m\rho_c = 2.66\times10^{-30}\,\text{g cm}^{-3}$, where $h = 0.678$ and $\Omega_m = 0.308$ were used. Then the mass contained within a volume of radius 156 Mpc is on average

$$M_s \sim \tfrac{4}{3}\pi(156\,\text{Mpc})^3\rho_m \simeq 6.3\times10^{17}M_\odot.$$

This is larger than the total mass for superclusters of galaxies and sets the minimal scale that must be analyzed to observe the effect of the baryon acoustic oscillations for clustering of visible matter.

20.14 From the last equation in Box 19.3 we have $d_L = (1+z)\ell(t_0)$, where $\ell(t_0)$ is the proper distance at the present time t_0. Thus, from the last equation in Box 20.2

$$(1+z)d_A = \frac{d_L}{1+z} = \ell(t_0),$$

and using Eq. (16.12) to relate scale factors to redshift,

$$d_A = \frac{\ell(t_0)}{1+z} = \ell(t_e),$$

where $\ell(t_e)$ is proper distance at the time of emission t_e.

21 Extending Classical Big Bang Theory

21.2 From Eqs. (19.5) and (17.6)

$$\frac{\Delta\rho}{\rho} = \frac{\rho - \rho_c}{\rho} = \frac{3k}{8\pi G a^2 \rho}.$$

But $\rho \propto a^{-4}$ for radiation dominated evolution, in which case $\Delta\rho/\rho \simeq a^2 \simeq t$, so the deviation from flatness grows smaller as time is extrapolated backwards. Take $t_0 \sim 4 \times 10^{17}$ s for today. Then (assuming radiation dominance to make the estimate simple), at the Planck time of $t \sim 10^{-44}$ s

$$\left(\frac{\Delta\rho}{\rho}\right)_t \Big/ \left(\frac{\Delta\rho}{\rho}\right)_{t_0} \simeq \frac{10^{-44}\,\text{s}}{4 \times 10^{17}\,\text{s}} \simeq 10^{-62}.$$

So unless the flatness is tuned to this precision at the Planck scale, the Universe does not evolve into one that is flat today.

21.3 First we make an unsophisticated argument. At the time of decoupling $t_d \sim 3 \times 10^5$ yr, corresponding to a redshift $z_d \sim 1100$. The size of the horizon at this time is $\ell_d \sim 2ct_d$ for radiation dominated and $\ell_d \sim 3ct_d$ for matter dominated cosmologies. (Let's assume radiation dominated for simplicity in estimates.) The size ℓ corresponding to this horizon in the current Universe would be stretched by the expansion according to $\ell/\ell_d = a_0/a_d$. However, $z_d = a_0/a_d - 1$, so $\ell \simeq 2ct_d(1+z_d)$ is the size of a causally connected region at decoupling in the present Universe. Assuming a flat Universe, one may take the distance to the last scattering surface to be close to the present horizon and given approximately by $c(t_0 - t_d)$. Thus, the approximate angular size of causally connected regions on the last scattering surface is

$$\theta \simeq \frac{2ct_d(1+z_d)}{c(t_0 - t_d)} \simeq 0.047\,\text{rad}.$$

Hence regions in the sky separated by more than a degree or so should not have been causally connected at any time in the past (in standard big bang cosmology).

A more sophisticated argument can be made by using the angular diameter distance discussed in Box 20.2 in the form $d_A = \ell_d/\Delta\theta$. In the standard cosmology the angular size distance corresponding to $z = 1100$ can be computed to be about 12.9 Mpc [5], and the horizon size at decoupling is about $\ell_d \sim 2ct_d \sim 0.184$ Mpc, assuming radiation dominance. Then

$$\Delta\theta = \frac{\ell_d}{d_A} = \frac{0.184\,\text{Mpc}}{12.9\,\text{Mpc}} = 0.014\,\text{rad} \sim 0.8^\circ.$$

Again we conclude that regions on the last scattering surface separated by more than a degree or so cannot have been in past causal contact in the standard cosmology, yet widely separated regions on the sky are observed to have the same cosmic microwave background temperature to one part in 10^5. This is the horizon problem.

