

$$E_m = \frac{M_m \omega_m^2}{2} q_m^2 + \frac{M_m}{2} \dot{q}_m^2 = \frac{M_m \omega_m^2}{2} q_m^2 + \frac{P_m^2}{2 M_m}$$

Ενέργεια
m τρόπου

q_m γεν. θέση
 \dot{q}_m γεν. ταχύτητα
 P_m γεν. ορμή

$$\hat{H}_{HM,m} = \frac{M_m \omega_m^2}{2} \hat{q}_m^2 + \frac{\hat{P}_m^2}{2 M_m}$$

Χαμιλτονιανή m τρόπου

$$E_{m,n_m} = \hbar \omega_m \left(n_m + \frac{1}{2} \right)$$

1 ιδιοενέργεια m τρόπου

$m \in \mathbb{N}^*$

$n_m \in \mathbb{N}$

$$\hat{q}_m = q_m$$

$$\hat{P}_m = -i \hbar \frac{\partial}{\partial q_m}$$

$$[\hat{q}_m, \hat{P}_m] = i \hbar$$

Εισάγουμε τους μετασχηματισμούς

$$\hat{a}_m = \frac{1}{\sqrt{2 M_m \hbar \omega_m}} (M_m \omega_m \hat{q}_m + i \hat{P}_m)$$

"καταστροφής"

$$\hat{a}_m^\dagger = \frac{1}{\sqrt{2 M_m \hbar \omega_m}} (M_m \omega_m \hat{q}_m - i \hat{P}_m)$$

"δημιουργίας"

1 ιδιότητα

$$[\hat{a}_m, \hat{a}_m^\dagger] = \hat{a}_m \hat{a}_m^\dagger - \hat{a}_m^\dagger \hat{a}_m = 1$$

Από

$$[\hat{a}, \hat{a}^\dagger] = \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} = \frac{1}{2 M \hbar \omega} (M \omega \hat{q} + i \hat{p})(M \omega \hat{q} - i \hat{p}) - \frac{1}{2 M \hbar \omega} (M \omega \hat{q} - i \hat{p})(M \omega \hat{q} + i \hat{p}) =$$

$$= \frac{1}{2 M \hbar \omega} \left(\cancel{M^2 \omega^2 \hat{q}^2} + \hat{p}^2 - M \omega \hat{q} i \hat{p} + i \hat{p} M \omega \hat{q} - \cancel{M^2 \omega^2 \hat{q}^2} - \hat{p}^2 + M \omega \hat{q} i \hat{p} - i \hat{p} M \omega \hat{q} \right)$$

$$= \frac{1}{2M\hbar\omega} (-2M\omega i \hat{q}\hat{p} + 2i\hat{p}M\omega\hat{q}) = \frac{1}{\hbar} (-i\hat{q}\hat{p} + i\hat{p}\hat{q})$$

$$= \frac{1}{\hbar} (-i)(\hat{q}\hat{p} - \hat{p}\hat{q}) = \frac{-i}{\hbar} [\hat{q}, \hat{p}] = \frac{-i}{\hbar} i\hbar = 1 \Rightarrow [\hat{a}, \hat{a}^\dagger] = 1$$

$$\hat{a}_m^\dagger + \hat{a}_m = \frac{1}{\sqrt{2M\omega\hbar}} 2M\omega\hat{q}_m = \sqrt{\frac{2M\omega\hbar}{\hbar}} \hat{q}_m \Rightarrow$$

$$\hat{q}_m = \sqrt{\frac{\hbar}{2M\omega}} (\hat{a}_m^\dagger + \hat{a}_m)$$

$$\hat{a}_m^\dagger - \hat{a}_m = \frac{1}{\sqrt{2M\omega\hbar}} (-2i)\hat{p}_m = (-i)\sqrt{\frac{2}{M\omega\hbar}} \hat{p}_m \Rightarrow$$

$$\hat{p}_m = i\sqrt{\frac{M\omega\hbar}{2}} (\hat{a}_m^\dagger - \hat{a}_m)$$

APA

$$\hat{H}_{HM,m} = \frac{M\omega^2}{2} \hat{q}_m^2 + \frac{\hat{p}_m^2}{2M} = \frac{M\omega^2}{2} \frac{\hbar}{2M\omega} (\hat{a}_m^\dagger + \hat{a}_m)^2 + \frac{1}{2M} \frac{\hbar}{2M\omega} (\hat{a}_m^\dagger - \hat{a}_m)^2$$

