

# ΣΥΝΟΨΗ ΧΑΜΙΛΤΟΝΙΑΝΩΝ

$$\hat{H}_{HM,m} = \hbar \omega_m (\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2}) = \hbar \omega (\hat{N}_m + \frac{1}{2})$$

η πρόθ.  $\hbar \omega$   
 η τρέπον ΗΜ πεδίου  
 στην κοιλότητα

και αγνοώντας τον όρο  $\frac{\hbar \omega_m}{2}$

$$\hat{H}_{HM,m} = \hbar \omega_m \hat{a}_m^\dagger \hat{a}_m = \hbar \omega_m \hat{N}_m$$

$$\hat{H}_{\Delta\Sigma} = E_2 \hat{S}_+ \hat{S}_- + E_1 \hat{S}_- \hat{S}_+ \quad E_2 - E_1 = \hbar \Omega$$

δισταθμικά σύστημα

και θέτουμε  $E_1 \equiv 0$

$$\hat{H}_{\Delta\Sigma} = \hbar \Omega \hat{S}_+ \hat{S}_-$$

$$\hat{U}_{\Sigma m} = \hbar g_m (\hat{S}_+ + \hat{S}_-) (\hat{a}_m^\dagger + \hat{a}_m) = \hat{H}_{\Delta\Sigma - HMm}$$

αλληλεπίδραση ΔΣ - η τρέπον ΗΜ πεδίου  
 στην κοιλότητα

( $\hat{H}_{AF}$  ατομική φυσική)  
 atom field

$$\hbar g_m := e x_{12} \left( \frac{\hbar \omega_m}{\epsilon_0 V} \right)^{1/2} \sin\left(\frac{m\pi z}{L}\right)$$

$$\hbar |g_m| = \underbrace{|e x_{12}|}_{|\mathcal{F}|} \underbrace{\left( \frac{4 \hbar \omega_m n_m}{\epsilon_0 V} \right)^{1/2}}_{E_{0m}(z)} \underbrace{\left| \sin\left(\frac{m\pi z}{L}\right) \right|}_{\frac{1}{2\sqrt{n_m}}}$$

$$\Omega_{Rm} = \frac{|\mathcal{F}| E_{0m}}{\hbar}$$

συντόνισμα  
 Rabi του  
 η τρέπον

$$\Omega_{\Sigma m} := 2 \sqrt{n_m} g_m$$

$$|\uparrow, n_m\rangle, |\downarrow, n_m\rangle \text{ οι } \frac{\delta E_0}{\sqrt{\text{καταστάσεις της}}} \text{ του } \hat{H}_{HM,m} + \hat{H}_{\Delta\Sigma}$$

Ενώ, η συνολική Χαμιλτονιανή γράφεται

$$\hat{H}_{RM} = \hbar \omega_m \hat{a}_m^\dagger \hat{a}_m + \hbar \Omega \hat{S}_+ \hat{S}_- + \hbar g_m (\hat{S}_+ + \hat{S}_-) (\hat{a}_m^\dagger + \hat{a}_m)$$

δημογραφία συχνά  
 Χαμιλτονιανή Rabi

Αγνοώντας τους όρους  $\hat{S}_+ \hat{a}_m^\dagger$  &  $\hat{S}_- \hat{a}_m$ , οι οποίοι γράφονται "παράλογοι", όταν υπάρχει  
 ένας μόνο τρόπος:

$$\hat{H}_{JCM} = \hbar \omega_m \hat{a}_m^\dagger \hat{a}_m + \hbar \Omega \hat{S}_+ \hat{S}_- + \hbar g_m (\hat{S}_+ \hat{a}_m + \hat{S}_- \hat{a}_m^\dagger)$$

δημογραφία συχνά  
 Χαμιλτονιανή  
 Jaynes-Cummings

ΑΣΚΗΣΗ (Παραδείνονται για χρησιμότητα των δεικτών των HM τριών m)

Θ

Α) Βρείτε τι κάνουν οι όροι

$$\hat{a}^{\dagger}\hat{a}, \hat{a}\hat{a}^{\dagger}, \hat{S}_+\hat{S}_-, \hat{S}_-\hat{S}_+, \hat{S}_+\hat{a}^{\dagger}, \hat{S}_+\hat{a}, \hat{S}_-\hat{a}^{\dagger}, \hat{S}_-\hat{a} \text{ επί καταστάσεις } |\uparrow, n\rangle \text{ ή } |\downarrow, n\rangle$$

Β) Υπολογιστείτε

$$\langle \hat{a}^{\dagger}\hat{a} \rangle, \langle \hat{a}\hat{a}^{\dagger} \rangle, \langle \hat{S}_+\hat{S}_- \rangle, \langle \hat{S}_-\hat{S}_+ \rangle, \langle \hat{S}_+\hat{a}^{\dagger} \rangle, \langle \hat{S}_+\hat{a} \rangle, \langle \hat{S}_-\hat{a}^{\dagger} \rangle, \langle \hat{S}_-\hat{a} \rangle$$