21.4 By analogy with the solution of Problem 21.3, the physical horizon size at the GUT transition may be estimated as $r_{\rm h} \simeq 2ct_{\rm GUT} \simeq 6 \times 10^{-26}$ cm, if $t_{\rm GUT} = 10^{-36}$ s. Assuming one monopole per horizon volume, the number density of monopoles at the GUT transition is then

$$n_{\rm M} \sim (r_{\rm h})^{-3} = 4.6 \times 10^{75}\,{\rm cm}^{-3},$$

and the energy density of monopoles will be

$$\varepsilon_{\rm M} \simeq 10^{15}\,{\rm GeV} \times n_{\rm M} \simeq 4.6 \times 10^{90}\,{\rm GeV\,cm}^{-3},$$

where we've assumed the average mass of a monopole to be the GUT scale. The temperature at the GUT transition is about 10^{28} K, so the energy density of radiation is

$$\varepsilon_{\rm r} = aT_{\rm GUT}^4 \sim 4.7 \times 10^{100}\,{\rm GeV\,cm}^{-3}.$$

This is 10 orders of magnitude larger than the energy density of monopoles, so at the GUT scale the Universe is highly radiation dominated. However, the radiation energy density scales as a^{-4} and the massive monopole energy density as a^{-3}. Thus, after the Universe expands to a scale factor approximately 10^{10} times that at the GUT scale, the Universe will begin to be dominated by the monopole energy density. From Eqs. (20.17) and (20.5), in the early Universe $T \sim a^{-1}$ and $t \sim a^{-2}$, so this transition will occur when the temperature has fallen to $10^{28}\,{\rm K} \times 10^{-10} \simeq 10^{18}$ K, at a time of $10^{36}\,{\rm s} \times (10^{-10})^2 \simeq 10^{-16}$ seconds after the Big Bang. Thus, the early Universe would have been strongly matter dominated, contradicting the observational evidence.

22 Gravitational Waves

22.1 Contract both sides of the Einstein equation (8.21) with the metric tensor,

$$g^{\mu\nu}R_{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}g_{\mu\nu}R = \frac{8\pi G}{c^4}g^{\mu\nu}T_{\mu\nu}.$$

The first term on the left reduces to R and the second term to $2R$, and the term on the right reduces to $(8\pi G/c^4)T^{\nu}{}_{\nu}$. Solving for R then gives

$$R = -\frac{8\pi G}{c^4}T^{\nu}{}_{\nu}.$$

Inserting this back in the original Einstein equation gives

$$R_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^{\alpha}{}_{\alpha}).$$

The vacuum Einstein equation $R_{\mu\nu} = 0$ then results from setting $T_{\mu\nu}$ and $T^{\alpha}{}_{\alpha}$ to zero.

22.2 (a) In linearized gravity $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Under a Lorentz transformation of the metric,

$$g'_{\mu\nu} = \Lambda^{\alpha}_{\mu'}\Lambda^{\beta}_{\nu'}g_{\alpha\beta} = \Lambda^{\alpha}_{\mu'}\Lambda^{\beta}_{\nu'}(\eta_{\alpha\beta} + h_{\alpha\beta}) = \eta'_{\mu\nu} + h'_{\mu\nu},$$

where $h'_{\mu\nu} \equiv \Lambda^{\alpha}_{\mu'}\Lambda^{\beta}_{\nu'}h_{\alpha\beta}$. Thus the field defined by $h_{\mu\nu}$ behaves as a rank-2 tensor in Minkowski space, for which indices may be raised or lowered by contraction with the Minkowski metric tensor $\eta_{\mu\nu}$.

(b) Since $g_{\mu\nu}$ and $g^{\mu\nu}$ must be matrix inverses of each other, this requires that $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ so that

$$g_{\mu\alpha}g^{\alpha\nu} = (\eta_{\mu\alpha} + h_{\mu\alpha})(\eta^{\alpha\nu} - h^{\alpha\nu}) = \delta^{\nu}_{\mu} + \mathscr{O}\left(h^2\right),$$

to first order in h.

