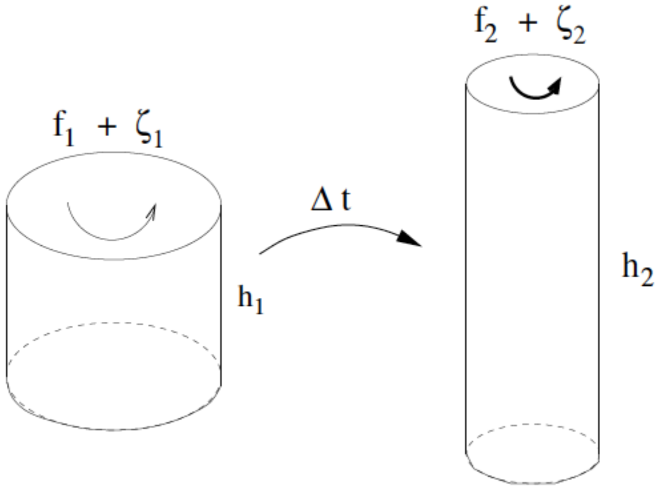
Οι εξισώσεις αβαθούς ωκεανού



5. Shallow water dynamics (barotropic and reduced gravity approximations)

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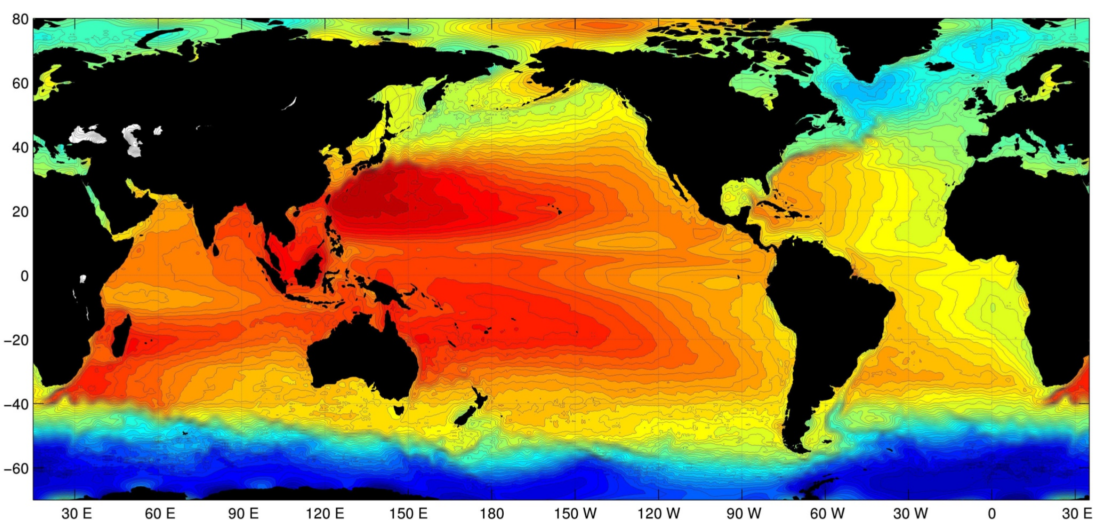
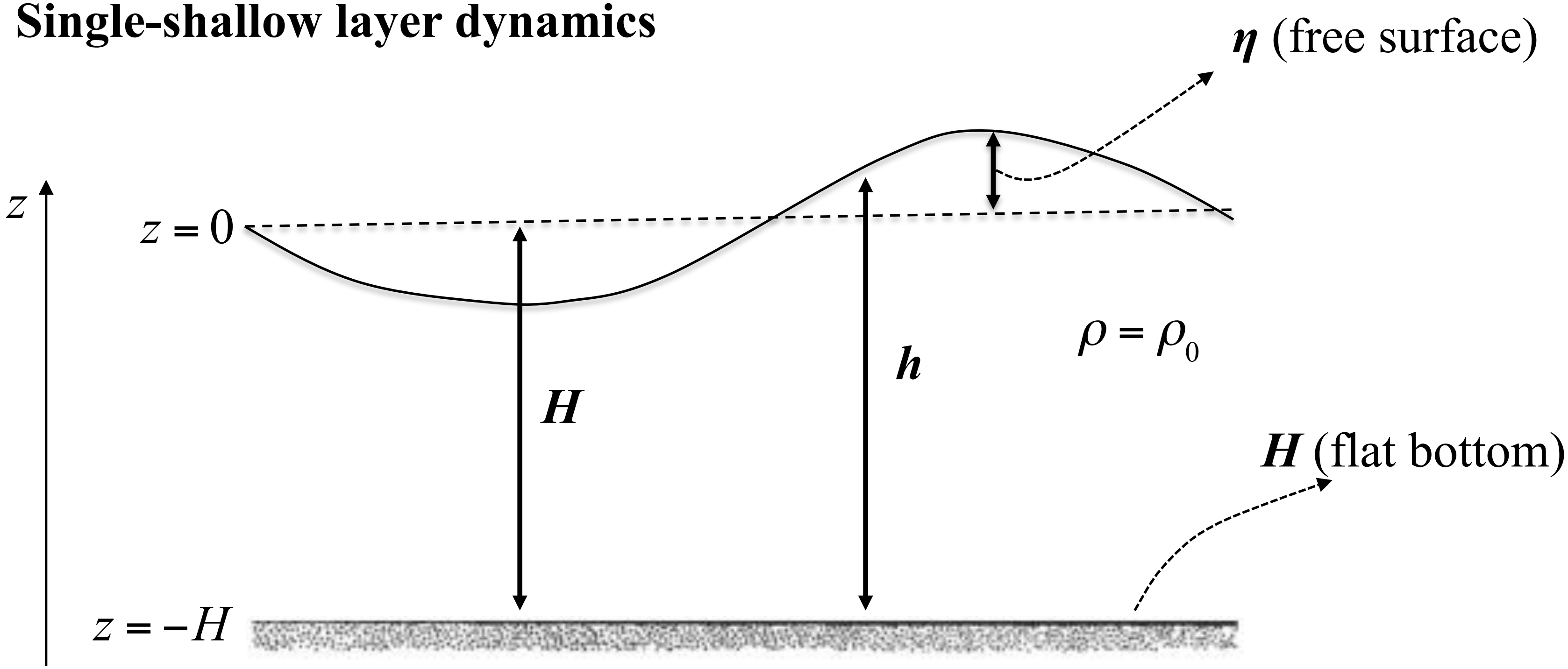


