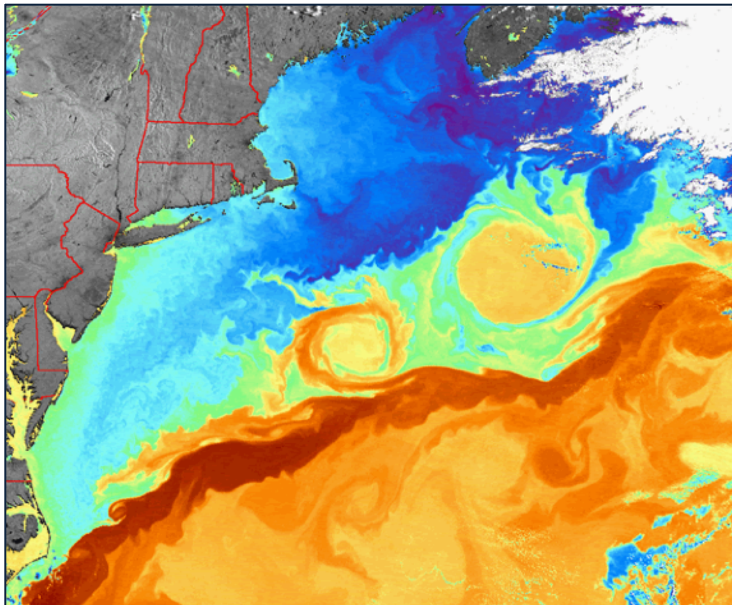
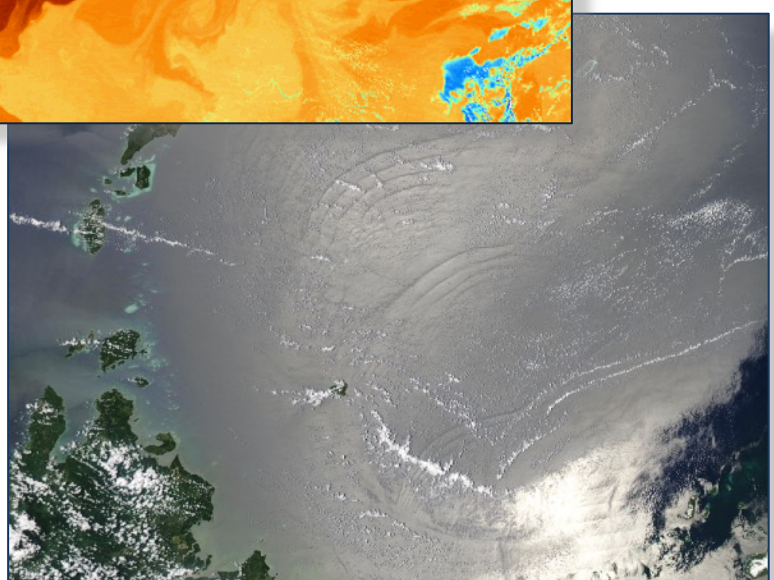
Department of Physics
University of Athens

4. Οι νόμοι διατήρησης στη Φυσική Ωκεανογραφία





4. The conservation laws in the ocean dynamics

- **Βασικοί Νόμοι - Fundamental dynamics and the equations for geophysical fluids**
- **Προσεγγίσεις - Oceanic approximations**
- **Περιστροφή - Rotation**
- **Διαστατική ανάλυση - Dominant scales in ocean dynamics**
- **Εφαρμογή της διαστατικής ανάλυσης – Scaling the ocean dynamics**
- **Μοντέλα διακριτών στρωμάτων**

Σ. Σοφιανός & Β. Βερβάτης

Α. Βασικές Εξισώσεις

i. Momentum Equation:

Pressure gradient

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla p + \nabla\Phi + \mathbf{F}$$

Material derivative

$$\frac{d\mathbf{u}}{dt} = \frac{\partial\mathbf{u}}{dt} + \mathbf{u}\cdot\nabla\mathbf{u}$$

advection

local rate of change

$$\Phi = -gz$$

Force potential

Non-conservative forces

e.g. $\mathbf{F} = \nu\nabla^2\mathbf{u}$

(see next transparency)

$$\boxed{\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p - g\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u}} \quad \textbf{(i)}$$

Non-linearity

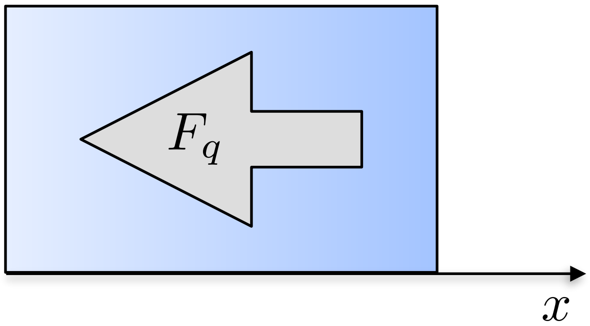
$$Re = \frac{\mathbf{u}\nabla\mathbf{u}}{\nu\nabla^2\mathbf{u}} = \frac{U^2L^2}{\nu UL} = \frac{UL}{\nu}$$

In the ocean, usually

$$Re \gg 1$$

Διάχυση ιδιοτήτων - ιξώδες

Τυχαίες (μοριακές) κινήσεις μεταφέρουν ιδιότητες (π.χ. q) στο ρευστό,



- **Flux=Force/Resistance** (De Groot, 1963)

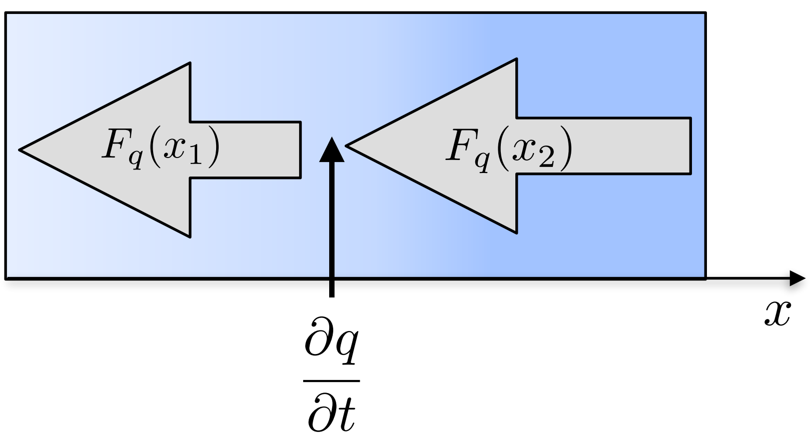
$$F_q = -\kappa_q \frac{\partial q}{\partial x}$$

Flux of q

medium Resistance

Force (gradient)

Τυχαίες (μοριακές) κινήσεις μεταφέρουν ιδιότητες (π.χ. q) στο ρευστό,

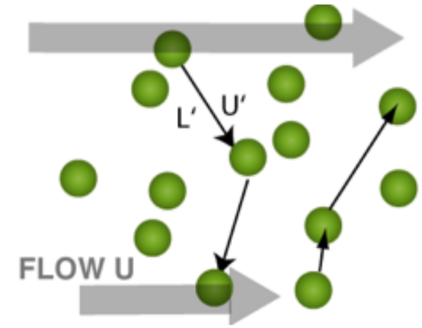


$$\frac{\partial q}{\partial t} = -\frac{\partial F_q}{\partial x} = \kappa_q \frac{\partial^2 q}{\partial x^2}$$

Για την ταχύτητα: resistance is μ (molecular viscosity) and $\nu=\mu/\rho$ (kinematic viscosity):

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

Why is viscosity a non-conservative force?



Ignoring any other force and density variations (and working only in one dimension for simplicity):

Multiplying by ρu

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \Rightarrow \rho u \frac{\partial u}{\partial t} = \rho u \nu \frac{\partial^2 u}{\partial x^2} \Rightarrow$$

$$\Rightarrow \frac{\partial \left(\rho u^2/2\right)}{\partial t} = \rho\nu \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right) - \rho\nu \left(\frac{\partial u}{\partial x}\right)^2 \Rightarrow$$

$$\Rightarrow \frac{\partial \left(\rho u^2/2\right)}{\partial t} = \nu \frac{\partial}{\partial x}\left(\frac{\partial \left(\rho u^2/2\right)}{\partial x}\right) - \rho\nu \left(\frac{\partial u}{\partial x}\right)^2 \Rightarrow$$

$$\Rightarrow \frac{\partial \left(\rho u^2/2\right)}{\partial t} = \nu \frac{\partial^2 \left(\rho u^2/2\right)}{\partial x^2} - \rho\nu \left(\frac{\partial u}{\partial x}\right)^2$$

In three-dimensions

$$\boxed{\frac{\partial \left(\rho u^2/2\right)}{\partial t} = \frac{\partial}{\partial z}\left(\nu \left(\frac{\partial \left(\rho u^2/2\right)}{\partial z}\right)\right) - \rho\nu \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]}$$

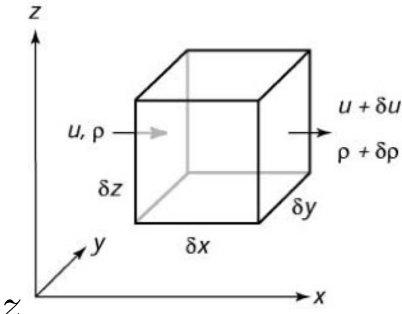
The heat added to the water increases its entropy at the rate of heat generation divided by absolute temperature (entropy source term).

Local rate of change of energy concentration

Energy Flux (Divergence of viscosity times the gradient of energy)

Loss of Mechanical Energy (Transformation to heat – always negative)

ii. Διατήρηση της μάζας (continuity equation):



Mass flux to the volume (x=direction) $= \rho u \delta y \delta z$

Mass flux from the volume (x=direction) $= \left(\rho + \frac{\partial \rho}{\partial x}\delta x\right)\left(u + \frac{\partial u}{\partial x}\delta x\right)\delta y \delta z$

$$\text{Accumulation} \quad = \left(u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x}\frac{\partial u}{\partial x}\delta x\right)\delta x \delta y \delta z = \left(\frac{\partial(\rho u)}{\partial x} + O(\cancel{\delta x})\right)\delta x \delta y \delta z$$

small terms

$$\text{In 3-D:} \quad = \left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)\delta x \delta y \delta z$$

This accumulation must be accompanied by increase of mass in the volume

$$\frac{\partial}{\partial t}(\text{density} \times \text{volume}) = \left(\frac{\partial \rho}{\partial t}\right)\delta x \delta y \delta z = -\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)\delta x \delta y \delta z$$

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \mathbf{u}\nabla\rho + \rho\nabla\mathbf{u} = 0$$

$$\boxed{\frac{d\rho}{dt} = -\rho\nabla\mathbf{u}} \quad \textbf{(ii)}$$

iii. Conservation of dissolved material concentration:

$$\frac{\partial(\rho q)}{\partial t} + \nabla(\rho q \mathbf{u}) = S^q$$

non conservative sources and sinks of q

e.g. $\kappa \nabla^2 q$

and using the continuity equation

$$\boxed{\frac{Dq}{Dt} = \kappa \nabla^2 q + S^q}$$

For the ocean (and the atmosphere) κ is of the order of ν, i.e. the ***Prandtl number***

$$\nu / \kappa = O(1)$$

thus advection dominates diffusion (as *Re* is very large).

