ΣΥΝΟΨΗ ΧΑΜΙΛΤΟΝΙΑΝΩΝ

προ θ

$$\hat{H}_{HM,m} = \hbar\omega_m\left(\hat{a}_m^{+}\hat{a}_m + \frac{1}{2}\right) = \hbar\omega\left(\hat{N}_m + \frac{1}{2}\right)$$

m τρόπος ΗΜ πεδίου στην κοιλότητα

κι αγνοώντας τον όρο $\frac{\hbar\omega_m}{2}$

$$\hat{H}_{HM,m} = \hbar\omega_m\hat{a}_m^{+}\hat{a}_m = \hbar\omega_m\hat{N}_m$$

$$\hat{H}_{\Delta\Sigma} = E_2\hat{S}_+\hat{S}_- + E_1\hat{S}_-\hat{S}_+ \qquad E_2 - E_1 = \hbar\Omega$$

δισταθμικό σύστημα

και θέτοντας $E_1 \equiv 0$

$$\hat{H}_{\Delta\Sigma} = \hbar\Omega\hat{S}_+\hat{S}_-$$

$$\hat{U}_{Em} = \hbar g_m(\hat{S}_+ + \hat{S}_-)(\hat{a}_m^{+} + \hat{a}_m) = \underset{\text{αλληλεπίδραση}}{\hat{H}_{\Delta\Sigma - HMm}}$$

αλληλεπίδραση ΔΣ - m τρόπου ΗΜ πεδίου στην κοιλότητα

($\hat{H}_{AF}$ ατομική φυσική)

atom field

$$\hbar g_m := ex_{12}\left(\frac{\hbar\omega_m}{\varepsilon_0 V}\right)^{1/2}\sin\left(\frac{m\pi z}{L}\right)$$

$$\hbar|g_m| = \underbrace{|ex_{12}|}_{|\wp|}\underbrace{\left(\frac{4\hbar\omega_m n_m}{\varepsilon_0 V}\right)^{1/2}\left|\sin\left(\frac{m\pi z}{L}\right)\right|}_{E_{0m}(z)}\frac{1}{2\sqrt{n_m}}$$

$$\Omega_{Rm} := \frac{|\wp|\,E_{0m}}{\hbar}$$

$$\Omega_{Rm} := 2\sqrt{n_m}\,g_m$$

συχνότητα Rabi του m τρόπου

$|\uparrow n_m\rangle, |\downarrow n_m\rangle$ οι ιδιοκαταστάσεις της $\underline{\hat{H}_{HMm} + \hat{H}_{\Delta\Sigma}}$

Ενώ, η συνολική Χαμιλτονιανή γράφεται

$$\hat{H}_{Rm} = \hbar\omega_m\hat{a}_m^{+}\hat{a}_m + \hbar\Omega\hat{S}_+\hat{S}_- + \hbar g_m(\hat{S}_+ + \hat{S}_-)(\hat{a}_m^{+} + \hat{a}_m)$$

ονομάζεται συχνά Χαμιλτονιανή Rabi

Αγνοούμε τους όρους $\hat{S}_+\hat{a}_m^{+}$ κ $\hat{S}_-\hat{a}_m$, οι οποίοι φαίνονται "παράλογοι" όταν υπάρχει ένας μόνο τρόπος!

$$\hat{H}_{JCm} = \underline{\hbar\omega_m\hat{a}_m^{+}\hat{a}_m + \hbar\Omega\hat{S}_+\hat{S}_-} + \hbar g_m(\hat{S}_+\hat{a}_m + \hat{S}_-\hat{a}_m^{+})$$

ονομάζεται συχνά Χαμιλτονιανή Jaynes-Cummings

ΑΣΚΗΣΗ (Παραλείπουμε για απλότητα τον δείκτη του ΗΜ τρόπου m) ⓗ

Ⓐ Βρείτε τι κάνουν οι όροι

$\hat{a}^+\hat{a}, \hat{a}\hat{a}^+, \hat{S}_+\hat{S}_-, \hat{S}_-\hat{S}_+, \hat{S}_+\hat{a}^+, \hat{S}_+\hat{a}, \hat{S}_-\hat{a}^+, \hat{S}_-\hat{a}$ στις καταστάσεις $|\uparrow, n_m\rangle$ κ $|\downarrow, n_m\rangle$

Ⓑ Υπολογίστε τα

$\langle\hat{a}^+\hat{a}\rangle, \langle\hat{a}\hat{a}^+\rangle, \langle\hat{S}_+\hat{S}_-\rangle, \langle\hat{S}_-\hat{S}_+\rangle, \langle\hat{S}_+\hat{a}^+\rangle, \langle\hat{S}_+\hat{a}\rangle \langle\hat{S}_-\hat{a}^+\rangle, \langle\hat{S}_-\hat{a}\rangle$ στι $|\uparrow n_m\rangle$, $|\downarrow n_m\rangle$