$$= \frac{\hbar\omega}{4} (\hat{a}_m^\dagger \hat{a}_m + \hat{a}_m^\dagger \hat{a}_m^\dagger + \hat{a}_m \hat{a}_m + \hat{a}_m \hat{a}_m^\dagger - \hat{a}_m^\dagger \hat{a}_m^\dagger - \hat{a}_m^\dagger \hat{a}_m - \hat{a}_m \hat{a}_m^\dagger + \hat{a}_m \hat{a}_m)$$

$$\hat{H}_{HM,m} = \frac{\hbar\omega}{4} (2\hat{a}_m^\dagger \hat{a}_m + 2\hat{a}_m \hat{a}_m^\dagger) = \frac{\hbar\omega}{2} (\hat{a}_m^\dagger \hat{a}_m + \hat{a}_m \hat{a}_m^\dagger)$$

Alle $[\hat{a}_m, \hat{a}_m^\dagger] = \hat{a}_m \hat{a}_m^\dagger - \hat{a}_m^\dagger \hat{a}_m = 1 \Rightarrow \hat{a}_m \hat{a}_m^\dagger = 1 + \hat{a}_m^\dagger \hat{a}_m$

$$\hat{H}_{HM,m} = \frac{\hbar\omega}{2} (2\hat{a}_m^\dagger \hat{a}_m + 1) \Rightarrow \hat{H}_{HM,m} = \hbar\omega \left(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2} \right)$$

ergibt $E_{n,m} = \hbar\omega \left(n_m + \frac{1}{2} \right)$

Αν θεωρήσουμε $|n_m\rangle$ την κατάσταση του ΗΜ πεδίου

με n_m αριθμό φωτονίων στον ΗΜ τρόπο m



$$\hat{H}_{HM,m} |n_m\rangle = E_{m,n_m} |n_m\rangle$$

$$\hbar\omega_m \left(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2} \right) |n_m\rangle = \hbar\omega_m \left(n_m + \frac{1}{2} \right) |n_m\rangle$$

$$\hat{a}_m^\dagger \hat{a}_m |n_m\rangle = n_m |n_m\rangle$$

Άρα ο τελεστής $\hat{N}_m = \hat{a}_m^\dagger \hat{a}_m$ μετρά τον αριθμό των φωτονίων στον ΗΜ τρόπο m .

$$[A+B, C] = [A, C] + [B, C]$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$\hat{H} = \kappa \hat{q}^2 + \lambda \hat{p}^2$$

$$\hat{a} = \mu \hat{q} + \nu \hat{p}$$

$$\hat{a}^\dagger = \gamma \hat{q} - \nu \hat{p}$$

Με τη βοήθεια των παραπάνω σχέσεων μπορεί να αποδειχτεί ότι

$$[\hat{H}, \hat{a}] = -\hbar\omega \hat{a}$$

$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega \hat{a}^\dagger$$

| | |
|--|---|
| $\hat{H} \hat{a} n\rangle - \hat{a} \hat{H} n\rangle = -\hbar\omega \hat{a} n\rangle$ $\hat{H} \hat{a} n\rangle - E_n \hat{a} n\rangle = -\hbar\omega \hat{a} n\rangle$ $\hat{H} \hat{a} n\rangle = (E_n - \hbar\omega) \hat{a} n\rangle$ <p>↓</p> <p>Ιδιοκατάσταση με ενέργεια κατεβαγμένη κατά $\hbar\omega$ (ένα φωτόνιο λιγότερο)</p> $\Rightarrow \hat{a} n\rangle = \xi n-1\rangle$ | $\hat{H} \hat{a}^\dagger n\rangle - \hat{a}^\dagger \hat{H} n\rangle = \hbar\omega \hat{a}^\dagger n\rangle$ $\hat{H} \hat{a}^\dagger n\rangle - E_n \hat{a}^\dagger n\rangle = \hbar\omega \hat{a}^\dagger n\rangle$ $\hat{H} \hat{a}^\dagger n\rangle = (E_n + \hbar\omega) \hat{a}^\dagger n\rangle$ <p>↓</p> <p>Ιδιοκατάσταση με ενέργεια ανεβασμένη κατά $\hbar\omega$ (ένα φωτόνιο περισσότερο)</p> $\Rightarrow \hat{a}^\dagger n\rangle = \rho n+1\rangle$ |
|--|---|

... πράξεις

$$[\hat{H}, \hat{a}] = -\hbar\omega \hat{a}$$

$$[\hat{H}, \hat{a}] |n\rangle = -\hbar\omega \hat{a} |n\rangle$$

$$\hat{H} \hat{a} |n\rangle - \hat{a} \hat{H} |n\rangle = -\hbar\omega \hat{a} |n\rangle$$

$$\hat{H} \hat{a} |n\rangle - \hat{a} E_n |n\rangle = -\hbar\omega \hat{a} |n\rangle$$

$$\hat{H} \hat{a} |n\rangle = (E_n - \hbar\omega) \hat{a} |n\rangle$$



? ιδιοκατάσταση με ενέργεια
κατεβαμένη κατά $\hbar\omega$
(ένα quantum λιγότερο)