22.3 Substituting Eq. (22.6) in Eq. (22.7) and noting that $h_{\mu\nu}$ is symmetric under exchange of indices,

$$\begin{aligned}
\delta R_{\mu\nu} &= \frac{\partial}{\partial x^{\gamma}}\left[\frac{1}{2}\eta^{\gamma\delta}\left(\frac{\partial h_{\delta\mu}}{\partial x^{\nu}} + \frac{\partial h_{\delta\nu}}{\partial x^{\mu}} - \frac{\partial h_{\mu\nu}}{\partial x^{\delta}}\right)\right] - \frac{\partial}{\partial x^{\nu}}\left[\frac{1}{2}\eta^{\gamma\delta}\left(\frac{\partial h_{\delta\mu}}{\partial x^{\gamma}} + \frac{\partial h_{\delta\gamma}}{\partial x^{\mu}} - \frac{\partial h_{\mu\gamma}}{\partial x^{\delta}}\right)\right] \\
&= \frac{1}{2}\partial_{\gamma}\left[\eta^{\gamma\delta}\left(\partial_{\nu}h_{\delta\mu} + \partial_{\mu}h_{\delta\nu} - \partial_{\delta}h_{\mu\nu}\right)\right] - \frac{1}{2}\partial_{\nu}\left[\eta^{\gamma\delta}\left(\partial_{\gamma}h_{\delta\mu} + \partial_{\mu}h_{\delta\gamma} - \partial_{\delta}h_{\mu\gamma}\right)\right] \\
&= \frac{1}{2}\left(-\partial_{\gamma}\partial_{\delta}\eta^{\gamma\delta}h_{\mu\nu} + \partial_{\mu}\partial_{\gamma}\eta^{\gamma\delta}h_{\delta\nu} - \partial_{\mu}\partial_{\nu}\eta^{\gamma\delta}h_{\delta\gamma} + \partial_{\nu}\partial_{\delta}\eta^{\gamma\delta}h_{\mu\gamma}\right).
\end{aligned}$$

Introducing the definitions

$$\Box \equiv \eta^{\gamma\delta}\partial_{\gamma}\partial_{\delta} \qquad V_{\nu} \equiv \partial_{\gamma}h^{\gamma}_{\nu} - \tfrac{1}{2}\partial_{\nu}h^{\gamma}_{\gamma} = \partial_{\gamma}\eta^{\gamma\delta}h_{\delta\nu} - \tfrac{1}{2}\partial_{\nu}\eta^{\gamma\delta}h_{\delta\gamma}$$

and noting that, for example,

$$\partial_\mu V_\nu = \partial_\mu \partial_\gamma \eta^{\gamma\delta} h_{\delta\nu} - \tfrac{1}{2}\partial_\mu \partial_\nu \eta^{\gamma\delta} h_{\delta\gamma},$$

the Ricci tensor to first order in h becomes

$$\delta R_{\mu\nu} = \tfrac{1}{2}\left(-\Box h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu\right).$$

Then to first order in the metric perturbation h the vacuum Einstein equation $\delta R_{\mu\nu} = 0$ is

$$\Box h_{\mu\nu} - \partial_\mu V_\nu - \partial_\nu V_\mu = 0,$$

which is Eq. (22.12).

22.4 Consider linearized gravity with the metric given by Eq. (22.2). Under the coordinate transformation (22.13), $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$, where it is assumed that ε^μ and $\partial\varepsilon^\mu/\partial x^\nu$ have magnitudes comparable to or smaller than $h_{\mu\nu}$, the metric tensor transforms as

$$\begin{aligned}
g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\
&\simeq \left(\delta^\alpha_\mu - \frac{\partial\varepsilon^\alpha}{\partial x^\mu}\right)\left(\delta^\beta_\nu - \frac{\partial\varepsilon^\beta}{\partial x^\nu}\right) g_{\alpha\beta} \\
&\simeq g_{\mu\nu} - g_{\mu\beta}\frac{\partial\varepsilon^\beta}{\partial x^\nu} - g_{\alpha\nu}\frac{\partial\varepsilon^\alpha}{\partial x^\mu},
\end{aligned}$$