Ⓐ $\hat{a}^+\hat{a}|\uparrow, n\rangle = n|\uparrow, n\rangle \qquad \hat{a}^+\hat{a}|\downarrow, n\rangle = n|\downarrow, n\rangle$

$[\hat{a}, \hat{a}^+] = 1 \iff \hat{a}\hat{a}^+ = 1 + \hat{a}^+\hat{a}$

$\hat{a}\hat{a}^+|\uparrow, n\rangle = (1 + \hat{a}^+\hat{a})|\uparrow, n\rangle = (n+1)|\uparrow, n\rangle$

$\hat{a}\hat{a}^+|\downarrow, n\rangle = (1 + \hat{a}^+\hat{a})|\downarrow, n\rangle = (n+1)|\downarrow, n\rangle$

$\hat{S}_+\hat{S}_-|\uparrow, n\rangle = |\uparrow, n\rangle \qquad \hat{S}_+\hat{S}_-|\downarrow, n\rangle = |0, n\rangle$

$\hat{S}_-\hat{S}_+|\uparrow, n\rangle = |0, n\rangle \qquad \hat{S}_-\hat{S}_+|\downarrow, n\rangle = |\downarrow, n\rangle$

$\hat{S}_+\hat{a}^+|\uparrow, n\rangle = \sqrt{n+1}\,|0, n+1\rangle \qquad \hat{S}_+\hat{a}^+|\downarrow n\rangle = \sqrt{n+1}\,|\uparrow, n+1\rangle$

$\hat{S}_+\hat{a}|\uparrow n\rangle = \sqrt{n}\,|0, n-1\rangle \qquad \hat{S}_+\hat{a}|\downarrow n\rangle = \sqrt{n}\,|\uparrow, n-1\rangle$

$\hat{S}_-\hat{a}^+|\uparrow n\rangle = \sqrt{n+1}\,|\downarrow, n+1\rangle \qquad \hat{S}_-\hat{a}^+|\downarrow n\rangle = \sqrt{n+1}\,|0, n+1\rangle$

$\hat{S}_-\hat{a}|\uparrow, n\rangle = \sqrt{n}\,|\downarrow, n-1\rangle \qquad \hat{S}_-\hat{a}|\downarrow, n\rangle = \sqrt{n}\,|0, n-1\rangle$

Ⓑ $\langle\uparrow n|\hat{a}^+\hat{a}|\uparrow n\rangle = n$

$\langle\downarrow n|\hat{a}^+\hat{a}|\downarrow n\rangle = n$

$\langle\uparrow n|\hat{a}\hat{a}^+|\uparrow n\rangle = n+1$

$\langle\downarrow n|\hat{a}\hat{a}^+|\downarrow n\rangle = n+1$

$\langle\uparrow n|\hat{S}_+\hat{S}_-|\uparrow n\rangle = 1$

$\langle\downarrow n|\hat{S}_+\hat{S}_-|\downarrow n\rangle = \langle\downarrow n|0\ n\rangle = 0$

$\langle\uparrow n|\hat{S}_-\hat{S}_+|\uparrow n\rangle = \langle\uparrow n|0\ n\rangle = 0$

$\langle\downarrow n|\hat{S}_-\hat{S}_+|\downarrow n\rangle = \langle\downarrow n|\downarrow n\rangle = 1$

συνεχίζεται...

$\langle \hat{a}^{+}\hat{a} \rangle_{\lvert\uparrow n\rangle} = n$	$\langle \hat{a}^{+}\hat{a} \rangle_{\lvert\downarrow n\rangle} = n$
$\langle \hat{a}\hat{a}^{+} \rangle_{\lvert\uparrow n\rangle} = n+1$	$\langle \hat{a}\hat{a}^{+} \rangle_{\lvert\downarrow n\rangle} = n+1$
$\langle \hat{S}_{+}\hat{S}_{-} \rangle_{\lvert\uparrow n\rangle} = 1$	$\langle \hat{S}_{+}\hat{S}_{-} \rangle_{\lvert\downarrow n\rangle} = 0$
$\langle \hat{S}_{-}\hat{S}_{+} \rangle_{\lvert\uparrow n\rangle} = 0$	$\langle \hat{S}_{-}\hat{S}_{+} \rangle_{\lvert\downarrow n\rangle} = 1$

$\langle \hat{S}_{+}\hat{a}^{+} \rangle_{\lvert\uparrow n\rangle} = 0$ $\qquad$ $\langle \hat{S}_{+}\hat{a}^{+} \rangle_{\lvert\downarrow n\rangle} = 0$

$\langle \hat{S}_{+}\hat{a} \rangle_{\lvert\uparrow n\rangle} = 0$ $\qquad$ $\langle \hat{S}_{+}\hat{a} \rangle_{\lvert\downarrow n\rangle} = 0$