επί  $|\uparrow, n\rangle, |\downarrow, n\rangle$

Α)  $\hat{a}^{\dagger}\hat{a}|\uparrow, n\rangle = n|\uparrow, n\rangle \quad \hat{a}^{\dagger}\hat{a}|\downarrow, n\rangle = n|\downarrow, n\rangle$

$$[\hat{a}, \hat{a}^{\dagger}] = 1 \Leftrightarrow \hat{a}\hat{a}^{\dagger} = 1 + \hat{a}^{\dagger}\hat{a}$$

$$\hat{a}\hat{a}^{\dagger}|\uparrow, n\rangle = (1 + \hat{a}^{\dagger}\hat{a})|\uparrow, n\rangle = (n+1)|\uparrow, n\rangle$$

$$\hat{a}\hat{a}^{\dagger}|\downarrow, n\rangle = (1 + \hat{a}^{\dagger}\hat{a})|\downarrow, n\rangle = (n+1)|\downarrow, n\rangle$$

$$\hat{S}_+\hat{S}_-|\uparrow, n\rangle = |\uparrow, n\rangle \quad \hat{S}_+\hat{S}_-|\downarrow, n\rangle = |\emptyset, n\rangle$$

$$\hat{S}_-\hat{S}_+|\uparrow, n\rangle = |\emptyset, n\rangle \quad \hat{S}_-\hat{S}_+|\downarrow, n\rangle = |\downarrow, n\rangle$$

$$\hat{S}_+\hat{a}^{\dagger}|\uparrow, n\rangle = \sqrt{n+1}|\emptyset, n+1\rangle \quad \hat{S}_+\hat{a}^{\dagger}|\downarrow, n\rangle = \sqrt{n+1}|\uparrow, n+1\rangle$$

$$\hat{S}_+\hat{a}|\uparrow, n\rangle = \sqrt{n}|\emptyset, n-1\rangle \quad \hat{S}_+\hat{a}|\downarrow, n\rangle = \sqrt{n}|\uparrow, n-1\rangle$$

$$\hat{S}_-\hat{a}^{\dagger}|\uparrow, n\rangle = \sqrt{n+1}|\downarrow, n+1\rangle \quad \hat{S}_-\hat{a}^{\dagger}|\downarrow, n\rangle = \sqrt{n+1}|\emptyset, n+1\rangle$$

$$\hat{S}_-\hat{a}|\uparrow, n\rangle = \sqrt{n}|\downarrow, n-1\rangle \quad \hat{S}_-\hat{a}|\downarrow, n\rangle = \sqrt{n}|\emptyset, n-1\rangle$$

Β)  $\langle \uparrow n | \hat{a}^{\dagger}\hat{a} | \uparrow n \rangle = n$

$$\langle \downarrow n | \hat{a}^{\dagger}\hat{a} | \downarrow n \rangle = n$$

$$\langle \uparrow n | \hat{a}\hat{a}^{\dagger} | \uparrow n \rangle = n+1$$

$$\langle \downarrow n | \hat{a}\hat{a}^{\dagger} | \downarrow n \rangle = n+1$$

$$\langle \uparrow n | \hat{S}_+\hat{S}_- | \uparrow n \rangle = 1$$

$$\langle \downarrow n | \hat{S}_+\hat{S}_- | \downarrow n \rangle = \langle \downarrow n | \emptyset n \rangle = 0$$

$$\langle \uparrow n | \hat{S}_-\hat{S}_+ | \uparrow n \rangle = \langle \uparrow n | \emptyset n \rangle = 0$$

$$\langle \downarrow n | \hat{S}_-\hat{S}_+ | \downarrow n \rangle = \langle \downarrow n | \downarrow n \rangle = 1$$

ΕΥΧΕΙΣΤΕ...

|                                                                      |                                                                        |
|----------------------------------------------------------------------|------------------------------------------------------------------------|
| $\langle \hat{a}^\dagger \hat{a} \rangle_{ \uparrow n\rangle} = n$   | $\langle \hat{a}^\dagger \hat{a} \rangle_{ \downarrow n\rangle} = n$   |
| $\langle \hat{a} \hat{a}^\dagger \rangle_{ \uparrow n\rangle} = n+1$ | $\langle \hat{a} \hat{a}^\dagger \rangle_{ \downarrow n\rangle} = n+1$ |
| $\langle \hat{S}_+ \hat{S}_- \rangle_{ \uparrow n\rangle} = 1$       | $\langle \hat{S}_+ \hat{S}_- \rangle_{ \downarrow n\rangle} = 0$       |
| $\langle \hat{S}_- \hat{S}_+ \rangle_{ \uparrow n\rangle} = 0$       | $\langle \hat{S}_- \hat{S}_+ \rangle_{ \downarrow n\rangle} = 1$       |

$$\begin{aligned} \hat{a}^\dagger |\uparrow n\rangle &= \sqrt{n+1} |\uparrow n+1\rangle \\ \hat{a} |\downarrow n\rangle &= \sqrt{n+1} |\downarrow n+1\rangle \\ \hat{a}^\dagger |\downarrow n\rangle &= \sqrt{n+1} |\uparrow n+1\rangle \\ \hat{a} |\uparrow n\rangle &= \sqrt{n} |\downarrow n-1\rangle \end{aligned}$$