- **Single-shallow layer dynamics**
- **Vorticity**
- **Reduced gravity models**

Single-shallow layer dynamics



$$\delta_a = \frac{H}{L} << 1 \qquad F_R = \frac{U}{\sqrt{gH}} << 1 \qquad E_k = \frac{A_H}{fL^2} << 1$$

Large scale, ocean interior and inviscid flow (for simplicity)

$$\rho_0 \frac{Dw}{Dt} \approx \rho_0 \frac{W}{T} \approx \rho_0 \frac{WU}{L} \approx \rho_0 \frac{U\frac{H}{L}U}{L} \approx \rho_0 \frac{U^2 H}{L^2}\cdot\frac{H}{H} \approx \rho_0 \frac{U^2 H^2}{HL^2}$$

$$\Rightarrow \frac{\rho_0 \frac{Dw}{Dt}}{g} \approx \rho_0 \frac{U^2 H^2}{gHL^2} = \rho_0 \cdot \left(F_R\right)^2 \cdot \left(\delta_a\right)^2 << 1$$

Hydrostatic approximation is valid

$$\Rightarrow \frac{\partial p}{\partial z} = -\rho_0 g$$

$$\Rightarrow dp = -\rho_0 g dz \Rightarrow \int_z^\eta dp = -\int_z^\eta \rho_0 g\, dz \Rightarrow p_\eta - p_z = -\rho_0 g\left(\eta - z\right) \Rightarrow$$

$$\Rightarrow p = \rho_0 g\left(\eta\left(x,y\right) - z\right) + p_{atm}$$

$$\Rightarrow \nabla_H p = -\rho_0 g \nabla_H \eta$$

Using the momentum equations (and u,υ independent of z):