where Eq. (22.13) was used, we have assumed that to first order $\partial\varepsilon^\mu/\partial x'^\nu = \partial\varepsilon^\mu/\partial x^\nu$, and terms higher-order in $\partial\varepsilon/\partial x$ have been neglected. Hence, from Eq. (22.2)

$$\begin{aligned}
h'_{\mu\nu} &= g'_{\mu\nu} - \eta_{\mu\nu} \\
&= h_{\mu\nu} - g_{\mu\beta}\frac{\partial\varepsilon^\beta}{\partial x^\nu} - g_{\alpha\nu}\frac{\partial\varepsilon^\alpha}{\partial x^\mu} \\
&= h_{\mu\nu} - \partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu,
\end{aligned}$$

which is Eq. (22.15).

22.6 The transversality condition $k^j\alpha_{ij} = 0$ from Eq. (22.24) may be written out explicitly as the set of equations

$$\begin{aligned}
k^1\alpha_{11} + k^2\alpha_{12} + k^3\alpha_{13} &= 0 \\
k^1\alpha_{21} + k^2\alpha_{22} + k^3\alpha_{23} &= 0 \\
k^1\alpha_{31} + k^2\alpha_{32} + k^3\alpha_{33} &= 0.
\end{aligned}$$

But from (22.26), $k^1 = k^2 = 0$, so $\alpha_{13} = \alpha_{23} = \alpha_{33} = 0$, and from (22.23), $\alpha_{0\mu} = 0$. Therefore, for the symmetric matrix $\alpha_{\mu\nu}$ the only nonvanishing components are α_{11}, $\alpha_{12} = \alpha_{21}$, and α_{22}, and these are further constrained by the trace requirement from Eq. (22.22), so $\alpha_{22} = -\alpha_{11}$.

22.8 From Eqs. (22.19) and (22.16)

$$\begin{aligned}\Box \bar{h}_{\mu\nu} &= \eta^{\lambda\sigma}\partial_\lambda\partial_\sigma\left(\alpha_{\mu\nu}e^{ik_\alpha x^\alpha}\right)\\ &= \eta^{\lambda\sigma}ik_\sigma\partial_\lambda\left(\alpha_{\mu\nu}e^{ik_\alpha x^\alpha}\right)\\ &= \eta^{\lambda\sigma}ik_\lambda ik_\sigma h_{\mu\nu}\\ &= -k_\lambda k^\lambda \bar{h}_{\mu\nu} = 0.\end{aligned}$$

But $\bar{h}_{\mu\nu}$ is not generally zero so a solution of the wave equation requires that k be a null vector, $k_\lambda k^\lambda = 0$.

22.9 The test particle is initially at rest with a 4-velocity $u^\mu = (c, 0, 0, 0)$. The geodesic equation (7.23) thus reduces to

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu_{00}(u^0)^2 = -c^2\Gamma^\mu_{00}.$$

From Eq. (22.6), to first order in h,

$$\Gamma^\mu_{00} = \tfrac{1}{2}\eta^{\mu\nu}(\partial_0 h_{\nu 0} + \partial_0 h_{\nu 0} - \partial_\nu h_{00}),$$

but from Eqs. (22.23) and (22.19), $h_{\nu 0} = h_{00} = 0$, so $\Gamma^\mu_{00} = 0$ and the initial 4-acceleration vanishes, $du^\mu/d\tau = 0$. (Note that we are in TT gauge where $h = \bar{h}$.) Thus, in TT gauge the particle is stationary with respect to the coordinate system as the gravitational wave passes.