$\langle \hat{S}_{-}\hat{a}_{m}^{+} \rangle_{\lvert\uparrow n\rangle} = 0$ $\qquad$ $\langle \hat{S}_{-}\hat{a}_{m}^{+} \rangle_{\lvert\downarrow n\rangle} = 0$

$\langle \hat{S}_{-}\hat{a} \rangle_{\lvert\uparrow n\rangle} = 0$ $\qquad$ $\langle \hat{S}_{-}\hat{a} \rangle_{\lvert\downarrow n\rangle} = 0$

... συνεχίζεται

$$\langle \uparrow n \rvert \hat{S}_{+}\hat{a}^{+} \lvert \uparrow n \rangle = \sqrt{n+1}\,\langle \uparrow n \vert 0, n+1 \rangle = 0$$

$$\langle \downarrow n \rvert \hat{S}_{+}\hat{a}^{+} \lvert \downarrow n \rangle = \sqrt{n+1}\,\langle \downarrow n \vert \uparrow n+1 \rangle = 0$$

$$\langle \uparrow n \rvert \hat{S}_{+}\hat{a} \lvert \uparrow n \rangle = \sqrt{n}\,\langle \uparrow n \vert 0, n-1 \rangle = 0$$

$$\langle \downarrow n \rvert \hat{S}_{+}\hat{a} \lvert \downarrow n \rangle = \sqrt{n}\,\langle \downarrow n \vert \uparrow n-1 \rangle = 0$$

$$\langle \uparrow n \rvert \hat{S}_{-}\hat{a}^{+} \lvert \uparrow n \rangle = \sqrt{n+1}\,\langle \uparrow n \vert \downarrow n+1 \rangle = 0$$

$$\langle \downarrow n \rvert \hat{S}_{-}\hat{a}^{+} \lvert \downarrow n \rangle = \sqrt{n+1}\,\langle \downarrow n \vert 0\ n+1 \rangle = 0$$

$$\langle \uparrow n \rvert \hat{S}_{-}\hat{a} \lvert \uparrow n \rangle = \sqrt{n}\,\langle \uparrow n \vert \downarrow n-1 \rangle = 0$$

$$\langle \downarrow n \rvert \hat{S}_{-}\hat{a} \lvert \downarrow n \rangle = \sqrt{n}\,\langle \downarrow n \vert 0, n-1 \rangle = 0$$

ΜΕΣΕΣ (ΑΝΑΜΕΝΟΜΕΝΕΣ) ΤΙΜΕΣ ΜΕΓΕΘΩΝ για τη Χαμιλτονιανή Jaynes – Cummings

Για την κατάσταση: Ⓐ $|\psi_A(t)\rangle = c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow, n-1\rangle = |A\rangle$

$$\langle \hat{a}^+\hat{a}\rangle_{|A\rangle} = \left(c_1^*(t)\langle\downarrow n| + c_2^*(t)\langle\uparrow n-1|\right)\hat{a}^+\hat{a}\left(c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle\right)$$

$$= |c_1(t)|^2\langle\downarrow n|\hat{a}^+\hat{a}|\downarrow n\rangle + c_1^*(t)c_2(t)\langle\downarrow n|\hat{a}^+\hat{a}|\uparrow n-1\rangle$$

$$+ c_2^*(t)c_1(t)\langle\uparrow n-1|\hat{a}^+\hat{a}|\downarrow n\rangle + |c_2(t)|^2\langle\uparrow n-1|\hat{a}^+\hat{a}|\uparrow n-1\rangle$$

$$= |c_1(t)|^2 n\,\cancelto{1}{\langle\downarrow n|\downarrow n\rangle} + c_1^*(t)c_2(t)(n-1)\,\cancelto{0}{\langle\downarrow n|\uparrow n-1\rangle}$$

$$+ c_2^*(t)c_1(t)\,n\,\cancelto{0}{\langle\uparrow n-1|\downarrow n\rangle} + |c_2(t)|^2(n-1)\,\cancelto{1}{\langle\uparrow n-1|\uparrow n-1\rangle}$$

$$= |c_1(t)|^2 n + |c_2(t)|^2 n - |c_2(t)|^2 \Rightarrow \boxed{\langle\hat{a}^+\hat{a}\rangle_{|A\rangle} = n - |c_2(t)|^2}$$

$$\langle\hat{a}\hat{a}^+\rangle_{|A\rangle} = \left(c_1^*(t)\langle\downarrow n| + c_2^*(t)\langle\uparrow n-1|\right)\hat{a}\hat{a}^+\left(c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle\right)$$

$$= |c_1(t)|^2\langle\downarrow n|\hat{a}\hat{a}^+|\downarrow n\rangle + c_1^*(t)c_2(t)\langle\downarrow n|\hat{a}\hat{a}^+|\uparrow n-1\rangle$$