$$\langle \hat{S}_+ \hat{a}^\dagger \rangle_{|\uparrow n\rangle} = 0 \quad \langle \hat{S}_+ \hat{a}^\dagger \rangle_{|\downarrow n\rangle} = 0$$

$$\langle \hat{S}_+ \hat{a} \rangle_{|\uparrow n\rangle} = 0 \quad \langle \hat{S}_+ \hat{a} \rangle_{|\downarrow n\rangle} = 0$$

$$\langle \hat{S}_- \hat{a}_m^\dagger \rangle_{|\uparrow n\rangle} = 0 \quad \langle \hat{S}_- \hat{a}_m^\dagger \rangle_{|\downarrow n\rangle} = 0$$

$$\langle \hat{S}_- \hat{a} \rangle_{|\uparrow n\rangle} = 0 \quad \langle \hat{S}_- \hat{a} \rangle_{|\downarrow n\rangle} = 0$$

... boundary

$$\langle \uparrow n | \hat{S}_+ \hat{a}^\dagger | \uparrow n \rangle = \sqrt{n+1} \langle \uparrow n | \emptyset, n+1 \rangle = 0$$

$$\langle \downarrow n | \hat{S}_+ \hat{a}^\dagger | \downarrow n \rangle = \sqrt{n+1} \langle \downarrow n | \uparrow n+1 \rangle = 0$$

$$\langle \uparrow n | \hat{S}_+ \hat{a} | \uparrow n \rangle = \sqrt{n} \langle \uparrow n | \emptyset, n-1 \rangle = 0$$

$$\langle \downarrow n | \hat{S}_+ \hat{a} | \downarrow n \rangle = \sqrt{n} \langle \downarrow n | \uparrow n-1 \rangle = 0$$

$$\langle \uparrow n | \hat{S}_- \hat{a}^\dagger | \uparrow n \rangle = \sqrt{n+1} \langle \uparrow n | \downarrow n+1 \rangle = 0$$

$$\langle \downarrow n | \hat{S}_- \hat{a}^\dagger | \downarrow n \rangle = \sqrt{n+1} \langle \downarrow n | \emptyset, n+1 \rangle = 0$$

$$\langle \uparrow n | \hat{S}_- \hat{a} | \uparrow n \rangle = \sqrt{n} \langle \uparrow n | \downarrow n-1 \rangle = 0$$

$$\langle \downarrow n | \hat{S}_- \hat{a} | \downarrow n \rangle = \sqrt{n} \langle \downarrow n | \emptyset, n-1 \rangle = 0$$

ΜΕΣΕΣ (ANAMENOMENES) TIMES ΜΕΓΕΘΩΝ  
για τη Χαμιλτονιανή Jaynes - Cummings

Για την κατάσταση: (A)  $|\psi_A(t)\rangle = c_1(t)|\downarrow, n\rangle + c_2(t)|\uparrow, n-1\rangle = |A\rangle$   
 (B)  $|\psi_B(t)\rangle = c_1(t)|\uparrow, n\rangle + c_2(t)|\downarrow, n\rangle = |B\rangle$

$$\begin{aligned} \langle \hat{a}^\dagger \hat{a} \rangle_{|A\rangle} &= \left( c_1(t)^* \langle \downarrow, n| + c_2(t)^* \langle \uparrow, n-1| \right) \hat{a}^\dagger \hat{a} \left( c_1(t) |\downarrow, n\rangle + c_2(t) |\uparrow, n-1\rangle \right) \\ &= |c_1(t)|^2 \langle \downarrow, n| \hat{a}^\dagger \hat{a} |\downarrow, n\rangle + c_1^*(t) c_2(t) \langle \downarrow, n| \hat{a}^\dagger \hat{a} |\uparrow, n-1\rangle \\ &\quad + c_2^*(t) c_1(t) \langle \uparrow, n-1| \hat{a}^\dagger \hat{a} |\downarrow, n\rangle + |c_2(t)|^2 \langle \uparrow, n-1| \hat{a}^\dagger \hat{a} |\uparrow, n-1\rangle \\ &= |c_1(t)|^2 n \langle \downarrow, n | \downarrow, n \rangle + c_1^*(t) c_2(t) (n-1) \langle \downarrow, n | \uparrow, n-1 \rangle \\ &\quad + c_2^*(t) c_1(t) n \langle \uparrow, n-1 | \downarrow, n \rangle + |c_2(t)|^2 (n-1) \langle \uparrow, n-1 | \uparrow, n-1 \rangle \\ &= |c_1(t)|^2 n + |c_2(t)|^2 n - |c_2(t)|^2 \Rightarrow \langle \hat{a}^\dagger \hat{a} \rangle_{|A\rangle} = n - |c_2(t)|^2 \end{aligned}$$