23 Weak Sources of Gravitational Waves

23.3 From Eq. (23.7) and being careless about numerical factors,

$$\dddot{I}_{ij} \simeq \frac{MR^2}{P^3} = \frac{MR^3}{RP^3} \sim \frac{Mv^3}{R},$$

where Eq. (23.9) has been used. Inserting this equation into Eq. (23.16) and eliminating M in favor of the Schwarzschild radius $r_S = 2M$ gives Eqs. (23.18)–(23.19) [a numerical factor has been omitted in (23.18)]. Utilizing Eq. (23.12), this also may be expressed as $L \simeq L_0 \left(r_S/R\right)^5$. The total energy emitted in one period P is

$$\Delta E \simeq LP \simeq \left(L_0 \frac{r_S^2}{R^2} \frac{v^6}{c^6} \right) P.$$

Utilizing $P = 2\pi R/v$, (23.12), (23.13), $r_s = 2M$, and (23.19), this gives Eq. (23.20),

$$\Delta E \simeq Mc^2 \left(\frac{r_S}{R}\right)^{7/2} = \varepsilon Mc^2,$$

where a factor of 2π has been dropped. Thus ε is a measure of the efficiency of converting mass to gravitational waves.

23.5 From Eq. (23.23) the non-zero components are

$$\begin{aligned}
I^{11} &= I^{xx} = \mu a^2 \cos^2 \omega t = \tfrac{1}{2}\mu a^2 (1 + \cos 2\omega t), \\
I^{12} &= I^{xy} = \mu a^2 \cos \omega t \sin \omega t = \tfrac{1}{2}\mu a^2 \sin 2\omega t, \\
I^{22} &= I^{yy} = \mu a^2 \sin^2 \omega t = \tfrac{1}{2}\mu a^2 (1 - \cos 2\omega t).
\end{aligned}$$

The trace-reversed amplitude is given by Eq. (23.4), which requires the second time derivatives. These are easily computed from the above equations. For example,

$$\begin{aligned}
\dot{I}^{xx}(t) &= \frac{d}{dt}\left(\tfrac{1}{2}\mu a^2 (1 + \cos 2\omega t)\right) = -\mu a^2 \omega \sin 2\omega t, \\
\ddot{I}^{xx}(t) &= -2\omega^2 \mu a^2 \cos 2\omega t, \\
\bar{h}^{xx} &= \frac{2}{r}\ddot{I}^{xx}(t - r) = \frac{-4\omega^2 \mu a^2}{r} \cos 2\omega (t - r).
\end{aligned}$$

Computing the second time derivatives for the other components in like manner gives

$$\bar{h}^{ij} = \frac{4\omega^2 \mu a^2}{r} \begin{pmatrix} -\cos 2\omega(t-r) & -\sin 2\omega(t-r) & 0 \\ -\sin 2\omega(t-r) & \cos 2\omega(t-r) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is Eq. (23.24). The triple time derivatives required to compute the gravitational wave power,

$$\dddot{I}^{xx}(t) = 4\omega^3 \mu a^2 \sin 2\omega t \qquad \dddot{I}^{yy}(t) = -4\omega^3 \mu a^2 \sin 2\omega t$$
$$\dddot{I}^{xy}(t) = \dddot{I}^{yx}(t) = -4\omega^3 \mu a^2 \cos 2\omega t \quad ,$$

may be found from the second derivatives computed above.

23.6 Differentiating both sides of Eq. (23.33) leads to

$$\frac{1}{P}\frac{dP}{dt} = \frac{3}{2a}\frac{da}{dt}.$$

Assume by energy conservation that the decay of the orbit causing the decrease in period results from emission of gravitational waves. The total energy of the binary orbital motion is given in Newtonian approximation by Eq. (23.32),

$$E = -\frac{Gm_1 m_2}{2a},$$

from which

$$\frac{1}{a}\frac{da}{dt} = -\frac{1}{E}\frac{dE}{dt},$$

and combining the first and third equations from above gives

$$\frac{1}{P}\frac{dP}{dt} = -\frac{3}{2}\frac{1}{E}\frac{dE}{dt}.$$

Equating the change in orbital energy with the energy carried off by gravitational waves, $dE/dt = -L$, and using Eq. (23.27) to specify L gives

$$\frac{1}{E}\frac{dE}{dt} = \frac{64}{5}\frac{G^3}{c^5}\frac{M^2\mu}{a^4}.$$

Therefore,

$$\frac{dP}{dt} = -\frac{3}{2}\frac{1}{E}\frac{dE}{dt}P = -\frac{96}{5}\frac{G^3}{c^5}\frac{M^2\mu}{a^4}P,$$

which is Eq. (23.34). The period P and the separation a are related by Kepler's 3rd law $a^3 = (GM/4\pi^2)P^2$, which can be used to eliminate a, giving an expression depending only on the period and masses

$$\frac{dP}{dt} = -\frac{192\pi}{5}\frac{G^{5/3}}{c^5}\frac{m_1 m_2}{M^{1/3}}\left(\frac{2\pi}{P}\right)^{5/3},$$

which is Eq. (23.35).