iv. Internal energy conservation:

$$\rho \frac{De}{Dt} = -p\nabla \mathbf{u} + \rho Q$$

Change of internal energy due to pressure variations

cooling/heating

and using the continuity equation

$$\boxed{\frac{De}{Dt} = -p \frac{D}{Dt}\left(\frac{1}{\rho}\right) + Q}$$

1st Law of Thermodynamics

v. Entropy conservation:

$$\boxed{T \frac{D\eta}{Dt} = Q - \sum_k \mu_k S^{(q_k)}}$$

Temperature $T(K)$

cooling/heating

μ_k chemical potential

2nd Law of Thermodynamics

vi. Equation of state:

$$\rho = \rho(T, p, q) \longrightarrow \text{ocean } q = S(\text{psu}) \longrightarrow$$

The equation for ρ is obtained in a sequence of steps. First, the density ρ_w of pure water ($S = 0$) is given by

$$\rho_w = 999.842594 + 6.793952 \times 10^{-2} t - 9.095290 \times 10^{-3} t^2 + 1.001685 \times 10^{-4} t^3 - 1.120083 \times 10^{-6} t^4 + 6.536332 \times 10^{-9} t^5. \quad \text{(A3.1)}$$

Second, the density at one standard atmosphere (effectively $p = 0$) is given by

$$\rho(S, t, 0) = \rho_w + S(0.824493 - 4.0899 \times 10^{-3} t + 7.6438 \times 10^{-5} t^2 - 8.2467 \times 10^{-7} t^3 + 5.3875 \times 10^{-9} t^4) + S^{3/2}(-5.72466 \times 10^{-3} + 1.0227 \times 10^{-4} t - 1.6546 \times 10^{-6} t^2) + 4.8314 \times 10^{-4} S^2. \quad \text{(A3.2)}$$

Finally, the density at pressure p is given by

$$\rho(S, t, p) = \rho(S, t, 0)/(1 - p/K(S, t, p)). \quad \text{(A3.3)}$$

where K is the secant bulk modulus. The pure water value K_w is given by

$$K_w = 19652.21 + 148.4206 t - 2.327105 t^2 + 1.360477 \times 10^{-2} t^3 - 5.155288 \times 10^{-5} t^4. \quad \text{(A3.4)}$$

The value at one standard atmosphere ($p = 0$) is given by

$$K(S, t, 0) = K_w + S(54.6746 - 0.603459 t + 1.09987 \times 10^{-2} t^2 - 6.1670 \times 10^{-5} t^3) + S^{3/2}(7.944 \times 10^{-2} + 1.6483 \times 10^{-2} t - 5.3009 \times 10^{-4} t^2) \quad \text{(A3.5)}$$

and the value at pressure p by

$$K(S, t, p) = K(S, t, 0) + p(3.239908 + 1.43713 \times 10^{-3} t + 1.16092 \times 10^{-4} t^2 - 5.77905 \times 10^{-7} t^3) + pS(2.2838 \times 10^{-3} - 1.0981 \times 10^{-5} t - 1.6078 \times 10^{-6} t^2) + 1.91075 \times 10^{-4} pS^{3/2} + p^2(8.50935 \times 10^{-5} - 6.12293 \times 10^{-6} t + 5.2787 \times 10^{-8} t^2) + p^2 S(-9.9348 \times 10^{-7} + 2.0816 \times 10^{-8} t + 9.1697 \times 10^{-10} t^2). \quad \text{(A3.6)}$$

Γραμμηκοποίηση

Boussinesq Equation of State

Typical values ρ_0=1028 *kg/m³*, T_0=10 °*C*=283 °*K*, S_0=35 ‰

$$\boxed{\rho = \rho_0 \left[1 - \alpha \left(T - T_0\right) + \beta \left(S - S_0\right)\right]}$$

$$\alpha = -\frac{1}{\rho}\frac{\partial \rho}{\partial T} \simeq 2 \times 10^{-4}\ \mathrm{K}^{-1}$$

Thermal expansion coefficient

$$\beta = \frac{1}{\rho}\frac{\partial \rho}{\partial S} \simeq 8 \times 10^{-4}\ \mathrm{ppt}^{-1}$$

Haline contraction coefficient

Neglecting possible
thermobaric instability
and ***cabelling instability***

In reality the thermal expansion coefficient depends on temperature (**cabelling**) and the non-linearity induced has the result that two water masses characterized by a different temperature and salinity but same density when mix together, the resulting mixed water can become denser and eventually sinks. Thermal expansion coefficient is also dependent on pressure (**thermobaricity**). For this reason two water masses with different salinity and temperature but same density at the surface, don't have the same density below the surface.

Oceanic approximations

Incompressibility

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla\cdot\mathbf{u}$$

Compressibility $\beta = -\frac{1}{V}\frac{DV}{DP} = -\left(\frac{1}{V}\frac{DV}{Dt}\right)\left(\frac{Dt}{DP}\right) = -\left(\frac{1}{V}\frac{DV}{Dt}\right)\Big/\frac{DP}{Dt}$

If the volume does not change under changes in pressure

$$\beta = 0; \quad \frac{1}{V}\frac{DV}{Dt} = 0$$

Using

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{V}{m}\frac{D}{Dt}\left(\frac{m}{V}\right) = -\frac{1}{V}\frac{DV}{Dt}$$

So, for incompressible fluid $\frac{1}{\rho}\frac{D\rho}{Dt} = 0$

$$\frac{1}{\rho}\frac{D\rho}{Dt} \sim \frac{1}{T}\frac{\delta\rho}{\rho} = \frac{U}{L}\frac{\delta\rho}{\rho} << \frac{U}{L} \sim \nabla\cdot\mathbf{u}$$

$$\frac{\delta\rho}{\rho} = O\left(10^{-3}\right)$$

Boussinesq Continuity Equation

$$\boxed{\nabla\cdot\mathbf{u} = 0 \text{ and } \frac{D\rho}{Dt} = 0}$$

Linearized equation of state under incompressibility

Typical values ρ_0=1028 kg/m^3, T_0=10 ^oC=283 ^oK, S_0=35 ‰

$$\boxed{\rho = \rho_0\left[1-\alpha\left(T-T_0\right)+\beta\left(S-S_0\right)\right]}$$

Boussinesq Equation of State

$$\alpha = -\frac{1}{\rho}\frac{\partial\rho}{\partial T} \approx 2\times10^{-4}\,\mathrm{K}^{-1}$$

Thermal expansion coefficient

$$\beta = +\frac{1}{\rho}\frac{\partial\rho}{\partial S} \approx 8\times10^{-4}\,\mathrm{ppt}^{-1}$$

Haline contraction coefficient

Neglecting possible ***thermobaric instability*** and ***cabelling instability***

Oceanic approximations

Boussinesq Equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p - g\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u} \ (1)$$

$$\nabla\mathbf{u} = 0 \ (2)$$

$$\frac{\partial S}{\partial t} + \mathbf{u}\nabla S = \kappa_S\nabla^2 S + S^S \ (3)$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\nabla T = \kappa_T\nabla^2 T + \frac{Q}{c_p\rho} \ (4)$$

$$\rho = \rho_0\left[1 - \alpha(T - T_0) + \beta(S - S_0)\right] \ (5)$$

$$c_p = 4\times10^{-3} \ J\,kg^{-1}\,\mathrm{K}^{-1}$$

- Entropy and density conservation equations become redundant due to the thermodynamic approximations

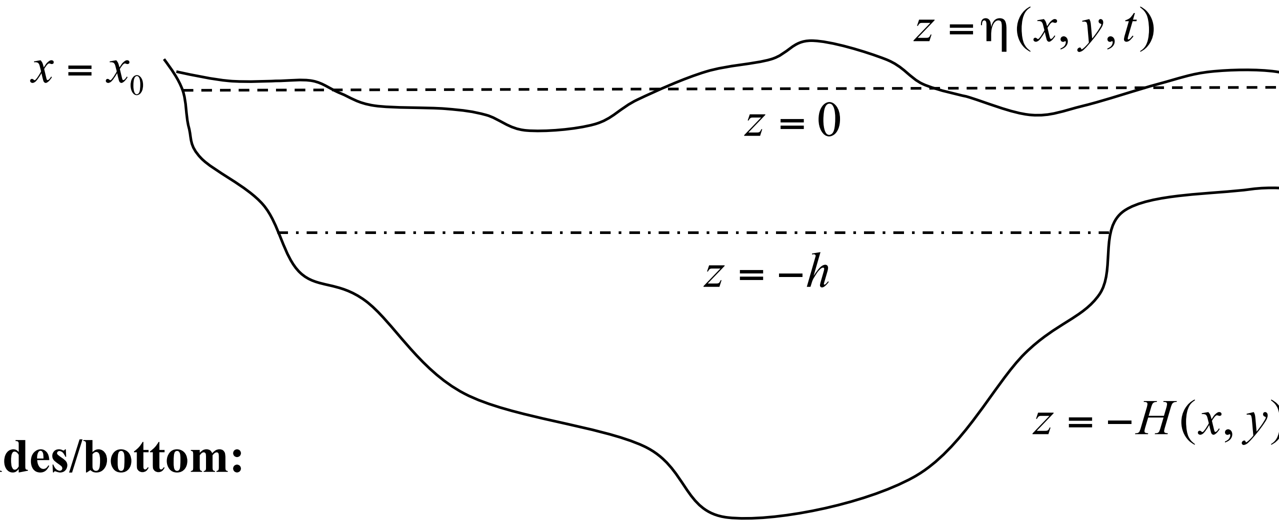
- Boussinesq approximations cancel acoustic and shock waves, it is a good approximation since

$$\text{Mach number}: \ M = \frac{U}{C_s} << 1$$

$$C_s \approx 1500\, m^{-1}\, s^{-1}, \quad U \approx 1\, m^{-1}\, s^{-1}$$

- When the right hand side of equations (3) and (4) is zero, the processes are called **adiabatic**.

Boundary conditions



Sides/bottom:

$$\mathbf{u} \cdot \hat{\mathbf{n}} = 0 \ \text{ at } \ z = -H(x, y) \ \& \ x = x_0$$