$$\begin{aligned} \langle \hat{a} \hat{a}^\dagger \rangle_{|A\rangle} &= \left( c_1(t)^* \langle \downarrow, n| + c_2(t)^* \langle \uparrow, n-1| \right) \hat{a} \hat{a}^\dagger \left( c_1(t) |\downarrow, n\rangle + c_2(t) |\uparrow, n-1\rangle \right) \\ &= |c_1(t)|^2 \langle \downarrow, n| \hat{a} \hat{a}^\dagger |\downarrow, n\rangle + c_1^*(t) c_2(t) \langle \downarrow, n| \hat{a} \hat{a}^\dagger |\uparrow, n-1\rangle \\ &\quad + c_2^*(t) c_1(t) \langle \uparrow, n-1| \hat{a} \hat{a}^\dagger |\downarrow, n\rangle + |c_2(t)|^2 \langle \uparrow, n-1| \hat{a} \hat{a}^\dagger |\uparrow, n-1\rangle = \\ &= |c_1(t)|^2 (n+1) + c_1^*(t) c_2(t) n \langle \downarrow, n | \uparrow, n-1 \rangle \\ &\quad + c_2^*(t) c_1(t) (n+1) \langle \uparrow, n-1 | \downarrow, n \rangle + |c_2(t)|^2 n = \\ &= |c_1(t)|^2 n + |c_1(t)|^2 + |c_2(t)|^2 n \Rightarrow \langle \hat{a} \hat{a}^\dagger \rangle_{|A\rangle} = n + |c_1(t)|^2 \end{aligned}$$

$$\langle \hat{a} \hat{a}^\dagger \rangle_{|A\rangle} - \langle \hat{a}^\dagger \hat{a} \rangle_{|A\rangle} = 1$$

$$\langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} = |c_2(t)|^2$$

$$\langle \hat{S}_- \hat{S}_+ \rangle_{|A\rangle} = |c_1(t)|^2$$

Συμμετρικά αντιστοιχίζουμε...

IB

$$\langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} + \langle \hat{S}_- \hat{S}_+ \rangle_{|A\rangle} = 1$$

$$\langle \hat{a}^\dagger \hat{a} \rangle_{|A\rangle} + \langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} = n$$

$$\langle \hat{S}_+ \hat{a} \rangle_{|A\rangle} = c_2^*(t) c_1(t) \sqrt{n}$$

$$\langle \hat{S}_- \hat{a}^\dagger \rangle_{|A\rangle} = c_1^*(t) c_2(t) \sqrt{n}$$

$$\langle \hat{S}_+ \hat{a}^\dagger \rangle_{|A\rangle} = 0$$

$$\langle \hat{S}_- \hat{a} \rangle_{|A\rangle} = 0$$

Για την κατάσταση (E)  $|\Psi_E(t)\rangle = c_1(t) |\uparrow n\rangle + c_2(t) |\downarrow n+1\rangle = |E\rangle$

Συμμετρικά αντιστοιχίζουμε

$$\langle \hat{a}^\dagger \hat{a} \rangle_{|E\rangle} =$$

$$\langle \hat{a} \hat{a}^\dagger \rangle_{|E\rangle} =$$

$$\langle \hat{a} \hat{a}^\dagger \rangle_{|E\rangle} - \langle \hat{a}^\dagger \hat{a} \rangle_{|E\rangle} =$$

$$\langle \hat{S}_+ \hat{S}_- \rangle_{|E\rangle} =$$

$$\langle \hat{S}_- \hat{S}_+ \rangle_{|E\rangle} =$$

$$\langle \hat{S}_+ \hat{S}_- \rangle_{|E\rangle} + \langle \hat{S}_- \hat{S}_+ \rangle_{|E\rangle} =$$

$$\langle \hat{a}^\dagger \hat{a} \rangle_{|E\rangle} + \langle \hat{S}_+ \hat{S}_- \rangle_{|E\rangle} =$$

$$\langle \hat{S}_+ \hat{a} \rangle_{|E\rangle} =$$

$$\langle \hat{S}_+ \hat{a}^\dagger \rangle_{|E\rangle} =$$

$$\langle \hat{S}_- \hat{a}^\dagger \rangle_{|E\rangle} =$$

$$\langle \hat{S}_- \hat{a} \rangle_{|E\rangle} =$$

$$\begin{aligned}
 \langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} &= (c_1^*(t) \langle \downarrow n | + c_2^*(t) \langle \uparrow n-1 |) \hat{S}_+ \hat{S}_- (c_1(t) | \downarrow n \rangle + c_2(t) | \uparrow n-1 \rangle) \\
 &= |c_1(t)|^2 \langle \downarrow n | \hat{S}_+ \hat{S}_- | \downarrow n \rangle + \\
 &\quad c_1^*(t) c_2(t) \langle \downarrow n | \hat{S}_+ \hat{S}_- | \uparrow n-1 \rangle + \\
 &\quad c_2^*(t) c_1(t) \langle \uparrow n-1 | \hat{S}_+ \hat{S}_- | \downarrow n \rangle + \\
 &\quad |c_2(t)|^2 \langle \uparrow n-1 | \hat{S}_+ \hat{S}_- | \uparrow n-1 \rangle = \\
 &= |c_1(t)|^2 \langle \downarrow n | \cancel{\uparrow n} \rangle + \\
 &\quad c_1^*(t) c_2(t) \langle \downarrow n | \cancel{\uparrow n-1} \rangle + \\
 &\quad c_2^*(t) c_1(t) \langle \uparrow n-1 | \cancel{\downarrow n} \rangle + \\
 &\quad |c_2(t)|^2 \langle \uparrow n-1 | \uparrow n-1 \rangle = |c_2(t)|^2
 \end{aligned}$$