$$+ c_2^*(t)c_1(t)\langle\uparrow n-1|\hat{a}\hat{a}^+|\downarrow n\rangle + |c_2(t)|^2\langle\uparrow n-1|\hat{a}\hat{a}^+|\uparrow n-1\rangle =$$

$$= |c_1(t)|^2(n+1) + c_1^*(t)c_2(t)\,n\,\cancel{\langle\downarrow n|\uparrow n-1\rangle}$$

$$+ c_2^*(t)c_1(t)(n+1)\cancel{\langle\uparrow n-1|\downarrow n\rangle} + |c_2(t)|^2 n =$$

$$= |c_1(t)|^2 n + |c_1(t)|^2 + |c_2(t)|^2 n \Rightarrow \boxed{\langle\hat{a}\hat{a}^+\rangle_{|A\rangle} = n + |c_1(t)|^2}$$

$$\boxed{\langle\hat{a}\hat{a}^+\rangle_{|A\rangle} - \langle\hat{a}^+\hat{a}\rangle_{|A\rangle} = 1}$$

$$\langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} = |c_2(t)|^2$$

$$\langle \hat{S}_- \hat{S}_+ \rangle_{|A\rangle} = |c_1(t)|^2$$

Ομοίως αποδεικνύονται ...

$$\langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} + \langle \hat{S}_- \hat{S}_+ \rangle_{|A\rangle} = 1$$

$$\langle \hat{a}^+ \hat{a} \rangle_{|A\rangle} + \langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} = n$$

$$\langle \hat{S}_+ \hat{a} \rangle_{|A\rangle} = c_2^*(t)\, c_1(t) \sqrt{n}$$

$$\langle \hat{S}_- \hat{a}^+ \rangle_{|A\rangle} = c_1^*(t)\, c_2(t) \sqrt{n}$$

$$\langle \hat{S}_+ \hat{a}^+ \rangle_{|A\rangle} = 0$$

$$\langle \hat{S}_- \hat{a} \rangle_{|A\rangle} = 0$$

Για την κατάσταση Ⓔ $|\psi_E(t)\rangle = c_1(t)|\uparrow n\rangle + c_2(t)|\downarrow n+1\rangle = |E\rangle$ Ομοίως αποδεικνύονται

$\langle \hat{a}^+ \hat{a} \rangle_{|E\rangle} =$

$\langle \hat{a} \hat{a}^+ \rangle_{|E\rangle} =$

$\langle \hat{a} \hat{a}^+ \rangle_{|E\rangle} - \langle \hat{a}^+ \hat{a} \rangle_{|E\rangle} =$

$\langle \hat{S}_+ \hat{S}_- \rangle_{|E\rangle} =$

$\langle \hat{S}_- \hat{S}_+ \rangle_{|E\rangle} =$

$\langle \hat{S}_+ \hat{S}_- \rangle_{|E\rangle} + \langle \hat{S}_- \hat{S}_+ \rangle_{|E\rangle} =$

$\langle \hat{a}^+ \hat{a} \rangle_{|E\rangle} + \langle \hat{S}_+ \hat{S}_- \rangle_{|E\rangle} =$

$\langle \hat{S}_+ \hat{a} \rangle_{|E\rangle} =$

$\langle \hat{S}_+ \hat{a}^+ \rangle_{|E\rangle} =$

$\langle \hat{S}_- \hat{a}^+ \rangle_{|E\rangle} =$

$\langle \hat{S}_- \hat{a} \rangle_{|E\rangle} =$

IB'

$$\langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} = \left(c_1^*(t) \langle \downarrow n| + c_2^*(t) \langle \uparrow n-1| \right) \hat{S}_+ \hat{S}_- \left(c_1(t) |\downarrow n\rangle + c_2(t) |\uparrow n-1\rangle \right)$$

$$= |c_1(t)|^2 \langle \downarrow n| \hat{S}_+ \hat{S}_- |\downarrow n\rangle +$$

$$c_1^*(t) c_2(t) \langle \downarrow n| \hat{S}_+ \hat{S}_- |\uparrow n-1\rangle +$$

$$c_2^*(t) c_1(t) \langle \uparrow n-1| \hat{S}_+ \hat{S}_- |\downarrow n\rangle +$$

$$|c_2(t)|^2 \langle \uparrow n-1| \hat{S}_+ \hat{S}_- |\uparrow n-1\rangle =$$

$$= |c_1(t)|^2 \cancel{\langle \downarrow n| 0 n\rangle} +$$

$$c_1^*(t) c_2(t) \cancel{\langle \downarrow n| \uparrow n-1\rangle} +$$

$$c_2^*(t) c_1(t) \cancel{\langle \uparrow n-1| 0 n\rangle} +$$

$$|c_2(t)|^2 \langle \uparrow n-1| \uparrow n-1\rangle = |c_2(t)|^2$$

$$\langle \hat{S}_- \hat{S}_+ \rangle_{|A\rangle} = \left(c_1^*(t) \langle \downarrow n| + c_2^*(t) \langle \uparrow n-1| \right) \hat{S}_- \hat{S}_+ \left(c_1(t) |\downarrow n\rangle + c_2(t) |\uparrow n-1\rangle \right)$$