Top:

$$w = \frac{D\eta}{Dt} \ \text{ at } \ z = \eta(x, y, t) \qquad \frac{D\eta}{Dt} = 0 \ \& \ p = 0 \ \text{ at } \ z = 0 \ \text{ Rigid-lid}$$

$$p = p_{\text{atm}}(x, y, t)$$

$$\downarrow$$

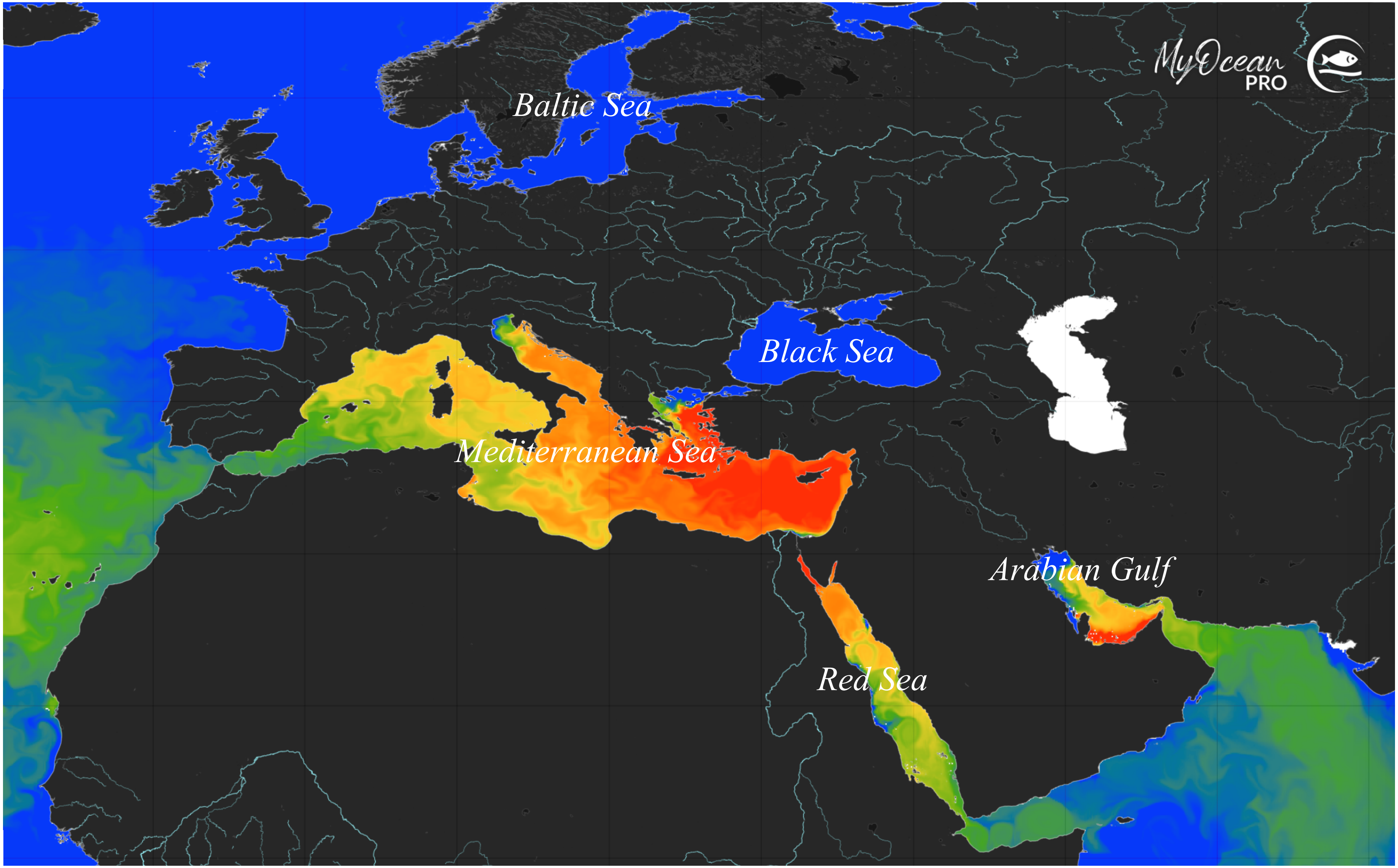
$$h \approx -\delta p_{\text{atm}} / g\rho_0$$

Inverse Barometer Effect
(no oceanic acceleration)

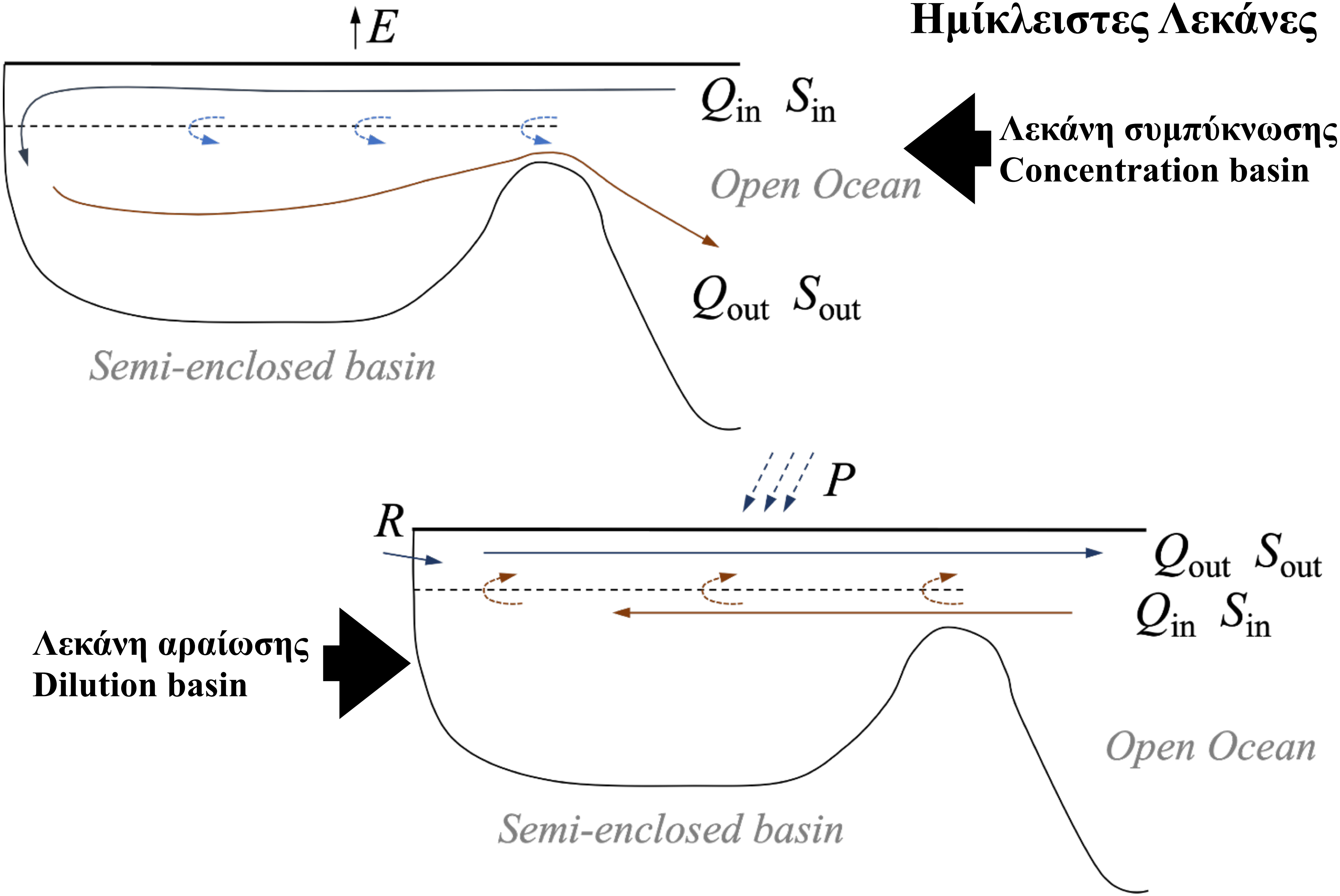
Interior:

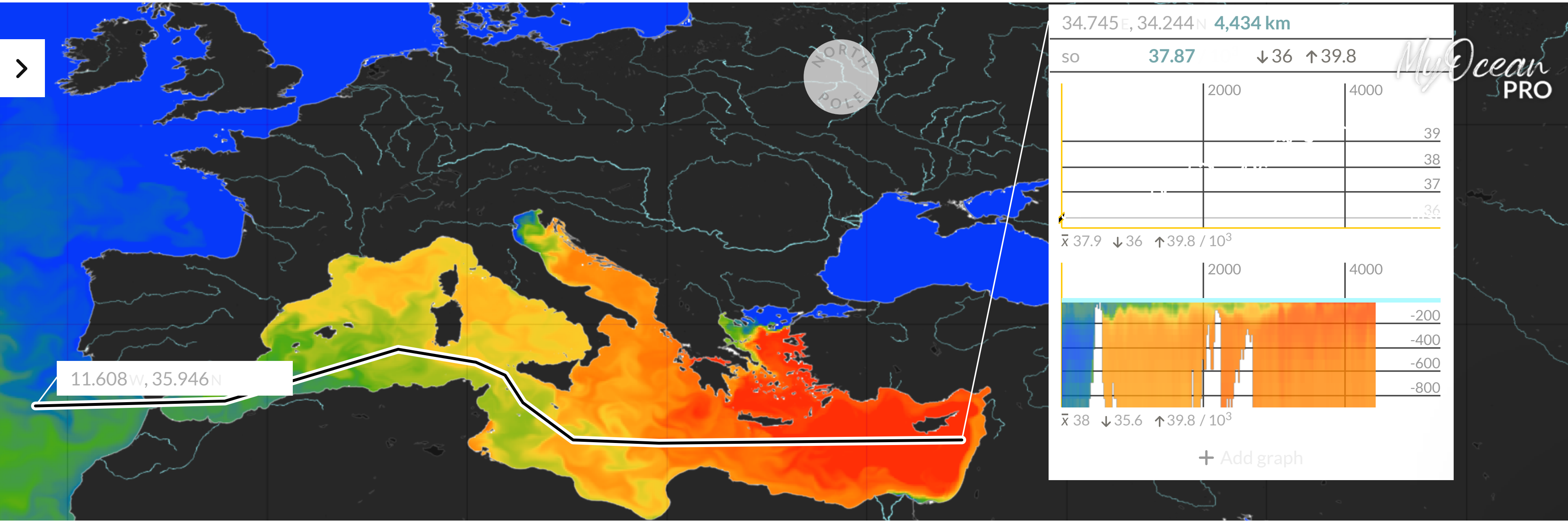
$$p_{\text{up}} = p_{\text{low}} \ \text{ at } \ z = -h$$

Εξισώσεις διατήρησης: Παράδειγμα: <u>Ημίκλειστες Λεκάνες</u>



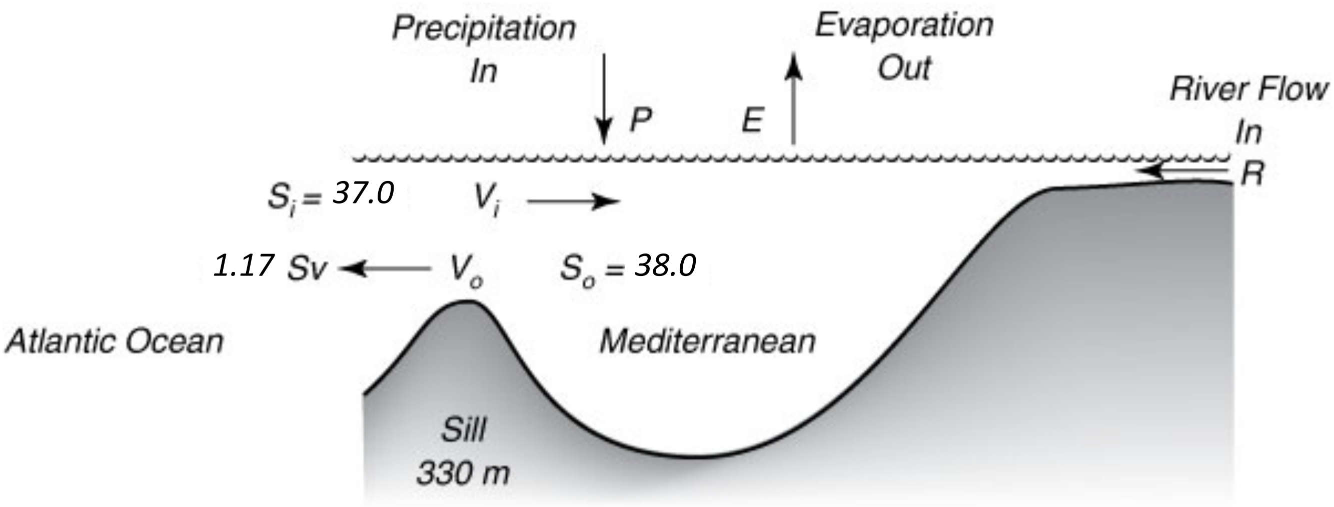
Ημίκλειστες Λεκάνες





11.608 W, 35.946 N
34.745 E, 34.244 N 4,434 km
SO 37.87 ↓36 ↑39.8
MyOcean PRO
NORTH POLE
2000
4000
39
38
37
36
x̄ 37.9 ↓36 ↑39.8 / 10³
-200
-400
-600
-800
x̄ 38 ↓35.6 ↑39.8 / 10³
+ Add graph

