the mathematical representation of putting in the atomic electron screening around a nucleus. Modifying the potential by putting in a screening factor of $e^{-x/a} \to e^{-s/a}$, where a is a distance of order a Bohr radius, the final integral becomes

$$2\pi \int ds (e^{(iq-1/a)s} - e^{(-iq-1/a)s})/iq, \tag{44}$$

or

$$\frac{2\pi}{iq}\left(\frac{e^{(iq-1/a)s}}{(iq-1/a)} - \frac{e^{(-iq-1/a)s}}{(-iq-1/a)}\right) \tag{45}$$

which with integration limits of 0 to ∞ evaluates to

$$\frac{2\pi}{iq}\left(\frac{-1}{(iq-1/a)} - \frac{-1}{(-iq-1/a)}\right). \tag{46}$$

Simplifying,

$$\frac{2\pi}{iq}\frac{2iq}{q^2+a^{-2}} = \frac{4\pi}{q^2+a^{-2}} \tag{47}$$

Now we can pull together an expression for the matrix element

$$V_{fi} = \frac{Ze^2}{4\pi}F(q)\frac{4\pi}{q^2+a^{-2}} = \frac{Ze^2F(q)}{q^2+a^{-2}} \tag{48}$$

For intermediate and high energy energy scattering, momentum transfers are usually 1 or more fm^{-1}. The range parameter a, being about a Bohr radius, would give $1/a^2 \approx 10^{-8}\mathrm{fm}^{-2}$. Since $q^2 >> 1/a^2$, we may drop the latter term. Thus

$$V_{fi} = \frac{Ze^2F(q)}{q^2} \tag{49}$$

The expression for the transition probability becomes

$$W = 2\pi V_{fi}^2\rho(E_f) = \frac{2\pi Z^2e^4F^2(q)}{q^4}\rho(E_f) \tag{50}$$

4 $F(q)$

Above we showed that the cross section directly measures the form factor (squared), which can be calculated from the charge distribution as

$$F(q) = \int d^3\vec{R}\rho(\vec{R})e^{i\vec{q}\cdot\vec{R}}. \tag{51}$$

We also pointed out that a point particle has a constant (or "hard") form factor,

$$F(q) = \int d^3\vec{R}\rho_0\delta(R)e^{i\vec{q}\cdot\vec{R}} = \rho_0 \int d^3\vec{R}\delta(R) = (2\pi)^3\rho_0. \tag{52}$$

The form factor is constant; since it does not decrease with momentum transfer, it is referred to as a "hard" form factor.

Usually the charge distribution is spherically symmetric, $\rho(\vec{R}) = \rho(R)$. In this case, the Fourier integral can be converted into an integral over R, $\cos\theta$, and an azimuthal angle, as was done above in the previous discussion. Similarly, by integrating over the two angle variables, we can reduce the Fourier integral to

$$F(q) = \int_0^\infty \rho(R)R^2 dR 2\pi \int_{-1}^{1} d\cos\theta e^{iqR\cos\theta}, \tag{53}$$

which becomes

$$F(q) = 2\pi \int_0^\infty \rho(R)R^2 dR \frac{e^{iqR} - e^{-iqR}}{iqR} \tag{54}$$

$$F(q) = 4\pi \int_0^\infty \rho(R)R^2 dR \frac{\sin qR}{qR}. \tag{55}$$

The latter form looks nicer, but the earlier form is often nicer for integration. In the following sections, we are generally going to leave out the integration limits for simplicity of notation, but we will evaluate the definite integrals in the end.

4.1 Yukawa distribution

There exist other charge distributions for which the Fourier transform can be calculated analytically. An algebraically simple example is the Yukawa distribution,

$$\rho(R) = \rho_0 e^{-R/R_0}/R. \tag{56}$$

The constant ρ_0 is found by normalizing the distribution to unity:

$$4\pi\rho_0 \int R^2 dR e^{-R/R_0}/R = 1 \tag{57}$$

$$4\pi\rho_0 \int R dR e^{-R/R_0} = 1. \tag{58}$$

Integrate by parts, letting $u = R$, $du = dR$, $dv = e^{-R/R_0}dR$, and $v = -R_0e^{-R/R_0}$. Then

$$-RR_0e^{-R/R_0} - \int -R_0e^{-R/R_0}dR = 1/4\pi\rho_0, \tag{59}$$

leading to

$$-RR_0e^{-R/R_0} - R_0^2e^{-R/R_0}dR = 1/4\pi\rho_0. \tag{60}$$

The upper limit at ∞ leads to 0 from the exponential, while the limit at 0 removes the first term. Thus