$$\begin{aligned}
 \langle \hat{S}_- \hat{S}_+ \rangle_{|A\rangle} &= (c_1^*(t) \langle \downarrow n | + c_2^*(t) \langle \uparrow n-1 |) \hat{S}_- \hat{S}_+ (c_1(t) | \downarrow n \rangle + c_2(t) | \uparrow n-1 \rangle) \\
 &= |c_1(t)|^2 \langle \downarrow n | \hat{S}_- \hat{S}_+ | \downarrow n \rangle + \\
 &\quad c_1^*(t) c_2(t) \langle \downarrow n | \hat{S}_- \hat{S}_+ | \uparrow n-1 \rangle + \\
 &\quad c_2^*(t) c_1(t) \langle \uparrow n-1 | \hat{S}_- \hat{S}_+ | \downarrow n \rangle + \\
 &\quad |c_2(t)|^2 \langle \uparrow n-1 | \hat{S}_- \hat{S}_+ | \uparrow n-1 \rangle = \\
 &= |c_1(t)|^2 \langle \downarrow n | \downarrow n \rangle + \\
 &\quad c_1^*(t) c_2(t) \langle \downarrow n | \cancel{\uparrow n-1} \rangle + \\
 &\quad c_2^*(t) c_1(t) \langle \uparrow n-1 | \cancel{\downarrow n} \rangle + \\
 &\quad |c_2(t)|^2 \langle \uparrow n-1 | \cancel{\downarrow n-1} \rangle = |c_1(t)|^2
 \end{aligned}$$

$$\begin{aligned}
 \langle \hat{S}_+ \hat{a} \rangle_{|A\rangle} &= (c_1^*(t) \langle \downarrow n | + c_2^*(t) \langle \uparrow n-1 |) \hat{S}_+ \hat{a} (c_1(t) |\downarrow n\rangle + c_2(t) |\uparrow n-1\rangle) \textcircled{\text{IB}} \\
 &= |c_1(t)|^2 \langle \downarrow n | \hat{S}_+ \hat{a} | \downarrow n \rangle + \\
 &\quad c_1^*(t) c_2(t) \langle \downarrow n | \hat{S}_+ \hat{a} | \uparrow n-1 \rangle + \\
 &\quad c_2^*(t) c_1(t) \langle \uparrow n-1 | \hat{S}_+ \hat{a} | \downarrow n \rangle + \\
 &\quad |c_2(t)|^2 \langle \uparrow n-1 | \hat{S}_+ \hat{a} | \uparrow n-1 \rangle = \\
 &= |c_1(t)|^2 \sqrt{n} \langle \downarrow n | \uparrow n-1 \rangle + \\
 &\quad c_1^*(t) c_2(t) \sqrt{n-1} \langle \downarrow n | \uparrow n-2 \rangle + \\
 &\quad c_2^*(t) c_1(t) \sqrt{n} \langle \uparrow n-1 | \uparrow n-1 \rangle + \\
 &\quad |c_2(t)|^2 \sqrt{n-1} \langle \uparrow n-1 | \uparrow n-2 \rangle = c_2^*(t) c_1(t) \sqrt{n}
 \end{aligned}$$

$$\begin{aligned}
 \langle \hat{S}_- \hat{a}^\dagger \rangle_{|A\rangle} &= (c_1(t) \langle \downarrow n | + c_2(t) \langle \uparrow n-1 |) \hat{S}_- \hat{a}^\dagger (c_1(t) |\downarrow n\rangle + c_2(t) |\uparrow n-1\rangle) \\
 &= |c_1(t)|^2 \langle \downarrow n | \hat{S}_- \hat{a}^\dagger | \downarrow n \rangle + \\
 &\quad c_1^*(t) c_2(t) \langle \downarrow n | \hat{S}_- \hat{a}^\dagger | \uparrow n-1 \rangle + \\
 &\quad c_2^*(t) c_1(t) \langle \uparrow n-1 | \hat{S}_- \hat{a}^\dagger | \downarrow n \rangle + \\
 &\quad |c_2(t)|^2 \langle \uparrow n-1 | \hat{S}_- \hat{a}^\dagger | \uparrow n-1 \rangle = \\
 &= |c_1(t)|^2 \sqrt{n+1} \langle \downarrow n | \uparrow n+1 \rangle + \\
 &\quad c_1^*(t) c_2(t) \sqrt{n} \langle \downarrow n | \downarrow n \rangle + \\
 &\quad c_2^*(t) c_1(t) \sqrt{n+1} \langle \uparrow n-1 | \uparrow n+1 \rangle + \\
 &\quad |c_2(t)|^2 \sqrt{n} \langle \uparrow n-1 | \downarrow n \rangle = c_1^*(t) c_2(t) \sqrt{n}
 \end{aligned}$$

$$\hat{H}_R = \hbar \omega_m \hat{a}_m^\dagger \hat{a}_m + \hbar \Omega \hat{S}_+ \hat{S}_- + \hbar g_m (\hat{S}_+ + \hat{S}_-) (\hat{a}_m^\dagger + \hat{a}_m)$$