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + f\upsilon + \cancel{A_H \frac{\partial^2 u}{\partial x^2}} + \cancel{A_H \frac{\partial^2 u}{\partial y^2}} + \cancel{A_V \frac{\partial^2 u}{\partial z^2}} \\ \frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - fu + \cancel{A_H \frac{\partial^2 \upsilon}{\partial x^2}} + \cancel{A_H \frac{\partial^2 \upsilon}{\partial y^2}} + \cancel{A_V \frac{\partial^2 \upsilon}{\partial z^2}} \end{cases}$$

substituting:

$$\left.\begin{aligned}&\Rightarrow \frac{1}{\rho}\frac{\partial P}{\partial x} = g\frac{\partial \eta}{\partial x}\\ &\Rightarrow \frac{1}{\rho}\frac{\partial P}{\partial x} = g\frac{\partial \eta}{\partial x}\end{aligned}\right\}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x} + f\upsilon$$

$$\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} = -g\frac{\partial \eta}{\partial y} - fu$$

Shallow Water Momentum Equations

The right-hand side of this equation is independent of the vertical coordinate **z (η=f(x,y))**. Thus, if the flow is initially independent of **z**, it must stay so.The velocities **u** and **v** are functions of **x, y** and **t** only.

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (u,\upsilon \not\propto z)$$

$$\frac{\partial u}{\partial x}\int_{-H}^{\eta} dz + \frac{\partial \upsilon}{\partial y}\int_{-H}^{\eta} dz + \frac{\partial}{\partial z}\int_{-H}^{\eta} w\, dz = 0$$

$$(\eta + H)\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) + w(\eta) - w(-H) = 0 \ (1)$$

$$w(\eta) = \frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + \upsilon\frac{\partial \eta}{\partial y} \ (2)$$

$$(1)\,\&\,(2) \Rightarrow \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + \upsilon\frac{\partial \eta}{\partial y} + (\eta + H)\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) = 0$$

$$\Rightarrow \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[u(\eta + H)\right] + \frac{\partial}{\partial y}\left[\upsilon(\eta + H)\right] = 0$$

$$\Rightarrow \frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) = 0 \quad \eta << H$$

Shallow Water Continuity Equation

Shallow-water equations (single layer model)

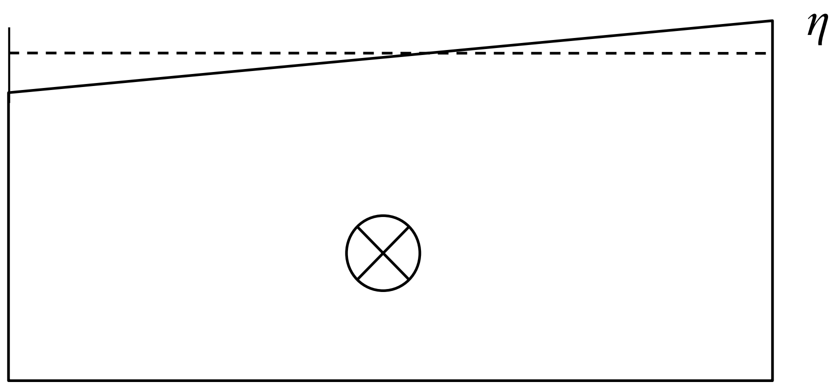
$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x} + fv$$

$$\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} = -g\frac{\partial \eta}{\partial y} - fu$$

Shallow water geostrophic approximation:

Now, additionally $R_o = \frac{U}{fL} << 1$

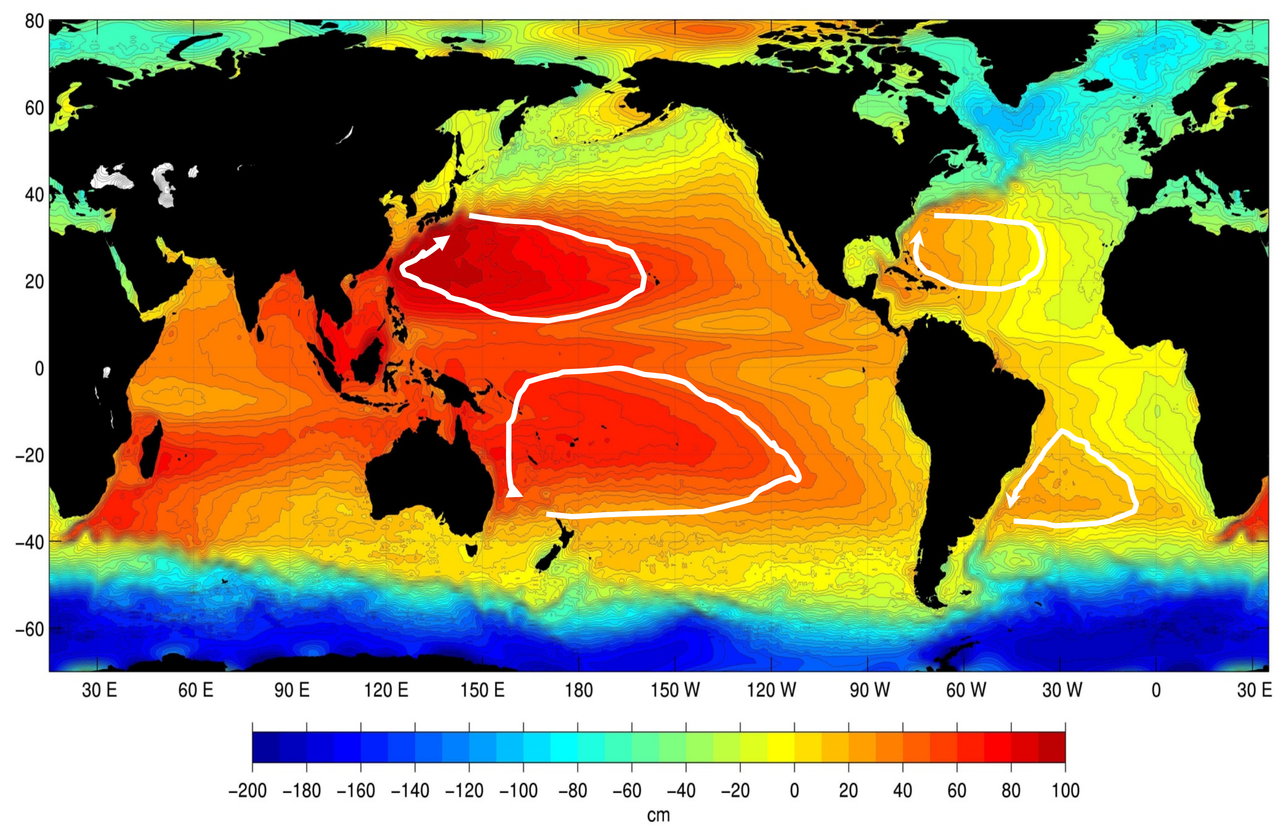


$$\begin{cases} \cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{\upsilon\frac{\partial u}{\partial y}} = -g\frac{\partial \eta}{\partial x} + f\upsilon \\ \cancel{\frac{\partial \upsilon}{\partial t}} + \cancel{u\frac{\partial \upsilon}{\partial x}} + \cancel{\upsilon\frac{\partial \upsilon}{\partial y}} = -g\frac{\partial \eta}{\partial y} - fu \end{cases}$$

$$\upsilon = \frac{g}{f}\frac{\partial \eta}{\partial x}$$

$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) = 0$$



Vorticity (στροβιλισμός)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-g\frac{\partial \eta}{\partial x}+f\upsilon \qquad (1)$$

$$\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}=-g\frac{\partial \eta}{\partial y}-fu \qquad (2)$$

$$\frac{1}{h}\frac{dh}{dt}+\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)=0 \qquad (3)$$

$$\frac{\partial}{\partial x}(2)-\frac{\partial}{\partial y}(1)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \upsilon}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial u}{\partial x}\frac{\partial \upsilon}{\partial x}+u\frac{\partial^2 \upsilon}{\partial x\partial x}+\frac{\partial \upsilon}{\partial x}\frac{\partial \upsilon}{\partial y}+\upsilon\frac{\partial^2 \upsilon}{\partial x\partial y}-$$

$$-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-u\frac{\partial^2 u}{\partial y\partial x}-\frac{\partial \upsilon}{\partial y}\frac{\partial u}{\partial y}-\upsilon\frac{\partial^2 u}{\partial y\partial y}=$$