23.7 The mass of the system contributing to gravitational wave radiation is assumed to be $\sim 0.5 M_\odot$ and the effective radius is taken to be $R \simeq 2R_\odot \simeq 14 \times 10^5$ km. From Eq. (23.13)

$$\varepsilon^{2/7} = \frac{r_s}{R} = \frac{2(G/c^2)M}{R} = 2.95\left(\frac{M}{M_\odot}\right)\left(\frac{\mathrm{km}}{R}\right),$$

which gives for 44 Boo an efficiency $\varepsilon^{2/7} \sim 10^{-6}$. Therefore, from Eq. (23.15) and the period of 6.4 hours the amplitude and frequency of the expected gravitational wave metric perturbation is

$$\bar{h} \simeq 7 \times 10^{-21} \qquad f \simeq 8.7 \times 10^{-5}\,\mathrm{Hz},$$

since [see Eq. (23.23)], the gravitational wave frequency is twice the binary frequency of revolution. Consulting Fig. 22.8, the expected gravitational wave frequency is outside the favorable response range of LIGO or Virgo, but within the frequency window for LISA. Thus, it is possible that space-based arrays may be able to detect gravitational radiation from some galactic binaries.

23.9 From the information in Section 10.4.1, assume that the average separation of the neutron stars is $R \sim 2R_\odot$, the effective mass entering into generation of gravitational waves is $M \sim 1\,M_\odot$, and the distance is $r = 6.4$ kpc. Then from Eq. (23.13)

$$\varepsilon^{2/7} = \frac{r_\mathrm{s}}{R} = \frac{2.95(M/M_\odot)}{2R_\odot} = 2.1 \times 10^{-6},$$

and from Eq. (23.15)

$$\bar{h} = 9.6 \times 10^{-17} \varepsilon^{2/7} \left(\frac{M}{M_\odot} \right) \left(\frac{\mathrm{kpc}}{r} \right) \simeq 3.2 \times 10^{-23}.$$

The period of the binary is 7.75 hours, implying an orbital frequency $3.6 \times 10^{-5}\,\mathrm{s}^{-1}$. The gravitational wave frequency is twice that, $f = 7.2 \times 10^{-5}\,\mathrm{s}^{-1}$. From Fig. 22.8 this is roughly in the LISA frequency window but the strain is several orders of magnitude too small to be measurable by LISA.

24 Strong Sources of Gravitational Waves

24.2 The chirp waveform in the bottom panel of Fig. 24.4 indicates a binary merger. The theoretical chirp mass

$$\mathscr{M} = \frac{\mu^{3/5}}{M^{2/5}}$$

from Eq. (24.2) is plotted as a function of m_1 for different values of m_2 in Fig. 24.1 [this document]. The chirp mass $\mathscr{M} \sim 28 \pm 2\ M_\odot$ (Table 24.1; see also Problem 24.5) determined observationally from the frequency and its time derivative of the gravitational wave is indicated by the dashed horizontal line and gray uncertainty box. By summing m_1 and m_2 at the intersections of the curves with the $\mathscr{M} = 28$ line, one sees that the minimum total mass of the binary consistent with the chirp mass is around $65 M_\odot - 70 M_\odot$. From Problem 24.1 the separation of centers at the time of maximum frequency was about 350 km. Only black holes or neutron stars are compact enough to be consistent with that. Assuming neutron stars to have an upper mass limit of $\sim 2\ M_\odot$, two neutron stars would have far too little chirp mass to account for the data. From the $m_2 = 2\ M_\odot$ curve, for a neutron star and black hole to give the observed value of $\mathscr{M}$ the mass of the black hole would have to be huge, giving a very large total mass for the system that would lead to a much lower gravitational wave frequency than observed. Thus a black hole and neutron star binary is