$$R_0^2 = 1/4\pi\rho_0, \tag{61}$$

and

$$\rho_0 = 1/4\pi R_0^2 \tag{62}$$

The rms (root mean square) charge radius is messier, since it starts from the integral

$$< r^2 >= 4\pi\rho_0 \int R^4dRe^{-R/R_0}/R. \tag{63}$$

We will have a series of integrations by parts, in which we know that the uv term will cancel when taking the limits. The progression is

$$< r^2 > = 4\pi\rho_0(3R_0)\int R^2dRe^{-R/R_0} \tag{64}$$

$$< r^2 > = 4\pi\rho_0(3R_0)(2R_0)\int RdRe^{-R/R_0} \tag{65}$$

$$< r^2 > = 4\pi\rho_0(3R_0)(2R_0)(R_0)\int dRe^{-R/R_0} \tag{66}$$

$$< r^2 > = 4\pi\rho_0(3R_0)(2R_0)(R_0)(R_0). \tag{67}$$

Putting in $\rho_0 = 1/4\pi R_0^2$, we see that $< r^2 >= 6R_0^2$, and $< r^2 >^{1/2}= \sqrt{6}R_0$.

The Fourier transform is best done using the exponential form given above,

$$F(q) = 2\pi\rho_0 \int (e^{-R/R_0}/R)R^2dR\frac{e^{iqR} - e^{-iqR}}{iqR} \tag{68}$$

$$F(q) = 2\pi\rho_0 \int dR\frac{e^{(iq-1/R_0)R} - e^{(-iq-1/R_0)R}}{iq}. \tag{69}$$

The integration yields

$$F(q) = 2\pi\rho_0 \frac{e^{(iq-1/R_0)R}}{iq(iq-1/R_0)} - \frac{e^{(-iq-1/R_0)R}}{iq(-iq-1/R_0)} \tag{70}$$

$$F(q) = 2\pi\rho_0 \frac{e^{-R/R_0}}{iq}\left(\frac{e^{iqR}}{iq-1/R_0} + \frac{e^{-iqR}}{iq+1/R_0}\right). \tag{71}$$

The exponential e^{-R/R_0} drives the expression to 0 at the limit of ∞, so we only need to work out its value at $R = 0$, at which all the exponentials go to 1, and there is an overall minus sign from the subtraction. Thus,

$$F(q) = -2\pi\rho_0 \frac{1}{iq}\left(\frac{1}{iq-1/R_0} + \frac{1}{iq+1/R_0}\right). \tag{72}$$

The fraction within the parentheses can be simplified, leading to

$$F(q) = -2\pi\rho_0 \frac{1}{iq}\left(\frac{iq+1/R_0+iq-1/R_0}{-q^2-1/R_0^2}\right), \tag{73}$$

and furthermore to

$$F(q) = 2\pi\rho_0\left(\frac{2}{q^2+1/R_0^2}\right), \tag{74}$$

Again substituting with $\rho_0 = 1/4\pi R_0^2$, we get

$$F(q) = \frac{1}{1+q^2R_0^2}, \tag{75}$$

which is a monopole form factor. We can express this also in terms of the rms charge radius as

$$F(q) = \frac{1}{1+q^2 < r^2 > /6}, \tag{76}$$

and expand, at least for small q^2, as

$$F(q) = 1 - q^2 < r^2 > /6 + O(q^4). \tag{77}$$

4.2 $F(q)$ for small q

It is possible to show that the form given above holds for any form factor at low momentum transfer. The technique is to expand the integrand in a Taylor series before doing the Fourier integral. Start from the expression

$$F(q) = 4\pi \int \rho(R) R^2 dR \frac{\sin qr}{qr}. \tag{78}$$

The Taylor series for $\sin x/x$ is

$$\sin x/x = \frac{x - x^3/3! + \cdots}{x} = 1 - x^2/3! + \cdots \tag{79}$$

The Fourier transform becomes

$$F(q) = 4\pi \int \rho(R) R^2 dR (1 - (qR)^2/6 + \cdots). \tag{80}$$

The first constant term is just the normalization integral, so it is "obviously" unity. The second term is

$$F_2(q) = 4\pi \frac{-q^2}{6} \int \rho(R) R^2 dR R^2. \tag{81}$$

Except for the factor $-q^2/6$, this is just the integral we used to determine the rms radius squared, $< R^2 >$. Thus, the Fourier transform becomes:

$$F(q) = 1 - \frac{q^2 < R^2 >}{6} + \cdots, \tag{82}$$

as we aimed to show. The factor of 6 comes from the coefficient of the second term in the expansion of $\sin x$.