$$= |c_1(t)|^2 \langle \downarrow n| \hat{S}_- \hat{S}_+ |\downarrow n\rangle +$$

$$c_1^*(t) c_2(t) \langle \downarrow n| \hat{S}_- \hat{S}_+ |\uparrow n-1\rangle +$$

$$c_2^*(t) c_1(t) \langle \uparrow n-1| \hat{S}_- \hat{S}_+ |\downarrow n\rangle +$$

$$|c_2(t)|^2 \langle \uparrow n-1| \hat{S}_- \hat{S}_+ |\uparrow n-1\rangle =$$

$$= |c_1(t)|^2 \langle \downarrow n| \downarrow n\rangle +$$

$$c_1^*(t) c_2(t) \cancel{\langle \downarrow n| 0 n-1\rangle} +$$

$$c_2^*(t) c_1(t) \cancel{\langle \uparrow n-1| \downarrow n\rangle} +$$

$$|c_2(t)|^2 \cancel{\langle \uparrow n-1| 0 n-1\rangle} = |c_1(t)|^2$$

$$\langle \hat{S}_+ \hat{a} \rangle_{|A\rangle} = \left(c_1^*(t) \langle \downarrow n | + c_2^*(t) | \uparrow n-1 \rangle \right) \hat{S}_+ \hat{a} \left(c_1(t) | \downarrow n \rangle + c_2(t) | \uparrow n-1 \rangle \right)$$ (IB)''

$$= |c_1(t)|^2 \langle \downarrow n | \hat{S}_+ \hat{a} | \downarrow n \rangle +$$
$$c_1^*(t) c_2(t) \langle \downarrow n | \hat{S}_+ \hat{a} | \uparrow n-1 \rangle +$$
$$c_2^*(t) c_1(t) \langle \uparrow n-1 | \hat{S}_+ \hat{a} | \downarrow n \rangle +$$
$$|c_2(t)|^2 \langle \uparrow n-1 | \hat{S}_+ \hat{a} | \uparrow n-1 \rangle =$$
$$= |c_1(t)|^2 \sqrt{n} \cancel{\langle \downarrow n | \uparrow n-1 \rangle} +$$
$$c_1^*(t) c_2(t) \sqrt{n-1} \cancel{\langle \downarrow n | 0 \, n-2 \rangle} +$$
$$c_2^*(t) c_1(t) \sqrt{n} \langle \uparrow n-1 | \uparrow n-1 \rangle +$$
$$|c_2(t)|^2 \sqrt{n-1} \cancel{\langle \uparrow n-1 | 0 \, n-2 \rangle} = c_2^*(t) c_1(t) \sqrt{n}$$

$$\langle \hat{S}_- \hat{a}^\dagger \rangle_{|A\rangle} = \left(c_1^*(t) \langle \downarrow n | + c_2^*(t) \langle \uparrow n-1 | \right) \hat{S}_- \hat{a}^\dagger \left(c_1(t) | \downarrow n \rangle + c_2(t) | \uparrow n-1 \rangle \right)$$

$$= |c_1(t)|^2 \langle \downarrow n | \hat{S}_- \hat{a}^\dagger | \downarrow n \rangle +$$
$$c_1^*(t) c_2(t) \langle \downarrow n | \hat{S}_- \hat{a}^\dagger | \uparrow n-1 \rangle +$$
$$c_2^*(t) c_1(t) \langle \uparrow n-1 | \hat{S}_- \hat{a}^\dagger | \downarrow n \rangle +$$
$$|c_2(t)|^2 \langle \uparrow n-1 | \hat{S}_- \hat{a}^\dagger | \uparrow n-1 \rangle =$$
$$= |c_1(t)|^2 \sqrt{n+1} \cancel{\langle \downarrow n | 0 \, n+1 \rangle}$$
$$c_1^*(t) c_2(t) \sqrt{n} \langle \downarrow n | \downarrow n \rangle$$
$$c_2^*(t) c_1(t) \sqrt{n+1} \cancel{\langle \uparrow n-1 | 0 \, n+1 \rangle}$$
$$|c_2(t)|^2 \sqrt{n} \cancel{\langle \uparrow n-1 | \downarrow n \rangle} = c_1^*(t) c_2(t) \sqrt{n}$$

√ ①

$$\hat{H}_R^m = \hbar\omega_m \hat{a}_m^{\dagger}\hat{a}_m + \hbar\Omega\hat{S}_+\hat{S}_- + \hbar g^m(\hat{S}_+ + \hat{S}_-)(\hat{a}_m^{\dagger} + \hat{a}_m)$$

Rabi Hamiltonian describing the interaction of a two level atom with a single-mode harmonic field