Μεσόγειος



Precipitation In
P
Evaporation Out
E
River Flow In
R
$S_i = 37.0$
V_i
1.17 Sv
V_o
$S_o = 38.0$
Atlantic Ocean
Mediterranean
Sill 330 m

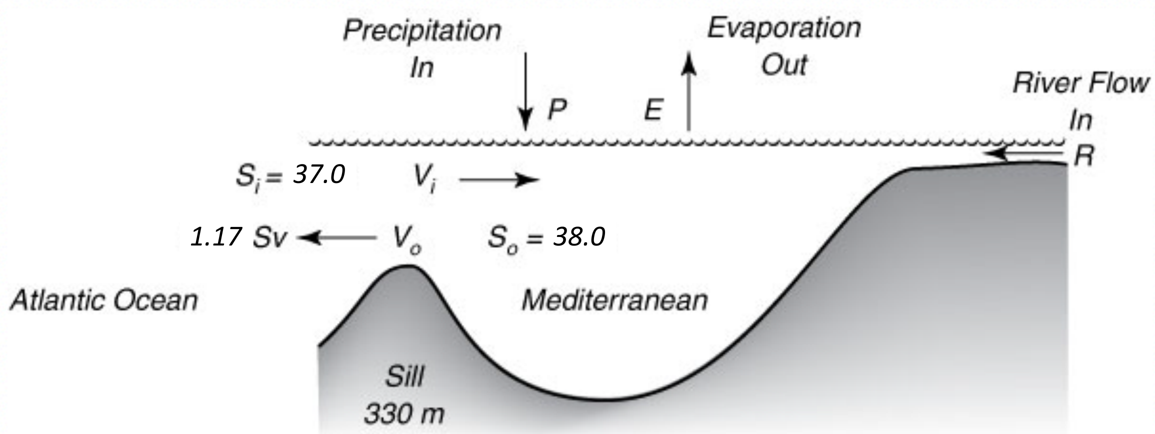
Μεσόγειος:

$$E - P - R = 1\text{m/year} \times 10^{12}\text{m}^2 = \frac{1\text{m}}{365 \cdot 86400\text{s}} \times 10^{12}\text{m}^2 = 3.17 \times 10^4 \text{m}^3\text{s}^{-1} = 0.0317\text{Sv}$$

Conservation of volume and salt:

$$V_i - V_o = E - P - R \quad (1)$$

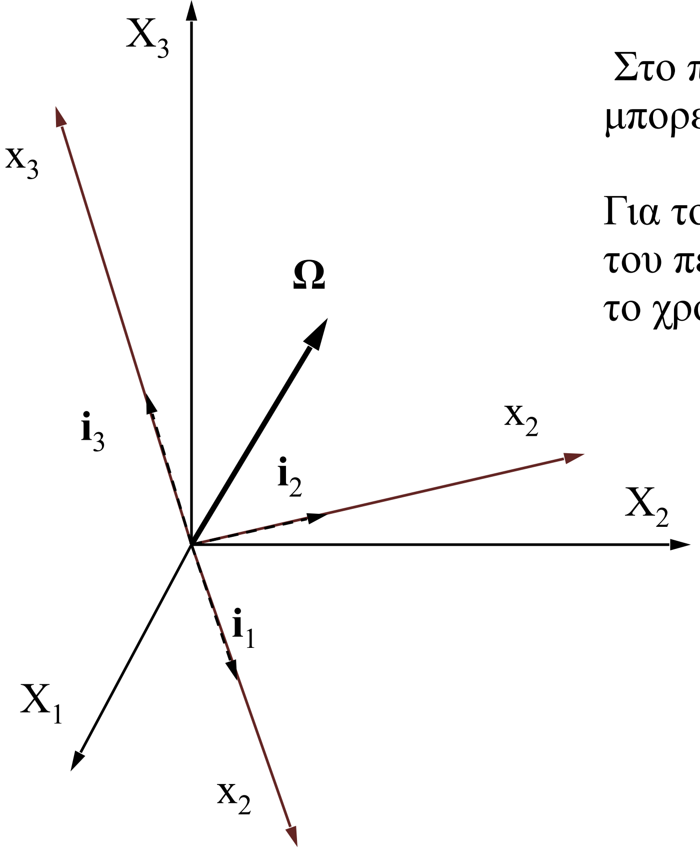
$$V_i S_i - V_o S_o = 0 \quad (2)$$



$$(2) \Rightarrow V_i = \frac{V_o S_o}{S_i}$$

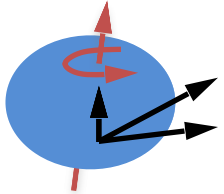
$$(1) \Rightarrow \frac{V_o S_o}{S_i} - V_o = E - P - R \Rightarrow V_o = \left(\frac{S_i}{S_o - S_i}\right) E - P - R$$

$$V_o = \frac{37}{1} 0.0317\text{Sv} = 1.17\text{Sv}$$



Στο περιστρεφόμενο σύστημα κάθε διάνυσμα μπορεί να γραφεί

$$\mathbf{P} = P_1\mathbf{i}_1 + P_2\mathbf{i}_2 + P_3\mathbf{i}_3$$



Για τον παρατηρητή στο σταθερό σύστημα τα μοναδιαία διανύσματα του περιστρεφόμενου συστήματος (**i**) μεταβάλουν τη θέση τους με το χρόνο. Άρα η χρονική μεταβολή του διανύσματος **P** είναι

$$\left(\frac{d\mathbf{P}}{dt}\right)_F = \frac{d}{dt}\left(P_1\mathbf{i}_1 + P_2\mathbf{i}_2 + P_3\mathbf{i}_3\right)$$

$$= \mathbf{i}_1\frac{dP_1}{dt} + \mathbf{i}_2\frac{dP_2}{dt} + \mathbf{i}_3\frac{dP_3}{dt} + P_1\frac{d\mathbf{i}_1}{dt} + P_2\frac{d\mathbf{i}_2}{dt} + P_3\frac{d\mathbf{i}_3}{dt}$$

Για τον παρατηρητή του περιστρεφόμενου συστήματος η μεταβολή του **P** είναι οι τρεις πρώτοι όροι, άρα

$$\left(\frac{d\mathbf{P}}{dt}\right)_F = \left(\frac{d\mathbf{P}}{dt}\right)_R + P_1\frac{d\mathbf{i}_1}{dt} + P_2\frac{d\mathbf{i}_2}{dt} + P_3\frac{d\mathbf{i}_3}{dt}$$

Σταθερό σύστημα (F) ———

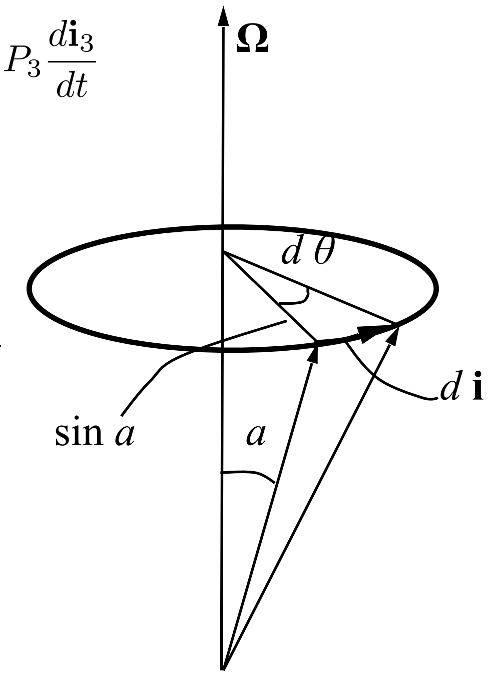
Περιστρεφόμενο σύστημα (R) ———

Η μεταβολή του **i** σε χρόνο dt είναι

$$|d\mathbf{i}| = \sin a \; d\theta \Rightarrow$$

$$\Rightarrow \frac{d\mathbf{i}}{dt} = \sin a \; \frac{d\theta}{dt} = \Omega \sin a = \Omega \times \mathbf{i}$$

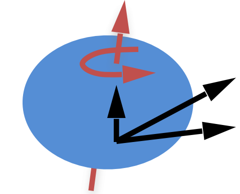
$$(\mathbf{a} \times \mathbf{b} = a \cdot b \cdot \sin\varphi \cdot \mathbf{n})$$

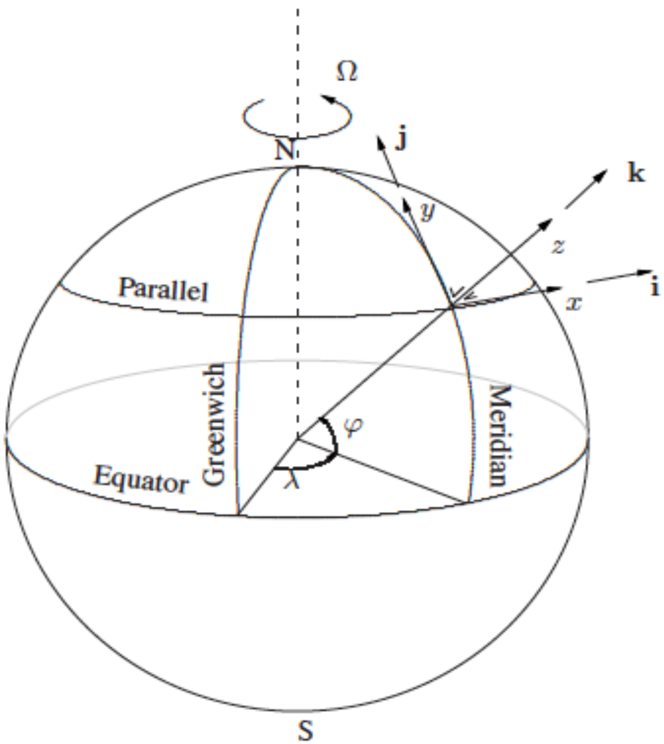


Άρα μπορούμε να γράψουμε

$$\left(\frac{d\mathbf{P}}{dt}\right)_F = \left(\frac{d\mathbf{P}}{dt}\right)_R + \Omega \times \mathbf{P} \qquad (1)$$

Η επιτάχυνση Coriolis





Χρησιμοποιώντας την (1)

$$\left(\frac{d\mathbf{r}}{dt}\right)_F = \left(\frac{d\mathbf{r}}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{r}$$

$$\mathbf{u}_F = \mathbf{u}_R + \boldsymbol{\Omega} \times \mathbf{r} \quad (2)$$

$$\left(\frac{d\mathbf{u}_F}{dt}\right)_F = \left(\frac{d\mathbf{u}_F}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{u}_F \qquad (3)$$

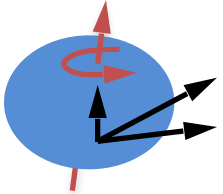
(2)&(3)

$$\begin{aligned}\left(\frac{d\mathbf{u}_F}{dt}\right)_F &= \left(\frac{d\mathbf{u}_R}{dt}\right)_R + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} + \boldsymbol{\Omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_R + \boldsymbol{\Omega} \times (\mathbf{u}_R + \boldsymbol{\Omega} \times \mathbf{r}) \\ &= \left(\frac{d\mathbf{u}_R}{dt}\right)_R + 2\boldsymbol{\Omega} \times \mathbf{u}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + \cancel{\frac{d\boldsymbol{\Omega}}{dt}} \times \mathbf{r}\end{aligned}$$