Rabi Hamiltonian  
describing the interaction  
of a two-level atom  
with a single-mode  
harmonic field

$$\hbar g_m (\hat{S}_+ \hat{a}_m^\dagger + \hat{S}_- \hat{a}_m) \text{ the so-called counter-rotating terms (approx.)}$$

$$\hat{H}_{JC} = \hbar \omega_m \hat{a}_m^\dagger \hat{a}_m + \hbar \Omega \hat{S}_+ \hat{S}_- + \hbar g_m (\hat{S}_+ \hat{a}_m + \hat{S}_- \hat{a}_m^\dagger)$$

Jaynes-Cummings  
Hamiltonian

Να υπολογίσουμε τα  $\langle \hat{a}_m^\dagger \hat{a}_m \rangle$ ,  $\langle \hat{S}_+ \hat{S}_- \rangle$ ,  $\langle \hat{S}_+ \hat{a}_m \rangle$ ,  $\langle \hat{S}_- \hat{a}_m^\dagger \rangle$  για τις καταστάσεις

Α  $|\Psi_A(t)\rangle = c_1(t) |\downarrow, n\rangle + c_2(t) |\uparrow, n-1\rangle$  θα χρησιμοποιηθούν  $c_1(0)=1, c_2(0)=0$

Β  $|\Psi_B(t)\rangle = c_1(t) |\downarrow, n+1\rangle + c_2(t) |\uparrow, n\rangle$  με αρχικές συνθήκες  $c_1(0)=0, c_2(0)=1$

Α

$$\begin{aligned} \langle \hat{a}_m^\dagger \hat{a}_m \rangle_A &= \langle \Psi_A(t) | \hat{a}_m^\dagger \hat{a}_m | \Psi_A(t) \rangle = \{ c_1^* \langle \downarrow, n | + c_2^* \langle \uparrow, n-1 | \} \hat{a}_m^\dagger \hat{a}_m \{ c_1 |\downarrow, n\rangle + c_2 |\uparrow, n-1\rangle \} \\ &= |c_1|^2 \langle \downarrow, n | \hat{a}_m^\dagger \hat{a}_m | \downarrow, n \rangle + c_1^* c_2 \langle \downarrow, n | \hat{a}_m^\dagger \hat{a}_m | \uparrow, n-1 \rangle \\ &\quad + c_2^* c_1 \langle \uparrow, n-1 | \hat{a}_m^\dagger \hat{a}_m | \downarrow, n \rangle + |c_2|^2 \langle \uparrow, n-1 | \hat{a}_m^\dagger \hat{a}_m | \uparrow, n-1 \rangle \\ &= |c_1|^2 \sqrt{n} \sqrt{n} \langle \downarrow, n | \downarrow, n \rangle + c_1^* c_2 \sqrt{n-1} \sqrt{n-1} \langle \downarrow, n | \uparrow, n-1 \rangle \\ &\quad + c_2^* c_1 \sqrt{n} \sqrt{n} \langle \uparrow, n-1 | \downarrow, n \rangle + |c_2|^2 \sqrt{n-1} \sqrt{n-1} \langle \uparrow, n-1 | \uparrow, n-1 \rangle = \\ &= n |c_1|^2 \cdot 1 + c_1^* c_2 (n-1) \cdot 0 + c_2^* c_1 n \cdot 0 + (n-1) |c_2|^2 \cdot 1 = \\ &= n |c_1|^2 + n |c_2|^2 - |c_2|^2 = n (|c_1|^2 + |c_2|^2) - |c_2|^2 = n - |c_2|^2 \Rightarrow \end{aligned}$$

$$\langle \hat{a}_m^\dagger \hat{a}_m \rangle_A = n - |c_2(t)|^2 \quad \checkmark$$

$$\begin{aligned} \langle \hat{S}_+ \hat{S}_- \rangle_A &= \langle \Psi_A(t) | \hat{S}_+ \hat{S}_- | \Psi_A(t) \rangle = \{ c_1^* \langle \downarrow, n | + c_2^* \langle \uparrow, n-1 | \} \hat{S}_+ \hat{S}_- \{ c_1 |\downarrow, n\rangle + c_2 |\uparrow, n-1\rangle \} \\ &= |c_1|^2 \langle \downarrow, n | \hat{S}_+ \hat{S}_- | \downarrow, n \rangle + c_1^* c_2 \langle \downarrow, n | \hat{S}_+ \hat{S}_- | \uparrow, n-1 \rangle \\ &\quad + c_2^* c_1 \langle \uparrow, n-1 | \hat{S}_+ \hat{S}_- | \downarrow, n \rangle + |c_2|^2 \langle \uparrow, n-1 | \hat{S}_+ \hat{S}_- | \uparrow, n-1 \rangle = \\ &= |c_1|^2 \cdot 0 + c_1^* c_2 \langle \downarrow, n | \uparrow, n-1 \rangle + c_2^* c_1 \cdot 0 + |c_2|^2 \langle \uparrow, n-1 | \uparrow, n-1 \rangle \Rightarrow \end{aligned}$$