$\hbar g^m(\hat{S}_+\hat{a}_m^{\dagger} + \hat{S}_-\hat{a}_m)$ the so-called counter-rotating terms (αγνοούμε)

$$\hat{H}_{JC}^m = \hbar\omega_m \hat{a}_m^{\dagger}\hat{a}_m + \hbar\Omega\hat{S}_+\hat{S}_- + \hbar g^m(\hat{S}_+\hat{a}_m + \hat{S}_-\hat{a}_m^{\dagger})$$

Jaynes-Cummings Hamiltonian

Να υπολογιστούν τα $\langle\hat{a}_m^{\dagger}\hat{a}_m\rangle$, $\langle\hat{S}_+\hat{S}_-\rangle$, $\langle\hat{S}_+\hat{a}_m\rangle$, $\langle\hat{S}_-\hat{a}_m^{+}\rangle$ για τις καταστάσεις

Ⓐ $|\psi_A(t)\rangle = c_1(t)|\downarrow, n\rangle + c_2(t)|\uparrow, n-1\rangle$ θα χρησιμοποιηθεί με αρχικές συνθήκες $c_1(0)=1, c_2(0)=0$

Ⓔ $|\psi_E(t)\rangle = c_1(t)|\downarrow, n+1\rangle + c_2(t)|\uparrow, n\rangle$ $c_1(0)=0, c_2(0)=1$

Ⓐ

$$\langle\hat{a}_m^{\dagger}\hat{a}_m\rangle_{Ⓐ} = \langle\psi_A(t)|\hat{a}_m^{\dagger}\hat{a}_m|\psi_A(t)\rangle = \{c_1^*\langle\downarrow,n| + c_2^*\langle\uparrow,n-1|\}\hat{a}_m^{\dagger}\hat{a}_m\{c_1|\downarrow,n\rangle + c_2|\uparrow,n-1\rangle\}$$

$$= |c_1|^2\langle\downarrow,n|\hat{a}_m^{\dagger}\hat{a}_m|\downarrow,n\rangle + c_1^*c_2\langle\downarrow,n|\hat{a}_m^{\dagger}\hat{a}_m|\uparrow,n-1\rangle$$

$$+ c_2^*c_1\langle\uparrow,n-1|\hat{a}_m^{\dagger}\hat{a}_m|\downarrow,n\rangle + |c_2|^2\langle\uparrow,n-1|\hat{a}_m^{\dagger}\hat{a}_m|\uparrow,n-1\rangle$$

$$= |c_1|^2\sqrt{n}\sqrt{n}\langle\downarrow,n|\downarrow,n\rangle + c_1^*c_2\sqrt{n-1}\sqrt{n-1}\langle\downarrow,n|\uparrow,n-1\rangle$$

$$+ c_2^*c_1\sqrt{n}\sqrt{n}\langle\uparrow,n-1|\downarrow,n\rangle + |c_2|^2\sqrt{n-1}\sqrt{n-1}\langle\uparrow,n-1|\uparrow,n-1\rangle =$$

$$= n\cdot|c_1|^2\cdot 1 + c_1^*c_2(n-1)\cdot 0 + c_2^*c_1\, n\cdot 0 + (n-1)|c_2|^2\cdot 1 =$$

$$= n|c_1|^2 + n|c_2|^2 - |c_2|^2 = n(|c_1|^2 + |c_2|^2) - |c_2|^2 = n - |c_2|^2 \Rightarrow$$

$$\boxed{\langle\hat{a}_m^{\dagger}\hat{a}_m\rangle_{Ⓐ} = n_m - |c_2(t)|^2} \checkmark$$

$$\langle\hat{S}_+\hat{S}_-\rangle_{Ⓐ} = \langle\psi_A(t)|\hat{S}_+\hat{S}_-|\psi_A(t)\rangle = \{c_1^*\langle\downarrow,n| + c_2^*\langle\uparrow,n-1|\}\hat{S}_+\hat{S}_-\{c_1|\downarrow,n\rangle + c_2|\uparrow,n-1\rangle\}$$

$$= |c_1|^2\langle\downarrow,n|\hat{S}_+\hat{S}_-|\downarrow,n\rangle + c_1^*c_2\langle\downarrow,n|\hat{S}_+\hat{S}_-|\uparrow,n-1\rangle$$

$$+ c_2^*c_1\langle\uparrow,n-1|\hat{S}_+\hat{S}_-|\downarrow,n\rangle + |c_2|^2\langle\uparrow,n-1|\hat{S}_+\hat{S}_-|\uparrow,n-1\rangle =$$

$$= |c_1|^2\cdot 0 + c_1^*c_2\underbrace{\langle\downarrow,n|\uparrow,n-1\rangle}_{0} + c_2^*c_1\cdot 0 + |c_2|^2\underbrace{\langle\uparrow,n-1|\uparrow,n-1\rangle}_{1} \Rightarrow$$

$$\boxed{\langle\hat{S}_+\hat{S}_-\rangle_{Ⓐ} = |c_2(t)|^2} \checkmark$$

ΑΡΑ $$\boxed{\langle\hat{a}_m^{\dagger}\hat{a}_m\rangle_{Ⓐ} + \langle\hat{S}_+\hat{S}_-\rangle_{Ⓐ} = n_m} \checkmark$$