$= 0$

Coriolis acceleration

Centrifugal acceleration



$$\left(\frac{d\mathbf{u}_F}{dt}\right)_F = \left(\frac{d\mathbf{u}_R}{dt}\right)_R + 2\Omega \times \mathbf{u}_R + \Omega \times (\Omega \times \mathbf{r}) + \cancel{\frac{d\Omega}{dt}} \times \mathbf{r}$$

= 0

Coriolis acceleration

small term (correction in g)

$$x \rightarrow -2\Omega \sin(\varphi) v + \cancel{2\Omega \cos(\varphi) w}$$

$$y \rightarrow 2\Omega \sin(\varphi) u$$

$$z \rightarrow \cancel{-2\Omega \cos(\varphi) u}$$

$$g_{effective} \simeq -g + \Omega \times (\Omega \times \mathbf{r})$$

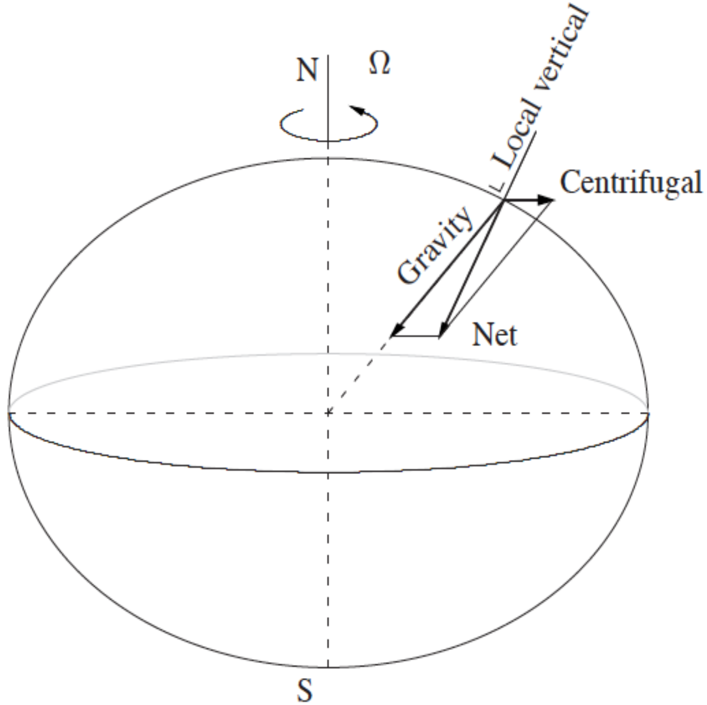
$$\boxed{f = 2\Omega \sin(\varphi)}$$

The β term

$$\beta = \frac{\partial f}{\partial y} \simeq \frac{\partial f}{R_{earth} \partial \varphi} =$$

$$= \frac{\partial \left(2\Omega \sin(\varphi)\right)}{R_{earth} \partial \varphi} = \frac{2\Omega \cos(\varphi)}{R_{earth}}$$

(for small changes in $\varphi : \sin(\varphi) \simeq \varphi$)



- Δεν υπάρχει επιτάχυνση Coriolis σε ακίνητα σώματα
- Στα γεωφυσικά ρευστά επικρατεί η οριζόντια εκτροπή (δεξιά ή αριστερά).
- Η δύναμη Coriolis δεν παράγει έργο (κάθετη στην κίνηση)..

The "mean" state

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \upsilon\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} = -\dfrac{1}{\rho_0}\dfrac{\partial P}{\partial x} + f\upsilon + \nu\dfrac{\partial^2 u}{\partial x^2} + \nu\dfrac{\partial^2 u}{\partial y^2} + \nu\dfrac{\partial^2 u}{\partial z^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial \upsilon}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \end{cases}$$

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial u\upsilon}{\partial y} + \frac{\partial uw}{\partial z} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + f\upsilon + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$\boxed{u = \overline{u} + u'; \quad \upsilon = \overline{\upsilon} + \upsilon'; \quad w = \overline{w} + w'; \quad P = \overline{P} + P'}$$

$$\overline{\frac{\partial \overline{u} + u'}{\partial t} + \frac{\partial(\overline{u} + u')(\overline{u} + u')}{\partial x} + \frac{\partial(\overline{u} + u')(\overline{\upsilon} + \upsilon')}{\partial y} + \frac{\partial(\overline{u} + u')(\overline{w} + w')}{\partial z}} =$$

$$\overline{-\frac{1}{\rho_0}\frac{\partial \overline{P} + P'}{\partial x} + f\left(\overline{\upsilon} + \upsilon'\right) + \nu\frac{\partial^2(\overline{u} + u')}{\partial x^2} + \nu\frac{\partial^2(\overline{u} + u')}{\partial y^2} + \nu\frac{\partial^2(\overline{u} + u')}{\partial z^2}}$$

$$\boxed{\begin{aligned} &\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{\upsilon}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} = \\ &-\frac{1}{\rho_0}\frac{\partial \overline{P}}{\partial x} + f\overline{\upsilon} + \nu\frac{\partial^2 \overline{u}}{\partial x^2} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} + \nu\frac{\partial^2 \overline{u}}{\partial z^2} - \frac{\partial}{\partial x}\overline{(u'u')} - \frac{\partial}{\partial y}\overline{(u'v')} - \frac{\partial}{\partial z}\overline{(u'w')} \end{aligned}}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p - 2\Omega \times \mathbf{u} - g\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u} \qquad \nabla\mathbf{u} = 0$$

$$\frac{\partial S}{\partial t} + \mathbf{u}\nabla S = \kappa_S\nabla^2 S + S^S \qquad \frac{\partial T}{\partial t} + \mathbf{u}\nabla T = \kappa_T\nabla^2 T + \frac{Q}{c_p\rho}$$

$$\rho = \rho_0\left[1 - \alpha(T - T_0) + \beta(S - S_0)\right]$$

WORKING EQUATIONS

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + A_H\frac{\partial^2 u}{\partial x^2} + A_H\frac{\partial^2 u}{\partial y^2} + A_V\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu + A_H\frac{\partial^2 v}{\partial x^2} + A_H\frac{\partial^2 v}{\partial y^2} + A_V\frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + A_H\frac{\partial^2 w}{\partial x^2} + A_H\frac{\partial^2 w}{\partial y^2} + A_V\frac{\partial^2 w}{\partial z^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} + w\frac{\partial S}{\partial z} = \kappa_S\frac{\partial^2 S}{\partial x^2} + \kappa_S\frac{\partial^2 S}{\partial y^2} + \kappa_S\frac{\partial^2 S}{\partial z^2} + S^S$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \kappa_T\frac{\partial^2 T}{\partial x^2} + \kappa_T\frac{\partial^2 T}{\partial y^2} + \kappa_T\frac{\partial^2 T}{\partial z^2} + \frac{Q}{c_p\rho}$$

$$\rho = \rho_0\left[1 - \alpha(T - T_0) + \beta(S - S_0)\right]$$

Reynold's decomposition and Turbulence:

$$\boxed{\begin{aligned}&\frac{\partial\overline{\upsilon}}{\partial t}+\overline{u}\frac{\partial\overline{\upsilon}}{\partial x}+\overline{\upsilon}\frac{\partial\overline{\upsilon}}{\partial y}+\overline{w}\frac{\partial\overline{\upsilon}}{\partial z}=\\&-\frac{1}{\rho_0}\frac{\partial\overline{P}}{\partial y}-f\overline{u}+\nu\frac{\partial^2\overline{\upsilon}}{\partial x^2}+\nu\frac{\partial^2\overline{\upsilon}}{\partial y^2}+\nu\frac{\partial^2\overline{\upsilon}}{\partial z^2}-\frac{\partial}{\partial x}\overline{(u'\upsilon')}-\frac{\partial}{\partial y}\overline{(\upsilon'\upsilon')}-\frac{\partial}{\partial z}\overline{(\upsilon'w')}\\&\frac{\partial\overline{w}}{\partial t}+\overline{u}\frac{\partial\overline{w}}{\partial x}+\overline{\upsilon}\frac{\partial\overline{w}}{\partial y}+\overline{w}\frac{\partial\overline{w}}{\partial z}=\\&-\frac{1}{\rho_0}\frac{\partial\overline{P}}{\partial z}-g+\nu\frac{\partial^2\overline{w}}{\partial x^2}+\nu\frac{\partial^2\overline{w}}{\partial y^2}+\nu\frac{\partial^2\overline{w}}{\partial z^2}-\frac{\partial}{\partial x}\overline{(u'w')}-\frac{\partial}{\partial y}\overline{(\upsilon'w')}-\frac{\partial}{\partial z}\overline{(w'w')}\end{aligned}}$$

The continuity equation:

$$\boxed{\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{\upsilon}}{\partial y}+\frac{\partial\overline{w}}{\partial z}=0}$$

Removing the "mean" state: $\boxed{\frac{\partial u'}{\partial x}+\frac{\partial\upsilon'}{\partial y}+\frac{\partial w'}{\partial z}=0}$

First order closure scheme:

$$\underbrace{\nu\frac{\partial^2\bar{u}}{\partial x^2}-\frac{\partial}{\partial x}\left(\overline{u'u'}\right)}_{A_H\frac{\partial^2\bar{u}}{\partial x^2}}+\underbrace{\nu\frac{\partial^2\bar{u}}{\partial y^2}-\frac{\partial}{\partial y}\left(\overline{u'v'}\right)}_{A_H\frac{\partial^2\bar{u}}{\partial y^2}}+\underbrace{\nu\frac{\partial^2\bar{u}}{\partial z^2}-\frac{\partial}{\partial z}\left(\overline{u'w'}\right)}_{A_V\frac{\partial^2\bar{u}}{\partial z^2}}$$

- **Molecular diffusion and viscocity**

  κ_T = 0.0014 cm^2/sec (temperature)

  κ_S = 0.000013 cm^2/sec (salinity)

  ν = 0.018 cm^2/sec at 0°C (0.010 at 20°C)

Turbulence has the same coefficients for temeprature, salinity and momentum but is anisotropic (vertical/horizontal)