$$=-g\frac{\partial^2 \eta}{\partial x\partial y}+g\frac{\partial^2 \eta}{\partial x\partial y}-f\frac{\partial u}{\partial x}-u\frac{\partial f}{\partial x}-f\frac{\partial \upsilon}{\partial y}-\upsilon\frac{\partial f}{\partial y}$$

$$\upsilon\frac{\partial f}{\partial y} \rightarrow =\frac{df}{dt}$$

$$(u,\upsilon \not\propto z)$$

$$f \not\propto x$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \upsilon}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial \upsilon}{\partial x}-\frac{\partial u}{\partial y}\right)+\upsilon\frac{\partial}{\partial y}\left(\frac{\partial \upsilon}{\partial x}-\frac{\partial u}{\partial y}\right)+$$

$$=\frac{d}{dt}\left(\frac{\partial \upsilon}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$+\frac{\partial u}{\partial x}\left(\frac{\partial \upsilon}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial \upsilon}{\partial y}\left(\frac{\partial \upsilon}{\partial x}-\frac{\partial u}{\partial y}\right)=$$

$$=\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)\left(\frac{\partial \upsilon}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$=-f\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)-\frac{df}{dt}$$

Defining **relative vorticity:** $\frac{\partial \upsilon}{\partial x}-\frac{\partial u}{\partial y}=\zeta$ (ocean velocity curl)

$$R_o=\frac{U}{fL} \ll 1$$

using (3):

$$h\frac{d}{dt}(f+\zeta)-(f+\zeta)\frac{dh}{dt}=0$$

or

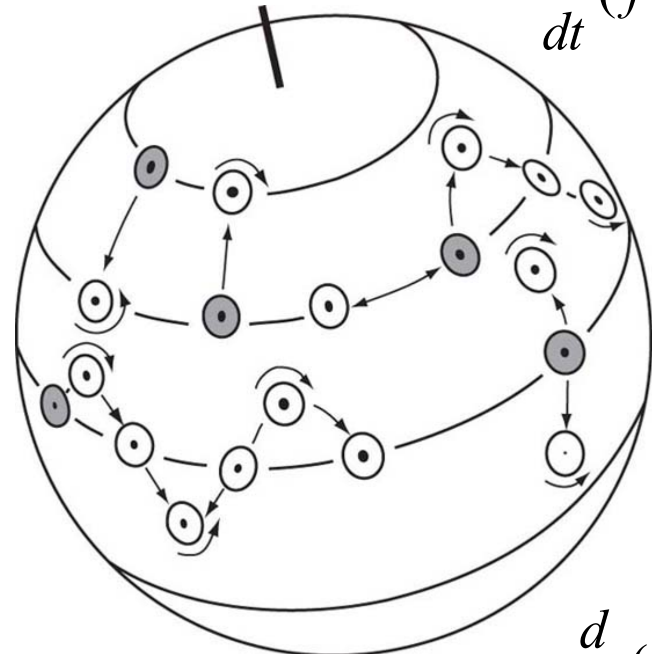
$$\boxed{\frac{d}{dt}\left(\frac{f+\zeta}{h}\right)=0}$$

Potential Vorticity (δυναμικός στροβιλισμός)

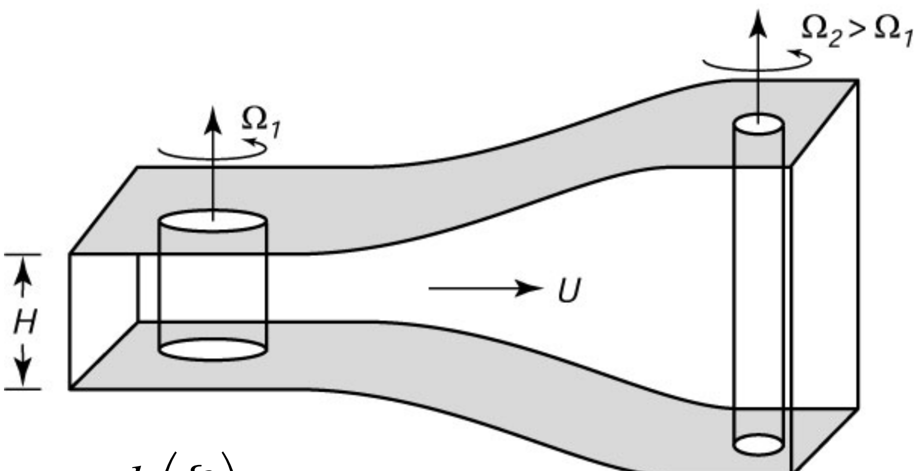
Earth's rotation ← f **Spinning** ← ζ **Stretching** ← h