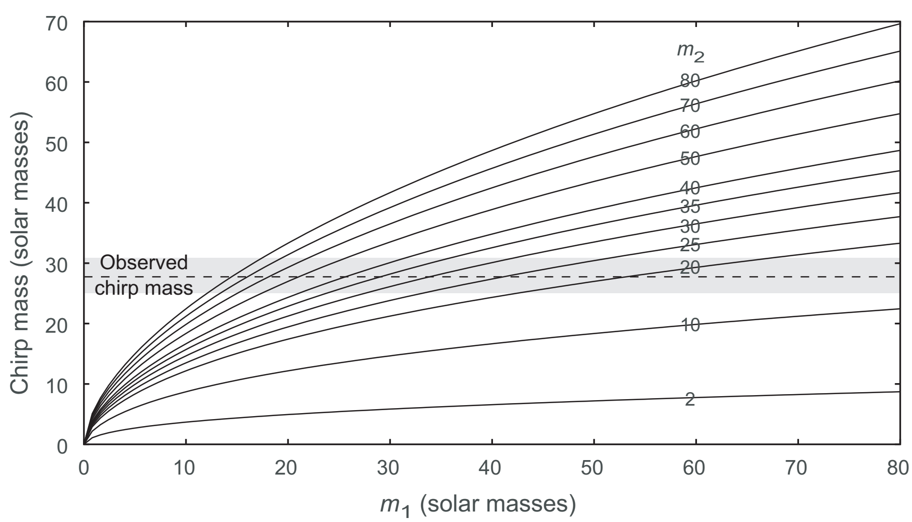


Fig. 24.1 Chirp mass for Problem 24.2 as a function of m_1 for different values of m_2.

ruled out, leaving merging black holes with a total summed mass near 70 $M_\odot$ as the only plausible explanation.

24.6 Assuming the validity of Newtonian mechanics and Newtonian gravity, the total (kinetic plus potential) energy is

$$E = \tfrac{1}{2}m_1 v_2^2 + \tfrac{1}{2}m_2 v_2^2 - \frac{Gm_1 m_2}{a} = \tfrac{1}{2}m_1 r_1^2 \omega^2 + \tfrac{1}{2}m_2 r_2^2 \omega^2 - \frac{Gm_1 m_2}{a},$$

where $v = \omega r$ has been used with $\omega \equiv 2\pi/P$. But

$$r_1 = \frac{m_2}{M}a \qquad r_2 = \frac{m_1}{M}a \qquad M \equiv m_1 + m_2,$$

allowing the total energy to be written

$$\begin{aligned} E &= \tfrac{1}{2}m_1 \left(\frac{m_2}{M}\right)^2 a^2 \omega^2 + \tfrac{1}{2}m_2 \left(\frac{m_1}{M}\right)^2 a^2 \omega^2 - \frac{Gm_1 m_2}{a} \\ &= \tfrac{1}{2}\mu a^2 \omega^2 - \frac{Gm_1 m_2}{a}, \end{aligned}$$

where $\mu \equiv m_1 m_2/M$ is the reduced mass. Eliminating the frequency ω using Kepler's 3rd law in the form $a^3 = GM/\omega^2$ then gives for the total orbital energy $E = -Gm_1 m_2/2a$.

25 Tests of General Relativity

25.1 Any viable relativistic gravitational theory should agree with the results of Newtonian gravity in the weak-field limit, as described in Section 8.1. There it was shown that the lowest-order relativistic correction to flat space modifies only the g_{00} component of the metric to $g_{00} = -(1 - 2GM/rc^2)$ [see Eq. (8.12)], with the other components unaltered to lowest order. Comparing with Eq. (25.1), agreement of general relativity with Newtonian gravity in the weak-field limit requires that to lowest order

$$A(r) = 1 - \frac{2GM}{rc^2} + \dots \qquad B(r) = 1 + \dots .$$

which is Eq. (25.3) to this order.