$$\langle \hat{S}_+ \hat{S}_- \rangle_A = |c_2(t)|^2 \quad \checkmark$$

ΑΡΑ  $\langle \hat{a}_m^\dagger \hat{a}_m \rangle_A + \langle \hat{S}_+ \hat{S}_- \rangle_A = n \quad \checkmark$

$$\begin{aligned}
 \langle \hat{S}_+ \hat{a}_m \rangle_A &= \langle \psi_A(t) | \hat{S}_+ \hat{a}_m | \psi_A(t) \rangle = \{ c_1^* \langle \downarrow, n | + c_2^* \langle \uparrow, n-1 | \} \hat{S}_+ \hat{a}_m \{ c_1 | \downarrow, n \rangle + c_2 | \uparrow, n-1 \rangle \} \\
 &= |c_1|^2 \langle \downarrow, n | \hat{S}_+ \hat{a}_m | \downarrow, n \rangle + c_1^* c_2 \langle \downarrow, n | \hat{S}_+ \hat{a}_m | \uparrow, n-1 \rangle + \\
 &\quad + c_2^* c_1 \langle \uparrow, n-1 | \hat{S}_+ \hat{a}_m | \downarrow, n \rangle + |c_2|^2 \langle \uparrow, n-1 | \hat{S}_+ \hat{a}_m | \uparrow, n-1 \rangle = \\
 &= |c_1|^2 \sqrt{n} \langle \downarrow, n | \uparrow, n-1 \rangle + c_1^* c_2 \cdot \sqrt{n-1} \cdot \langle \downarrow, n | \hat{S}_+ | \uparrow, n-2 \rangle \\
 &\quad + c_2^* c_1 \sqrt{n} \langle \uparrow, n-1 | \uparrow, n-1 \rangle + |c_2|^2 \sqrt{n-1} \langle \uparrow, n-1 | \hat{S}_+ | \uparrow, n-2 \rangle \Rightarrow \\
 \langle \hat{S}_+ \hat{a}_m \rangle_A &= c_2^*(t) c_1(t) \cdot \sqrt{n}
 \end{aligned}$$

$$\begin{aligned}
 \langle \hat{S}_- \hat{a}_m^\dagger \rangle_A &= \langle \psi_A(t) | \hat{S}_- \hat{a}_m^\dagger | \psi_A(t) \rangle = \{ c_1^* \langle \downarrow, n | + c_2^* \langle \uparrow, n-1 | \} \hat{S}_- \hat{a}_m^\dagger \{ c_1 | \downarrow, n \rangle + c_2 | \uparrow, n-1 \rangle \} \\
 &= |c_1|^2 \langle \downarrow, n | \hat{S}_- \hat{a}_m^\dagger | \downarrow, n \rangle + c_1^* c_2 \langle \downarrow, n | \hat{S}_- \hat{a}_m^\dagger | \uparrow, n-1 \rangle + \\
 &\quad + c_2^* c_1 \langle \uparrow, n-1 | \hat{S}_- \hat{a}_m^\dagger | \downarrow, n \rangle + |c_2|^2 \langle \uparrow, n-1 | \hat{S}_- \hat{a}_m^\dagger | \uparrow, n-1 \rangle = \\
 &= |c_1|^2 \sqrt{n+1} \langle \downarrow, n | \hat{S}_- | \downarrow, n+1 \rangle + c_1^* c_2 \sqrt{n} \langle \downarrow, n | \downarrow, n \rangle \\
 &\quad + c_2^* c_1 \langle \uparrow, n-1 | \hat{S}_- | \uparrow, n+1 \rangle \sqrt{n+1} + |c_2|^2 \langle \uparrow, n-1 | \downarrow, n \rangle \sqrt{n} \Rightarrow \\
 \langle \hat{S}_- \hat{a}_m^\dagger \rangle_A &= c_1^*(t) c_2(t) \sqrt{n}
 \end{aligned}$$