✓

$$\langle \hat{S}_+\hat{a}_m\rangle_{\textcircled{A}} = \langle \Psi_A(t)|\hat{S}_+\hat{a}_m|\Psi_A(t)\rangle = \{c_1^*\langle\downarrow,n| + c_2^*\langle\uparrow,n-1|\}\hat{S}_+\hat{a}_m\{c_1|\downarrow,n\rangle + c_2|\uparrow,n-1$$

$$= |c_1|^2\langle\downarrow,n|\hat{S}_+\hat{a}_m|\downarrow,n\rangle + c_1^*c_2\langle\downarrow,n|\hat{S}_+\hat{a}_m|\uparrow,n-1\rangle +$$

$$+ c_2^*c_1\langle\uparrow,n-1|\hat{S}_+\hat{a}_m|\downarrow,n\rangle + |c_2|^2\langle\uparrow,n-1|\hat{S}_+\hat{a}_m|\uparrow,n-1\rangle =$$

$$= |c_1|^2\sqrt{n}\langle\downarrow,n|\uparrow,n-1\rangle + c_1^*c_2\cdot\sqrt{n-1}\cdot\langle\downarrow,n|\hat{S}_+|\uparrow,n-2\rangle$$

$$+ c_2^*c_1\sqrt{n}\langle\uparrow,n-1|\uparrow,n-1\rangle + |c_2|^2\sqrt{n-1}\langle\uparrow,n-1|\underbrace{\hat{S}_+|\uparrow,n-2\rangle}_{0} \Rightarrow$$

$$\boxed{\langle\hat{S}_+\hat{a}_m\rangle_A = c_2^*(t)c_1(t)\cdot\sqrt{n}}$$

$$\langle\hat{S}_-\hat{a}_m^+\rangle_{\textcircled{A}} = \langle\Psi_A(t)|\hat{S}_-\hat{a}_m^+|\Psi_A(t)\rangle = \{c_1^*\langle\downarrow,n| + c_2^*|\uparrow,n-1\rangle\}\hat{S}_-\hat{a}_m^+\{c_1|\downarrow,n\rangle + c_2|\uparrow,n-1\rangle\}$$

$$= |c_1|^2\langle\downarrow,n|\hat{S}_-\hat{a}_m^+|\downarrow,n\rangle + c_1^*c_2\langle\downarrow,n|\hat{S}_-\hat{a}_m^+|\uparrow,n-1\rangle +$$

$$c_2^*c_1\langle\uparrow,n-1|\hat{S}_-\hat{a}_m^+|\downarrow,n\rangle + |c_2|^2\langle\uparrow,n-1|\hat{S}_-\hat{a}_m^+|\uparrow,n-1\rangle =$$

$$= |c_1|^2\sqrt{n+1}\langle\downarrow,n|\underbrace{\hat{S}_-|\downarrow,n+1\rangle}_{0} + c_1^*c_2\sqrt{n}\langle\downarrow,n|\downarrow,n\rangle$$

$$+ c_2^*c_1\langle\uparrow,n-1|\underbrace{\hat{S}_-|\downarrow,n+1\rangle}_{0}\sqrt{n+1} + |c_2|^2\langle\uparrow,n-1|\downarrow,n\rangle\sqrt{n} \Rightarrow$$

$$\boxed{\langle\hat{S}_-\hat{a}_m^+\rangle_{\textcircled{A}} = c_1^*(t)c_2(t)\sqrt{n}}$$

$\textcircled{E}$

$$\langle \hat a_m^{+} \hat a_m \rangle_{\textcircled{E}} = \langle \psi_E(t) | \hat a_m^{+} \hat a_m | \psi_E(t) \rangle = \{ c_1^{*} \langle \downarrow, n+1 | + c_2^{*} \langle \uparrow, n | \} | \hat a_m^{+} \hat a_m | \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \}$$

$$= |c_1|^2 \langle \downarrow, n+1 | \hat a_m^{+} \hat a_m | \downarrow, n+1 \rangle + c_1^{*} c_2 \langle \downarrow, n+1 | \hat a_m^{+} \hat a_m | \uparrow, n \rangle$$

$$+ c_2^{*} c_1 \langle \uparrow, n | \hat a_m^{+} \hat a_m | \downarrow, n+1 \rangle + |c_2|^2 \langle \uparrow, n | \hat a_m^{+} \hat a_m | \uparrow, n \rangle$$