- **Eddy diffusivity and viscosity**

  A_H = (horizontal) = 1 to 10^4 m^2/sec

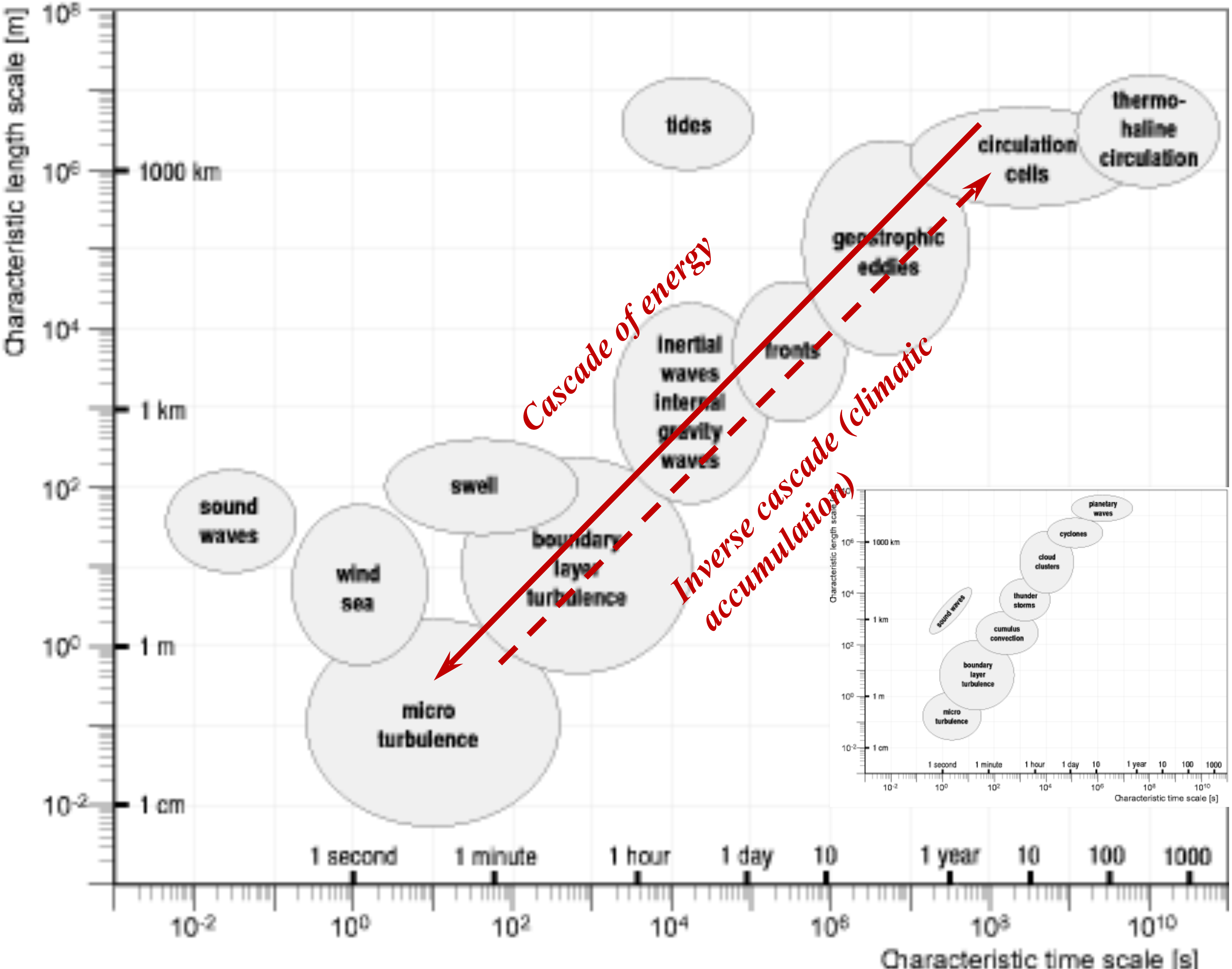
  A_V = (vertical) = 10^{-5} to 10^{-4} m^2/sec

-

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+fv+A_H\frac{\partial^2 u}{\partial x^2}+A_H\frac{\partial^2 u}{\partial y^2}+A_V\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}-fu+A_H\frac{\partial^2 v}{\partial x^2}+A_H\frac{\partial^2 v}{\partial y^2}+A_V\frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}-g+A_H\frac{\partial^2 w}{\partial x^2}+A_H\frac{\partial^2 w}{\partial y^2}+A_V\frac{\partial^2 w}{\partial z^2}$$



Characteristic length scale [m]
Characteristic time scale [s]
1000 km
1 km
1 m
1 cm
1 second
1 minute
1 hour
1 day
10
1 year
10
100
1000
tides
circulation cells
thermo-haline circulation
geostrophic eddies
fronts
inertial waves internal gravity waves
swell
sound waves
wind sea
boundary layer turbulence
micro turbulence
Cascade of energy
Inverse cascade (climatic accumulation)
planetary waves
cyclones
cloud clusters
thunder storms
cumulus convection
sound waves
boundary layer turbulence
micro turbulence

v. Ανάλυση κλίμακας - Scaling

Ορισμός φαινομένου προς μελέτη → **Επιλογή κατάλληλων κλιμάκων** → **Απλοποίηση εξισώσεων εργασίας** → **Λύση – μελέτη βασικών ισορροπιών**

Οι βασικές εξισώσεις περιλαμβάνουν μεγάλο αριθμό διαδικασιών (από μοριακή κλίμακα μέχρι παγκόσμια κλίμακα). Ανάλογα με το φαινόμενο που θέλουμε να μελετήσουμε, μπορούμε να επιλέξουμε τις κατάλληλες κλίμακες για αυτό το φαινόμενο.

$$u, v \rightarrow U \qquad x, y \rightarrow L$$

$$w \rightarrow W \qquad z \rightarrow H$$

$$t \rightarrow T \sim \frac{L}{U} \sim \frac{H}{W}$$

$$f \rightarrow f(y)$$

$$\rho \rightarrow \rho_0$$

$$\partial\rho \rightarrow \Delta\rho$$

$$g \rightarrow g$$

$$\dots$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + fv + A_H\frac{\partial^2 u}{\partial x^2} + A_H\frac{\partial^2 u}{\partial y^2} + A_V\frac{\partial^2 u}{\partial z^2}$$

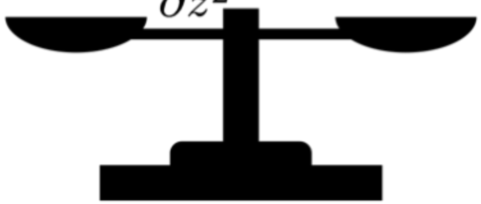
SCALING

$$\frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{U^2}{L} \qquad \frac{UW}{H} \qquad \frac{\Delta P}{\rho L} \qquad fU \qquad A_H\frac{U}{L^2} \qquad A_H\frac{U}{L^2} \qquad A_V\frac{U}{H^2}$$

Scaling numbers

Ekman Number

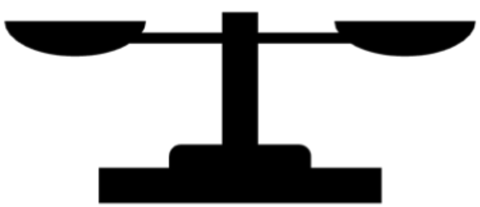
$$A_H \nabla_H^2 \mathbf{u} + A_V \frac{\partial^2 \mathbf{u}}{\partial z^2} \qquad 2\Omega \times \mathbf{u} \rightarrow fU$$



$$E_k = \frac{\text{Viscosity Terms}}{\text{Coriolis Term}} = A_H \frac{U}{L^2} \frac{1}{fU} = \frac{A_H}{fL^2} \quad \left(\frac{A_V}{fH^2} \right)$$

Rossby Number

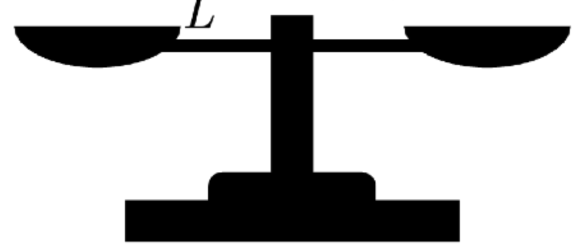
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \nabla \mathbf{u} \rightarrow \frac{U^2}{L} \qquad 2\Omega \times \mathbf{u} \rightarrow fU$$



$$R_o = \frac{\text{Non - linear Term}}{\text{Coriolis Term}} = \frac{U^2}{L} \frac{1}{fU} = \frac{U}{fL}$$

Reynolds Number

$$\mathbf{u} \nabla \mathbf{u} \rightarrow \frac{U^2}{L} \qquad \nu \nabla^2 \mathbf{u} \rightarrow \nu U$$



$$R_e = \frac{\text{Non-linear terms}}{\text{Molecular Viscosity}} = \frac{UL}{\nu}$$

Γεωστροφική/υδροστατική προσέγγιση (μεγάλες κλίμακες)

Για σύστημα που χαρακτηρίζεται από:

$$Ro = \frac{U}{fL} \ll 1 \qquad Ek = \frac{A_H}{fL^2} \ll 1 \qquad \left(\frac{H}{L} \ll 1\right)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + f\upsilon + A_H\frac{\partial^2 u}{\partial x^2} + A_H\frac{\partial^2 u}{\partial y^2} + A_V\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu + A_H\frac{\partial^2 \upsilon}{\partial x^2} + A_H\frac{\partial^2 \upsilon}{\partial y^2} + A_V\frac{\partial^2 \upsilon}{\partial z^2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g + A_H\frac{\partial^2 w}{\partial x^2} + A_H\frac{\partial^2 w}{\partial y^2} + A_V\frac{\partial^2 w}{\partial z^2}$$

$$\boxed{f\upsilon = \frac{1}{\rho_0}\frac{\partial p}{\partial x}, \ (1) \quad fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \ (2) \quad \frac{1}{\rho_0}\frac{\partial p}{\partial z} = -g \ (3)}$$

The geostrophic – hydrostatic limit

The "Taylor Columns"

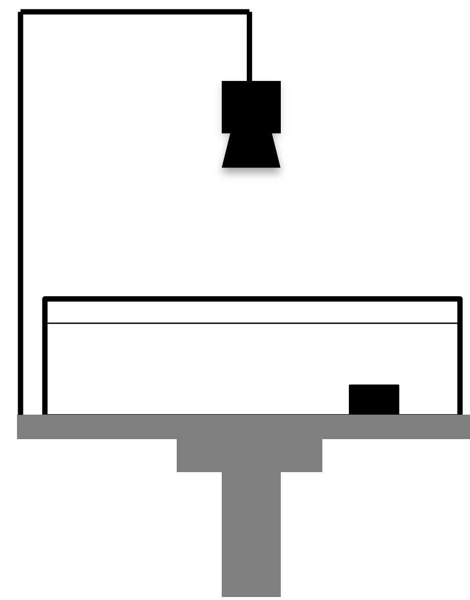
$$\frac{\partial}{\partial y}(1) + \frac{\partial}{\partial x}(2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = \frac{1}{\rho_0 f}\underbrace{\cancel{\left[\frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 P}{\partial x \partial y}\right]}}_{=0}$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0}$$

Non-divergent horizontal flow (2D)