26 Beyond Standard Models

26.1 This solution follows an example in Zwiebach [257]. For a square well in two variables $(x,y) \sim (x,y+2\pi R)$, the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial x^2}\right)=E\psi.$$

Substituting $\psi(x,y)=\psi(x)\varphi(y)$, the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2}-\frac{\hbar^2}{2m}\frac{1}{\varphi(y)}\frac{d^2\varphi(y)}{dy^2}=E.$$

The solutions of this equation are

$$\psi_k(x)=c_k\sin\left(\frac{k\pi x}{a}\right) \qquad \varphi_\ell(y)=a_\ell\sin\left(\frac{\ell y}{R}\right)+b_\ell\cos\left(\frac{\ell y}{R}\right)$$
$$E_{k,\ell}=\frac{\hbar^2}{2m}\left[\left(\frac{k\pi}{a}\right)^2+\left(\frac{\ell}{R}\right)^2\right] \qquad (k=1,2,3,\ldots\infty;\ \ \ell=0,1,2,\ldots\infty),$$

where $\ell=0$ is allowed because of the boundary conditions in the y direction. As shown in Problem 26.2, if R is small the new states introduced by the compactified y dimension will be very high in energy.

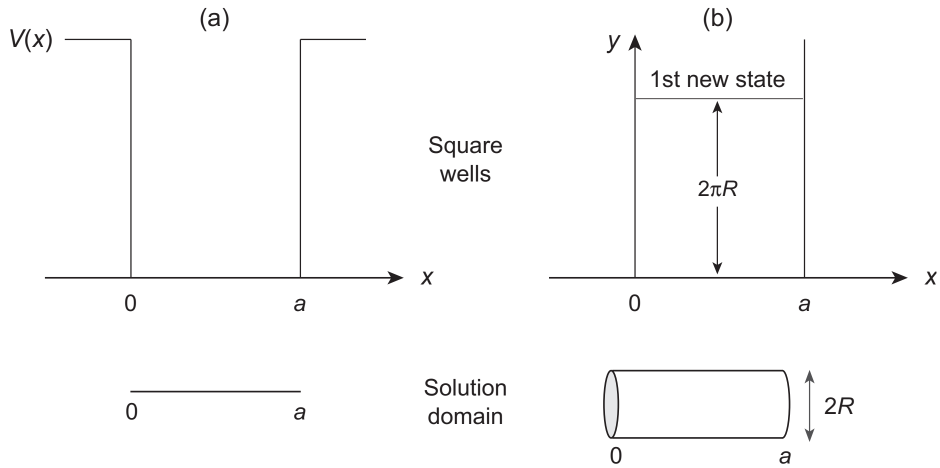
26.2 From the spectrum obtained in the solution of Problem 26.1, if $\ell=0$

$$E_{k,\ell}=\frac{\hbar^2}{2m}\left[\left(\frac{k\pi}{a}\right)^2+\left(\frac{\ell}{R}\right)^2\right] \quad\longrightarrow\quad \frac{\hbar^2}{2m}\left(\frac{k\pi}{a}\right)^2,$$

which is the spectrum of the 1-dimensional square well. Thus the states with $\ell=0$ are the old states of the 1-dimensional square well. The lowest-energy new state corresponds to $k=1$ and $\ell=1$, giving

$$E_{\min}^{\text{new}}=E_{1,1}=\frac{\hbar^2}{2m}\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{1}{R}\right)^2\right]\simeq\frac{\hbar^2}{2m}\left(\frac{1}{R}\right)^2,$$

where in the last step $R \ll a$ was assumed. This state has the energy of a state with $k=a/\pi R \gg 1$ in the original 1-dimensional spectrum, so it is very high in energy. The following figure illustrates schematically how the compactified dimension changes the spectrum.



In (a) the particle is restricted to a line between 0 and a. In (b) with the added compactified dimension the particle is restricted to the surface of a cylinder of length a and circumference $2\pi R$. If $R \ll a$, particle vibrations in the y direction are strongly restricted, which by uncertainty principle arguments means that they represent states having very high energy compared with those associated with motion in the x direction. Thus, if the radius of compactification is small compared with the characteristic length scales of a physical problem, the compactified dimensions have negligible influence on the low-energy spectrum.