$$= |c_1|^2 \sqrt{n+1} \sqrt{n+1} \langle \downarrow, n+1 | \downarrow, n+1 \rangle + c_1^{*} c_2 \, n \cancelto{0}{\langle \downarrow, n+1 | \uparrow, n \rangle}$$

$$+ c_2^{*} c_1 (n+1) \cancelto{0}{\langle \uparrow, n | \downarrow, n+1 \rangle} + |c_2|^2 n \langle \uparrow, n | \uparrow, n \rangle =$$

$$= |c_1|^2 (n+1) + n |c_2|^2 = n \left(|c_1|^2 + |c_2|^2 \right) + |c_1|^2 \Rightarrow$$

$$\boxed{\langle \hat a_m^{+} \hat a_m \rangle_{\textcircled{E}} = n + |c_1(t)|^2} \checkmark \qquad (4.60)$$

$$\langle \hat S_+ \hat S_- \rangle_{\textcircled{E}} = \langle \psi_E(t) | \hat S_+ \hat S_- | \psi_E(t) \rangle = \{ c_1^{*} \langle \downarrow, n+1 | + c_2^{*} \langle \uparrow, n | \} \hat S_+ \hat S_- \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \}$$

$$= |c_1|^2 \cdot 0 + c_1^{*} c_2 \cancelto{0}{\langle \downarrow, n+1 | \uparrow, n \rangle} + c_2^{*} c_1 \cdot 0 + |c_2|^2 \langle \uparrow, n | \uparrow, n \rangle \Rightarrow$$

$$\boxed{\langle \hat S_+ \hat S_- \rangle_{\textcircled{E}} = |c_2(t)|^2} \checkmark \qquad (4.60)$$

$$\langle \hat S_+ \hat a_m \rangle_{\textcircled{E}} = \langle \psi_E(t) | \hat S_+ \hat a_m | \psi_E(t) \rangle = \{ c_1^{*} \langle \downarrow, n+1 | + c_2^{*} \langle \uparrow, n | \} \hat S_+ \hat a_m \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \}$$

$$= |c_1|^2 \langle \downarrow, n+1 | \hat S_+ \hat a_m | \downarrow, n+1 \rangle + c_1^{*} c_2 \langle \downarrow, n+1 | \hat S_+ \hat a_m | \uparrow, n \rangle +$$

$$c_2^{*} c_1 \langle \uparrow, n | \hat S_+ \hat a_m | \downarrow, n+1 \rangle + |c_2|^2 \langle \uparrow, n | \hat S_+ \hat a_m | \uparrow, n \rangle$$

$$= |c_1|^2 \cancelto{0}{\langle \downarrow, n+1 | \uparrow, n \rangle} \sqrt{n+1} + c_1^{*} c_2 \cdot 0 + c_2^{*} c_1 \langle \uparrow, n | \uparrow, n \rangle \sqrt{n+1} + |c_2|^2 0$$

$$\boxed{\langle \hat S_+ \hat a_m \rangle_{\textcircled{E}} = c_2^{*}(t)\, c_1(t) \cdot \sqrt{n+1}}$$

$$\langle \hat S_- \hat a_m^{+} \rangle_{\textcircled{E}} = \langle \psi_E(t) | \hat S_- \hat a_m^{+} | \psi_E(t) \rangle = \{ c_1^{*} \langle \downarrow, n+1 | + c_2^{*} \langle \uparrow, n | \} \hat S_- \hat a_m^{+} \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \}$$

$$= |c_1|^2 \langle \downarrow, n+1 | \hat S_- \hat a_m^{+} | \downarrow, n+1 \rangle + c_1^{*} c_2 \langle \downarrow, n+1 | \hat S_- \hat a_m^{+} | \uparrow, n \rangle$$

$$+ c_2^{*} c_1 \langle \uparrow, n | \hat S_- \hat a_m | \downarrow, n+1 \rangle + |c_2|^2 \langle \uparrow, n | \hat S_- \hat a_m^{+} | \uparrow, n \rangle =$$

$$= |c_1|^2 \cancelto{0}{\langle \downarrow, n+1 | \uparrow, n+2 \rangle} \sqrt{n+2} + c_1^{*} c_2 \langle \downarrow, n+1 | \downarrow, n+1 \rangle \sqrt{n+1}$$

$$+ c_2^{*} c_1 \cdot 0 + |c_2|^2 \cancelto{0}{\langle \uparrow, n | \downarrow, n+1 \rangle} \sqrt{n+1} \Rightarrow$$

$$\boxed{\langle \hat S_- \hat a_m^{+} \rangle_{\textcircled{E}} = c_1^{*}(t) \cdot c_2(t) \cdot \sqrt{n+1}}$$

APA $\boxed{\langle \hat a_m^{+} \hat a_m \rangle_{\textcircled{E}} + \langle \hat S_+ \hat S_- \rangle_{\textcircled{E}} = n+1} \checkmark \qquad (4.61)$