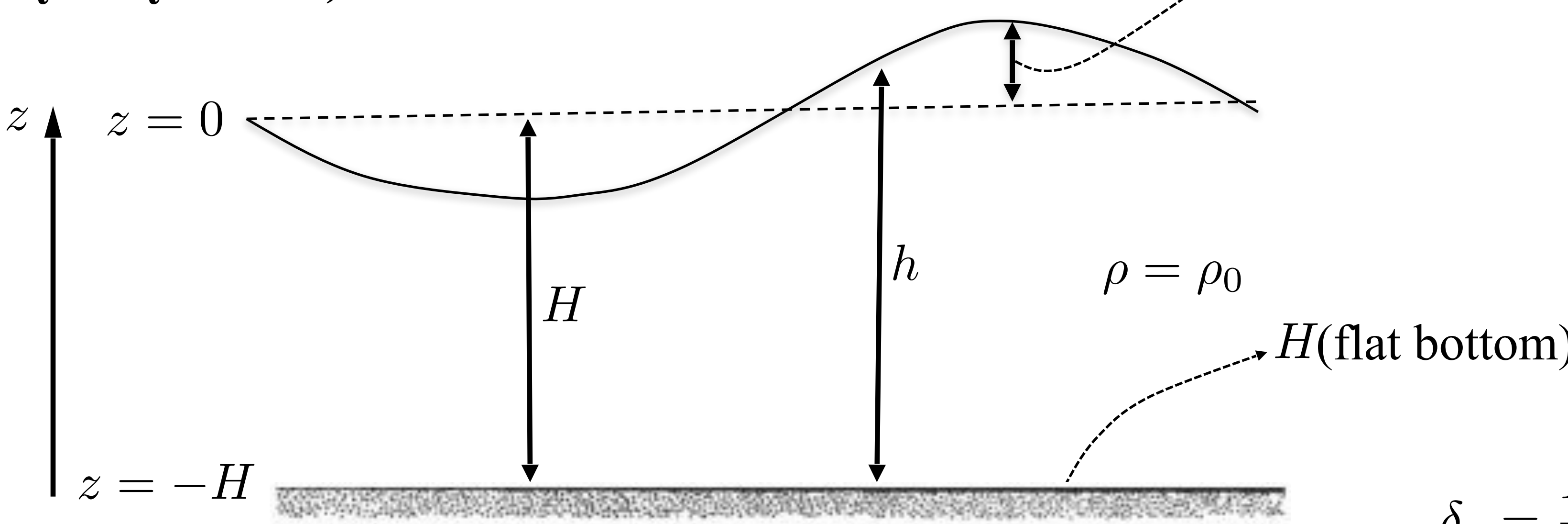
Boussinesq continuity $\Rightarrow \dfrac{\partial w}{\partial z} = 0$ or $w = 0$



Μοντέλα διακριτών στρωμάτων

***Δυναμική των Ρευστών**

Εξισώσεις αβαθούς ωκεανού (Single-shallow layer dynamics)



$$\delta_a = \frac{H}{L} \ll 1$$

$$F_R = \frac{U}{\sqrt{gH}} \ll 1$$

$$\rho_0 \frac{dw}{dt} \sim \rho_0 \frac{W}{T} \sim \rho_0 \frac{WU}{L} \sim \rho_0 \frac{U\frac{H}{L}U}{L} \sim \rho_0 \frac{UH^2}{L^2}\frac{H}{H} \sim \rho_0 \frac{U^2H^2}{HL^2}$$

$$\Rightarrow \frac{\rho_0 \frac{dw}{dt}}{g} \sim \rho_0 \frac{U^2H^2}{gHL^2} = \rho_0 F_R^2 \delta_a^2 \ll 1$$

hydrostatic approximation holds

$$\Rightarrow \frac{\partial p}{\partial z} = -\rho_0 g$$

$$\Rightarrow dp = -\rho_0 g\ dz \Rightarrow \int_z^\eta dp = -\int_z^\eta \rho_0 g\ dz \Rightarrow p(\eta) - p(z) = -\rho_0 g(\eta - z)$$

$$\Rightarrow p = \rho_0 g\,(\eta(x,y) - z) + p_{atm}$$

$$\Rightarrow \nabla_H p = \rho_0 g\ \nabla_H\ \eta$$

Ξεκινώντας από

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+f\upsilon+A_H\frac{\partial^2 u}{\partial x^2}+A_H\frac{\partial^2 u}{\partial y^2}+A_V\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}+w\frac{\partial \upsilon}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}-fu+A_H\frac{\partial^2 \upsilon}{\partial x^2}+A_H\frac{\partial^2 \upsilon}{\partial y^2}+A_V\frac{\partial^2 \upsilon}{\partial z^2}$$

αντικαθιστώντας $\left.\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial x}&=g\frac{\partial \eta}{\partial x}\\ \frac{1}{\rho}\frac{\partial p}{\partial y}&=g\frac{\partial \eta}{\partial y}\end{aligned}\right\}$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-g\frac{\partial \eta}{\partial x}+f\upsilon+A_H\frac{\partial^2 u}{\partial x^2}+A_H\frac{\partial^2 u}{\partial y^2}+A_V\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}+w\frac{\partial \upsilon}{\partial z}=-g\frac{\partial \eta}{\partial y}-fu+A_H\frac{\partial^2 \upsilon}{\partial x^2}+A_H\frac{\partial^2 \upsilon}{\partial y^2}+A_V\frac{\partial^2 \upsilon}{\partial z^2}$$

Εξισώσεις διατήρησης της ορμής

Τώρα $\eta = f(x,y)$ ανεξάρτητο από το *z* και άρα οι ταχύτητες *u* και *υ* είναι ανεξάρτητες από το *z,* δηλ. (*u*,*υ*) = *f* (*x*,*y*,*t*).

Εξίσωση διατήρησης της μάζας

$$\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}+\frac{\partial z}{\partial x}=0 \text{ where } u,\upsilon \propto x,y$$

$$\Rightarrow \frac{\partial}{\partial x}\int_{-H}^{\eta} u\ dz+\frac{\partial}{\partial y}\int_{-H}^{\eta} \upsilon\ dz+\frac{\partial}{\partial z}\int_{-H}^{\eta} w\ dz=0$$

$$\Rightarrow \frac{\partial}{\partial x}\left[u\left(\eta+H\right)\right]+\frac{\partial}{\partial y}\left[\upsilon\left(\eta+H\right)\right]+w(\eta)-w\left(H\right)=0$$

$$\text{but: } w\left(H\right)=0,\ w(\eta)=\frac{\partial \eta}{\partial t},\ \eta \ll H$$

$$\Rightarrow \frac{\partial \eta}{\partial t}+H\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)=0$$

Shallow-water equations (single layer model)

$$\frac{\partial \eta}{\partial t}+H\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-g\frac{\partial \eta}{\partial x}+fv$$

$$\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}=-g\frac{\partial \eta}{\partial y}-fu$$

Vorticity (στροβιλισμός)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x} + f\upsilon \qquad (1)$$

$$\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} = -g\frac{\partial \eta}{\partial y} - fu \qquad (2)$$

$$\frac{1}{h}\frac{dh}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) = 0 \qquad (3)$$

$\frac{\partial}{\partial x}(2) - \frac{\partial}{\partial y}(1)$ and using (3) $\longrightarrow$ **Potential Vorticity (δυναμικός στροβιλισμός)**

$$\boxed{\frac{d}{dt}\left(\frac{f+\zeta}{h}\right) = 0}$$

Earth's rotation

Spinning

Stretching

Potential Vorticity (δυναμικός στροβιλισμός)

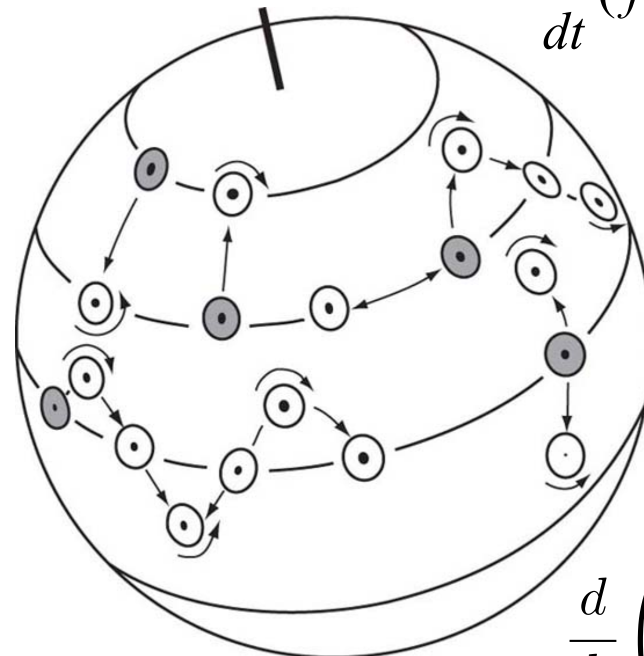
Earth's rotation

Spinning

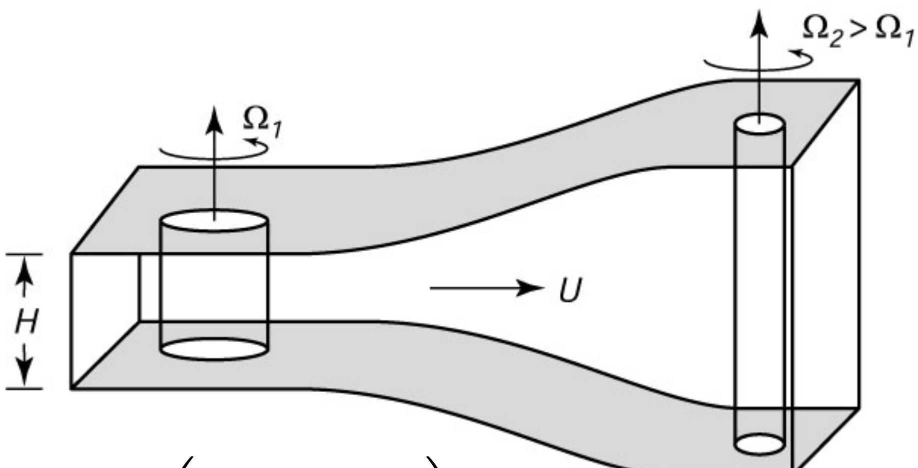
$$\boxed{\frac{d}{dt}\left(\frac{f+\zeta}{h}\right)=0}$$

Stretching

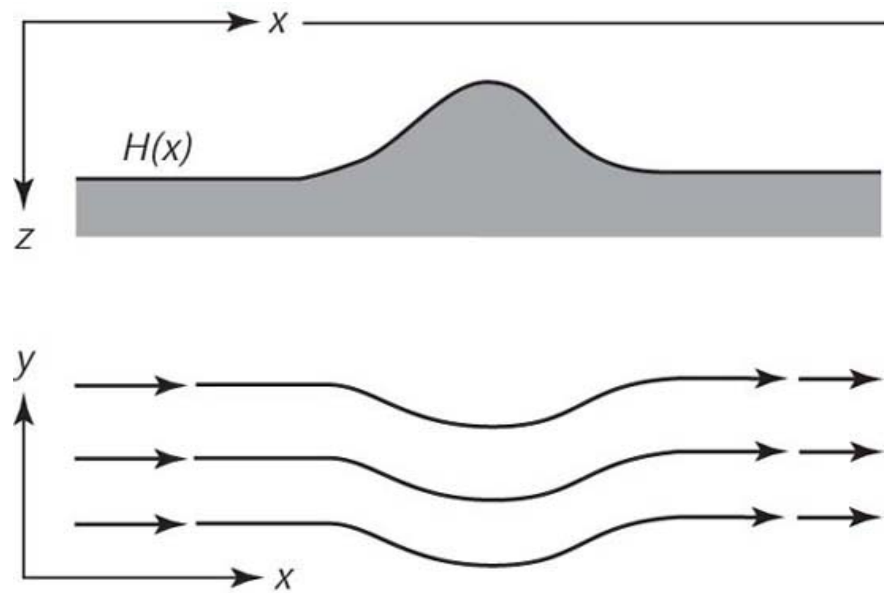
$$\frac{d}{dt}(f+\zeta)=0$$



$$\frac{d}{dt}\left(\frac{\zeta_0+f}{h}\right)$$



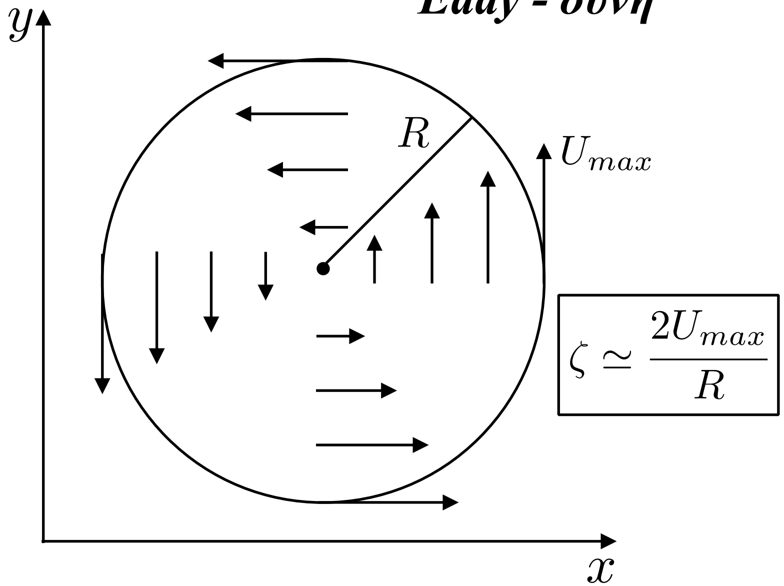
$$\frac{d}{dt}\left(\frac{\zeta+f_{const}}{h}\right)$$