$$\frac{d}{dt}\left(\frac{f+\zeta}{h}\right)=0$$

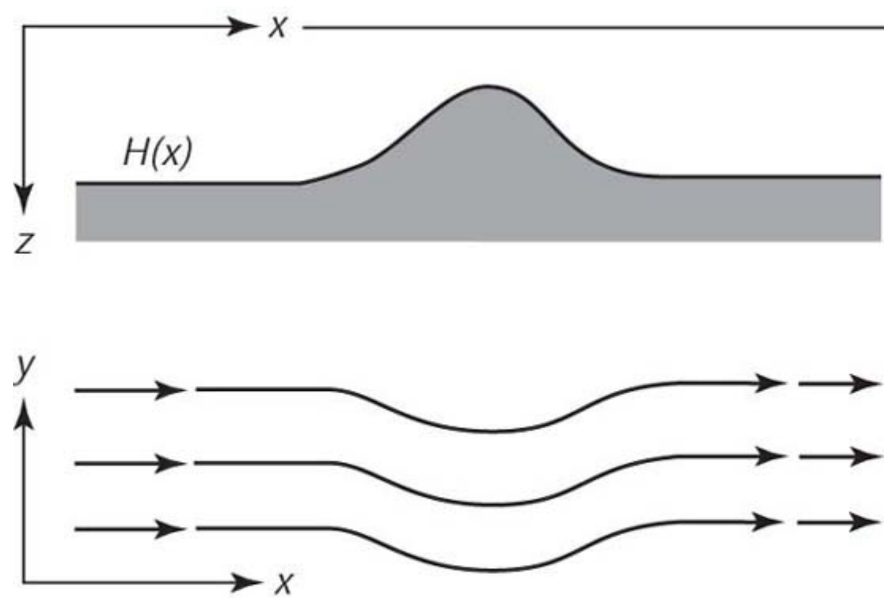
$$\frac{d}{dt}(f+\zeta)=0$$



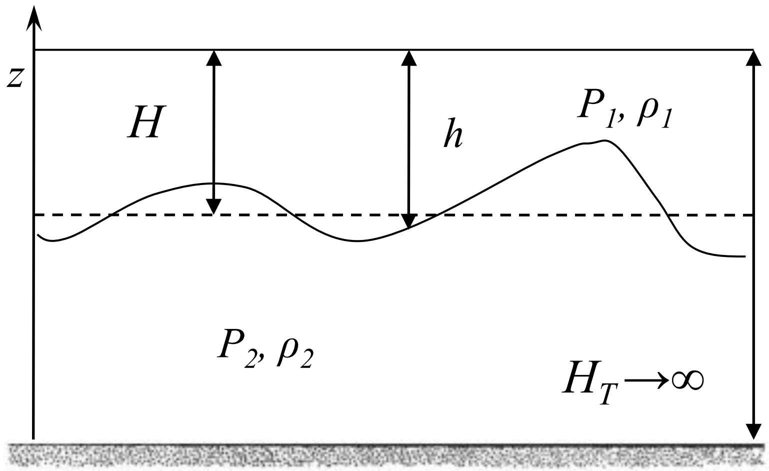
$$\frac{d}{dt}\left(\frac{f+\zeta}{H}\right)=0$$