4.3 Hard sphere distribution

As another example, let's consider a distribution of charge that is constant up to a radius r, and 0 outside that radius. The distribution, with unit normalization, is

$$\rho(R) = \frac{3}{4\pi r^3} \Theta(r - R). \tag{83}$$

The rms charge radius is calculated from

$$< R^2 >= \frac{3}{4\pi r^3} 4\pi \int_0^\infty \Theta(r-R) R^2 R^2 dR, \tag{84}$$

which becomes

$$< R^2 >= \frac{3}{r^3}\frac{r^5}{5} = \frac{3}{5}r^2. \tag{85}$$

Now let's carry out the Fourier integration. The Θ function acts to change the limits of integration so that

$$F(q) = 4\pi \frac{3}{4\pi r^3} \int_0^\infty \Theta(r-R) R^2 dR \frac{\sin qR}{qR} \tag{86}$$

becomes

$$F(q) = \frac{3}{qr^3} \int_0^r dR R \sin(qR). \tag{87}$$

Integrate by parts, with $u = R$, $du = dR$, $dv = \sin(qR)dR$, and $v = \cos(qR)/(-q)$. This gives

$$F(q) = \frac{3}{qr^3}\left(\frac{R\cos(qR)}{-q}|_0^r - \int_0^r dR\cos(qR)/(-q)\right) \tag{88}$$

$$F(q) = \frac{3}{qr^3}\left(\frac{r\cos(qr)}{-q} - \sin(qR)/(-q^2)|_0^r\right) \tag{89}$$

$$F(q) = \frac{3}{qr^3}\left(\frac{r\cos(qr)}{-q} - \sin(qr)/(-q^2)\right) \tag{90}$$

$$F(q) = \frac{-3}{(qr)^2}\left(\cos(qr) - \frac{\sin(qr)}{qr}\right) \tag{91}$$

This expression is not obviously finite at the origin, so we expand the sinusoids to get

$$F(q) = \frac{-3}{(qr)^2}\left(1 - (qr)^2/2 + \cdots - 1 + (qr)^2/6 - \cdots\right), \tag{92}$$

leading to

$$F(q) = \frac{-3}{(qr)^2}\left(\frac{-(qr)^2}{3}\right) + \cdots \tag{93}$$

which you can see is equal to unity in leading order! This is exactly what we should get for the leading order term, if we have normalized the charge distribution to unity. The next to leading order term should give $-q^2 < R^2 > /6 = -q^2r^2/10$, where again r is the radius of the charge distribution.

We close this section with the comment that, experimentally, one can measure the cross section and determine $F(q)$. Then, by performing the inverse Fourier transform, the charge distribution can be calculated. This procedure has been followed for many nuclei. Instead of this "model-independent" procedure, one may also use a parameterized form for the charge distribution, such as a Gaussian or Fermi function, and then fit parameters of that model.

5 $\rho(E_f)$

The density of states is the number of states at a total energy in the final state of E_f. One of the nicer discussions I have seen on this topic is in the old Frauenfelder and Henley.

For a single particle, we recall from statistical mechanics that there is one state per $(2\pi)^3$ volume in phase space - neglecting spin degrees of freedom and recalling we are in units of $h = 2\pi\hbar = 2\pi$. Then the density of states for an energy E_F is

$$\rho(E_f) = \frac{dN}{dE_f} = \frac{V}{(2\pi)^3}\frac{d}{dE}\int p^2 dp d\Omega. \tag{94}$$

We are free to choose the volume V for normalization, since it must cancel in the end. For simplicity, we choose $V = 1$. Using $E^2 = p^2 + m^2$, we see that $EdE = pdp$, or $\frac{d}{dE} = \frac{E}{p}\frac{d}{dp}$. The momentum derivative will cancel the integration, leaving

$$\rho(E_f) = \frac{1}{(2\pi)^3}\frac{E}{p}p^2 d\Omega = \frac{Ep}{(2\pi)^3}d\Omega. \tag{95}$$

For a relativistic particle, we may further assume that $E = p$.

The calculation above is appropriate for a single particle, which implies a potential scattering problem, or a very heavy non-recoiling target. For two particles with the target recoiling, a similar derivation can be simply done in the center of mass of the reaction.