(E)

$$\begin{aligned}
 \langle \hat{a}_m^\dagger \hat{a}_m \rangle_E &= \langle \psi_E(t) | \hat{a}_m^\dagger \hat{a}_m | \psi_E(t) \rangle = \{ c_1^* \langle \downarrow, n+1 | + c_2^* \langle \uparrow, n | \} | \hat{a}_m^\dagger \hat{a}_m | \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \} \\
 &= |c_1|^2 \langle \downarrow, n+1 | \hat{a}_m^\dagger \hat{a}_m | \downarrow, n+1 \rangle + c_1^* c_2 \langle \downarrow, n+1 | \hat{a}_m^\dagger \hat{a}_m | \uparrow, n \rangle \\
 &\quad + c_2^* c_1 \langle \uparrow, n | \hat{a}_m^\dagger \hat{a}_m | \downarrow, n+1 \rangle + |c_2|^2 \langle \uparrow, n | \hat{a}_m^\dagger \hat{a}_m | \uparrow, n \rangle \\
 &= |c_1|^2 \sqrt{n+1} \sqrt{n+1} \langle \downarrow, n+1 | \downarrow, n+1 \rangle + c_1^* c_2 n \langle \downarrow, n+1 | \uparrow, n \rangle \\
 &\quad + c_2^* c_1 (n+1) \langle \uparrow, n | \downarrow, n+1 \rangle + |c_2|^2 n \langle \uparrow, n | \uparrow, n \rangle = \\
 &= |c_1|^2 (n+1) + n |c_2|^2 = n (|c_1|^2 + |c_2|^2) + |c_1|^2 \Rightarrow
 \end{aligned}$$

$$\langle \hat{a}_m^\dagger \hat{a}_m \rangle_E = n + |c_1(t)|^2 \quad (4.60)$$

$$\begin{aligned}
 \langle \hat{S}_+ \hat{S}_- \rangle_E &= \langle \psi_E(t) | \hat{S}_+ \hat{S}_- | \psi_E(t) \rangle = \{ c_1^* \langle \downarrow, n+1 | + c_2^* \langle \uparrow, n | \} \hat{S}_+ \hat{S}_- \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \} \\
 &= |c_1|^2 \cdot 0 + c_1^* c_2 \langle \downarrow, n+1 | \uparrow, n \rangle + c_2^* c_1 \cdot 0 + |c_2|^2 \langle \uparrow, n | \uparrow, n \rangle \Rightarrow
 \end{aligned}$$

$$\langle \hat{S}_+ \hat{S}_- \rangle_E = |c_2(t)|^2 \quad (4.60)$$

$$\begin{aligned}
 \langle \hat{S}_+ \hat{a}_m \rangle_E &= \langle \psi_E(t) | \hat{S}_+ \hat{a}_m | \psi_E(t) \rangle = \{ c_1^* \langle \downarrow, n+1 | + c_2^* \langle \uparrow, n | \} \hat{S}_+ \hat{a}_m \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \} \\
 &= |c_1|^2 \langle \downarrow, n+1 | \hat{S}_+ \hat{a}_m | \downarrow, n+1 \rangle + c_1^* c_2 \langle \downarrow, n+1 | \hat{S}_+ \hat{a}_m | \uparrow, n \rangle + \\
 &\quad - c_2^* c_1 \langle \uparrow, n | \hat{S}_+ \hat{a}_m | \downarrow, n+1 \rangle + |c_2|^2 \langle \uparrow, n | \hat{S}_+ \hat{a}_m | \uparrow, n \rangle \\
 &= |c_1|^2 \langle \downarrow, n+1 | \uparrow, n \rangle \sqrt{n+1} + c_1^* c_2 \cdot 0 + c_2^* c_1 \langle \uparrow, n | \uparrow, n \rangle \sqrt{n+1} + |c_2|^2 \cdot 0
 \end{aligned}$$

$$\langle \hat{S}_+ \hat{a}_m \rangle_E = c_2^*(t) c_1(t) \cdot \sqrt{n+1}$$

$$\begin{aligned}
 \langle \hat{S}_- \hat{a}_m^\dagger \rangle_E &= \langle \psi_E(t) | \hat{S}_- \hat{a}_m^\dagger | \psi_E(t) \rangle = \{ c_1^* \langle \downarrow, n+1 | + c_2^* \langle \uparrow, n | \} \hat{S}_- \hat{a}_m^\dagger \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \} \\
 &= |c_1|^2 \langle \downarrow, n+1 | \hat{S}_- \hat{a}_m^\dagger | \downarrow, n+1 \rangle + c_1^* c_2 \langle \downarrow, n+1 | \hat{S}_- \hat{a}_m^\dagger | \uparrow, n \rangle \\
 &\quad + c_2^* c_1 \langle \uparrow, n | \hat{S}_- \hat{a}_m^\dagger | \downarrow, n+1 \rangle + |c_2|^2 \langle \uparrow, n | \hat{S}_- \hat{a}_m^\dagger | \uparrow, n \rangle = \\
 &= |c_1|^2 \langle \downarrow, n+1 | \uparrow, n+2 \rangle \sqrt{n+2} + c_1^* c_2 \langle \downarrow, n+1 | \downarrow, n+1 \rangle \sqrt{n+1} \\
 &\quad + c_2^* c_1 \cdot 0 + |c_2|^2 \langle \uparrow, n | \downarrow, n+1 \rangle \sqrt{n+1} \Rightarrow
 \end{aligned}$$

$$\langle \hat{S}_- \hat{a}_m^\dagger \rangle_E = c_1^*(t) c_2(t) \cdot \sqrt{n+1}$$

$$\text{APA} \quad \langle \hat{a}_m^\dagger \hat{a}_m \rangle_E + \langle \hat{S}_+ \hat{S}_- \rangle_E = n+1 \quad (4.61)$$