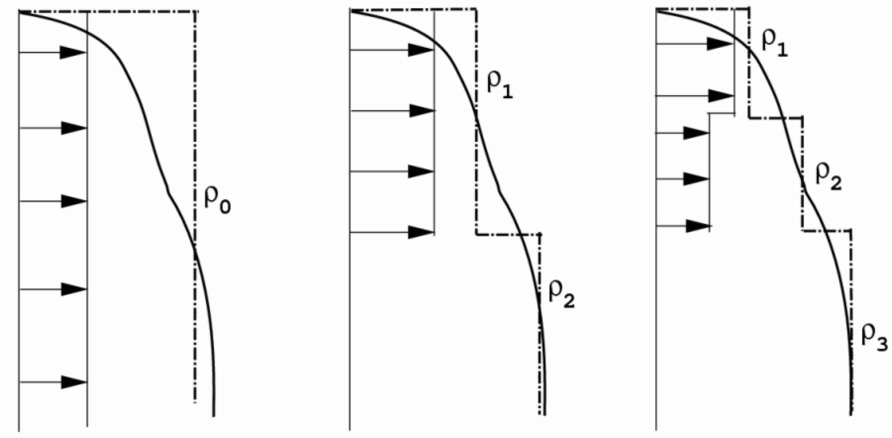


$$\frac{d}{dt}\left(\frac{\zeta}{h}\right)=0$$



Reduced gravity models





$$P_1 = P_a - \rho_1 g z; \quad P_2 = P_a + \rho_1 g h - \rho_2 g (z+h)$$

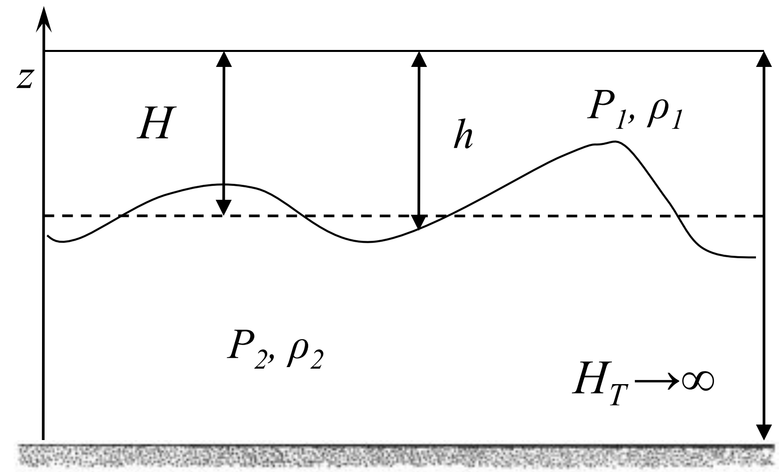
$$\nabla_H P_2 = 0$$

$$\nabla_H P_a + \rho_1 g \nabla_H h - \rho_2 g \nabla_H h = 0$$

$$\Rightarrow \nabla_H P_a = (\rho_2 - \rho_1) g \nabla_H h$$

$$\Rightarrow \frac{1}{\rho_0} \nabla_H P_1 = \frac{(\rho_2 - \rho_1)}{\rho_0} g \nabla_H h = g' \nabla_H h$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \upsilon\dfrac{\partial u}{\partial y} - f\upsilon = -g'\dfrac{\partial h}{\partial x} \\ \dfrac{\partial \upsilon}{\partial t} + u\dfrac{\partial \upsilon}{\partial x} + \upsilon\dfrac{\partial \upsilon}{\partial y} + fu = -g'\dfrac{\partial h}{\partial y} \end{cases}$$



$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial}{\partial x}\int_{-h}^{0} u\,dz + \frac{\partial}{\partial y}\int_{-h}^{o} \upsilon\,dz + \frac{\partial}{\partial z}\int_{-h}^{0} w\,dz = 0$$

$$\frac{\partial}{\partial x}\left[uh\right] + \frac{\partial}{\partial y}\left[\upsilon h\right] + w(0) - w(-h) = 0; \quad w(-h) = -\frac{\partial h}{\partial t}$$

$$\Rightarrow \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(hu\right) + \frac{\partial}{\partial y}\left(h\upsilon\right) = 0$$

$$g' = g\frac{\Delta\rho}{\rho_0}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} - f\upsilon = -g'\frac{\partial h}{\partial x}$$

$$\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + fu = -g'\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(hu\right) + \frac{\partial}{\partial y}\left(h\upsilon\right) = 0$$

Reduced gravity