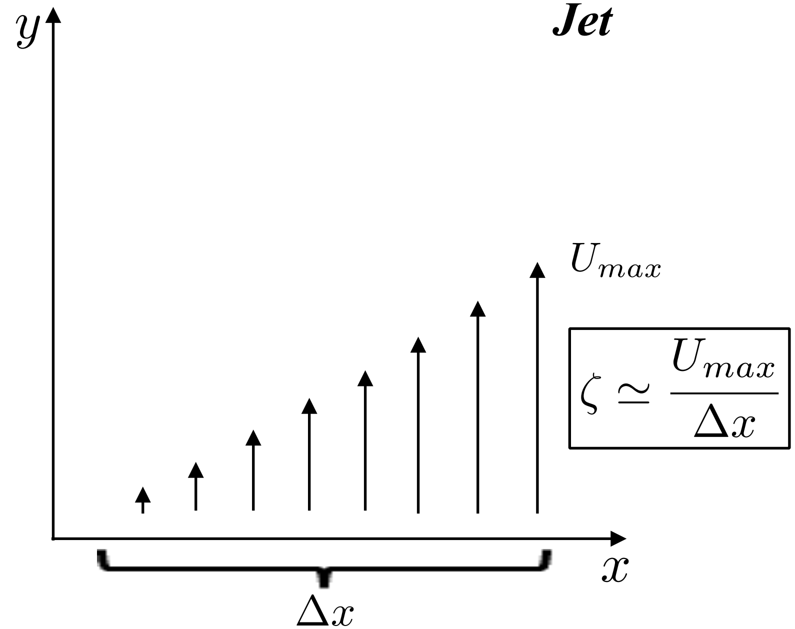
Relative vorticity of geophysical flows

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

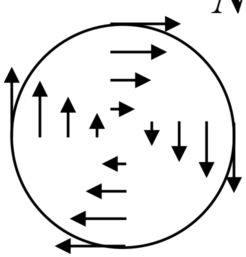
Eddy - δύνη



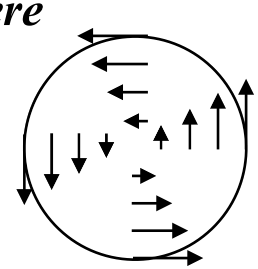
Jet



North hemisphere

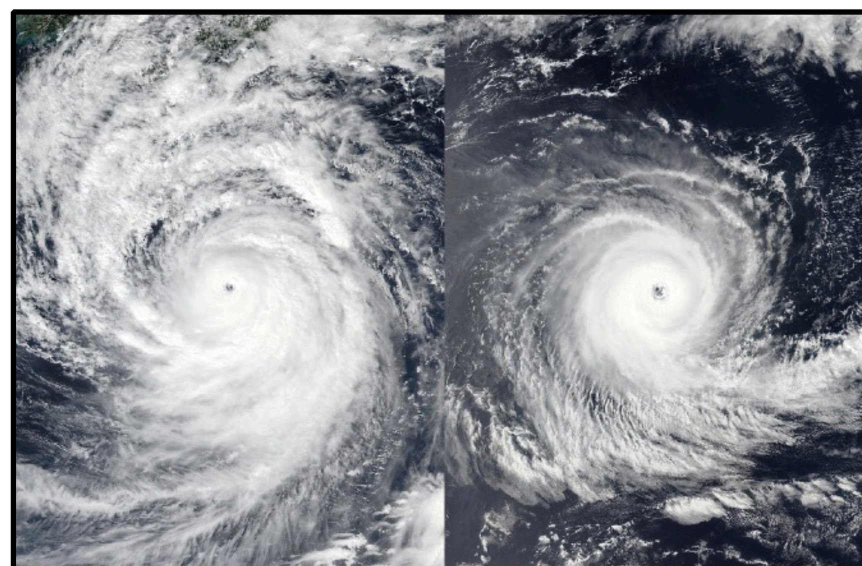


Anticyclonic
(αντικυκλωνική)
$\zeta < 0$

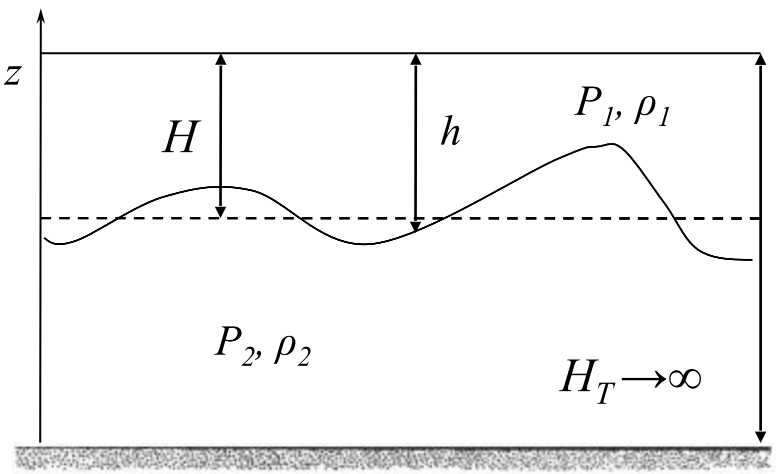


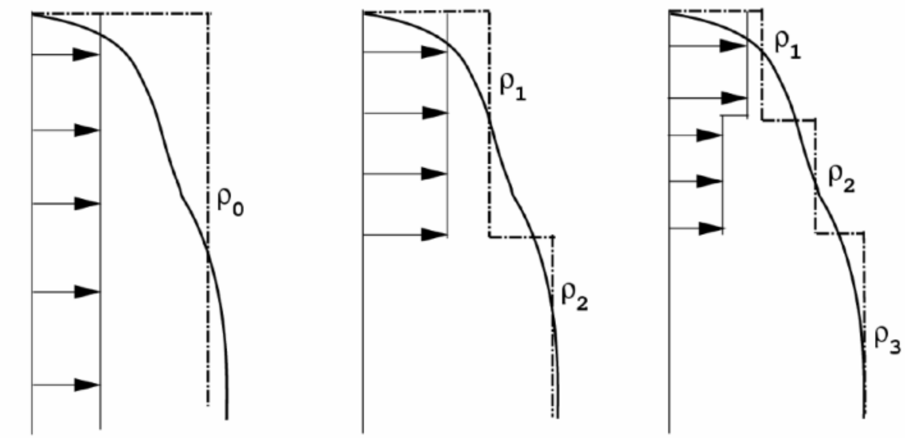
Cyclonic
(κυκλωνική)
$\zeta > 0$

**** opposite in southern hemisphere***



Μοντέλα στρωμάτων (Reduced gravity models)





Homogeneous $\left(1\frac{1}{2}\right)$ **layers** $\left(2\frac{1}{2}\right)$ **layers**

$$P_1 = P_a - \rho_1 g z \text{ and } P_2 = Pa + \rho_1 g h - \rho_2 g\left(z+h\right)$$

$$\begin{aligned}
&\nabla_H P_2 = 0\\
&\Rightarrow \nabla_H P_a + \rho_1 g \nabla_H h - \rho_2 g \nabla_H h = 0\\
&\Rightarrow \nabla_H P_a = \left(\rho_2 - \rho_1\right) g \nabla_H h\\
&\Rightarrow \frac{1}{\rho_0}\nabla_H P_1 = \frac{\left(\rho_2 - \rho_1\right)}{\rho_0} g \nabla_H h = g' \nabla_H h
\end{aligned}$$

$$\left\{\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - f\upsilon = -g'\frac{\partial h}{\partial x}\\
&\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z} + fu = -g'\frac{\partial h}{\partial y}
\end{aligned}\right.$$

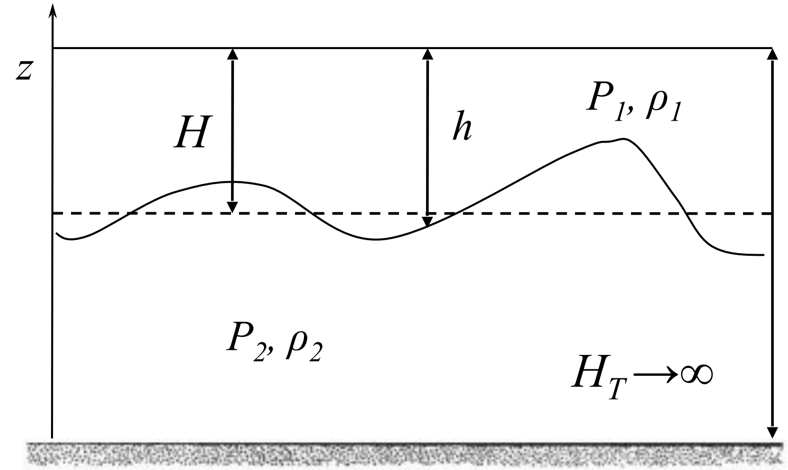
$$\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}+\frac{\partial z}{\partial x}=0 \text{ where } u,\upsilon \propto x,y$$

$$\Rightarrow \frac{\partial}{\partial x}\int_{-h}^{0} u\ dz+\frac{\partial}{\partial y}\int_{-h}^{0} \upsilon\ dz+\frac{\partial}{\partial z}\int_{-h}^{0} w\ dz=0$$

$$\Rightarrow \frac{\partial}{\partial x}(uH)+\frac{\partial}{\partial y}(\upsilon h)+w(0)-w(-h)=0$$

but: $w(0)=0,\ w(-h)=\dfrac{\partial h}{\partial t}$

$$\Rightarrow \frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu)+\frac{\partial}{\partial y}(h\upsilon)=0$$



$$g'=g\frac{\Delta\rho}{\rho_0}$$

$$\boxed{\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-f\upsilon=-g'\frac{\partial h}{\partial x}\\
&\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}+w\frac{\partial \upsilon}{\partial z}+fu=-g'\frac{\partial h}{\partial y}\\
&\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu)+\frac{\partial}{\partial y}(h\upsilon)=0
\end{aligned}}$$

Reduced gravity equations

Τώρα μπορούμε να γράψουμε τη συχνότητα Brunt Vaissala:

$$N^2=-\frac{g}{\rho_0}\frac{\partial\rho}{\partial z}\sim\frac{g}{\rho_0}\frac{\Delta\rho}{H}=\frac{g'}{H}$$

Και η Internal Rossby Radius of deformation:

$$R=\frac{NH}{f}=\frac{\sqrt{g'H}}{f}$$