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$$

$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega \hat{a}^\dagger$$

$$[\hat{H}, \hat{a}^\dagger] |n\rangle = \hbar\omega \hat{a}^\dagger |n\rangle$$

$$\hat{H} \hat{a}^\dagger |n\rangle - \hat{a}^\dagger \hat{H} |n\rangle = \hbar\omega \hat{a}^\dagger |n\rangle$$

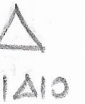
$$\hat{H} \hat{a}^\dagger |n\rangle - \hat{a}^\dagger E_n |n\rangle = \hbar\omega \hat{a}^\dagger |n\rangle$$

$$\hat{H} \hat{a}^\dagger |n\rangle = (E_n + \hbar\omega) \hat{a}^\dagger |n\rangle$$



ιδιοκατάσταση με ενέργεια
άνεβαμένη κατά $\hbar\omega$
(ένα quantum περισσότερο)

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$



$$\begin{aligned}
 [\hat{H}, \hat{a}] &= [\hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}), \hat{a}] = \hbar\omega [\hat{a}^\dagger\hat{a} + \frac{1}{2}, \hat{a}] = \\
 &= \hbar\omega \left([\hat{a}^\dagger\hat{a}, \hat{a}] + [\frac{1}{2}, \hat{a}] \right) = \hbar\omega [\hat{a}^\dagger\hat{a}, \hat{a}] = \\
 &= \hbar\omega \left(\underset{-1}{[\hat{a}^\dagger, \hat{a}]\hat{a}} + \underset{0}{\hat{a}^\dagger[\hat{a}, \hat{a}]} \right) = -\hbar\omega\hat{a}
 \end{aligned}$$

$$\begin{aligned}
 [\hat{H}, \hat{a}^\dagger] &= [\hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}), \hat{a}^\dagger] = \hbar\omega [\hat{a}^\dagger\hat{a} + \frac{1}{2}, \hat{a}^\dagger] = \\
 &= \hbar\omega \left([\hat{a}^\dagger\hat{a}, \hat{a}^\dagger] + [\frac{1}{2}, \hat{a}^\dagger] \right) = \hbar\omega [\hat{a}^\dagger\hat{a}, \hat{a}^\dagger] \\
 &= \hbar\omega \left(\hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a} \right) = \hbar\omega\hat{a}^\dagger
 \end{aligned}$$



(1) The lowest energy state is the ground state. It is the state with the lowest energy and is the most stable. It is the state with the lowest energy and is the most stable. It is the state with the lowest energy and is the most stable.

(2) The energy levels are discrete. They are separated by a fixed amount of energy. This is called quantization of energy.

(3) The energy levels are equally spaced. The difference between adjacent energy levels is constant.

(4) The energy levels are symmetric about the zero energy level.



$$\hat{a}|n\rangle = \xi|n-1\rangle \quad \left\{ \begin{array}{l} \Rightarrow \langle n|a^\dagger a|n\rangle = |\xi|^2 \langle n-1|n-1\rangle \Rightarrow \\ \langle n|\hat{a}^\dagger = \xi^* \langle n-1| \end{array} \right. \quad n \langle n|n\rangle = |\xi|^2 \langle n-1|n-1\rangle \Rightarrow |\xi|^2 = n \Rightarrow \overset{\text{p.x.}}{\xi} = \sqrt{n}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

$$[a, a^\dagger] = 1 \Rightarrow aa^\dagger = 1 + a^\dagger a$$

$$\hat{a}^\dagger|n\rangle = \rho|n+1\rangle \quad \left\{ \begin{array}{l} \Rightarrow \langle n|aa^\dagger|n\rangle = |\rho|^2 \langle n+1|n+1\rangle \\ \langle n|a = \rho^* \langle n+1| \end{array} \right. \quad \langle n|(1 + a^\dagger a)|n\rangle = |\rho|^2 \langle n+1|n+1\rangle$$

$$(1+n) = |\rho|^2 \quad \Rightarrow \text{p.x. } \rho = \sqrt{n+1}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$\hat{a}_m^\dagger |n_m\rangle = \sqrt{n_m+1} |n_m+1\rangle$$

$$\hat{a}_m |n_m\rangle = \sqrt{n_m} |n_m-1\rangle$$

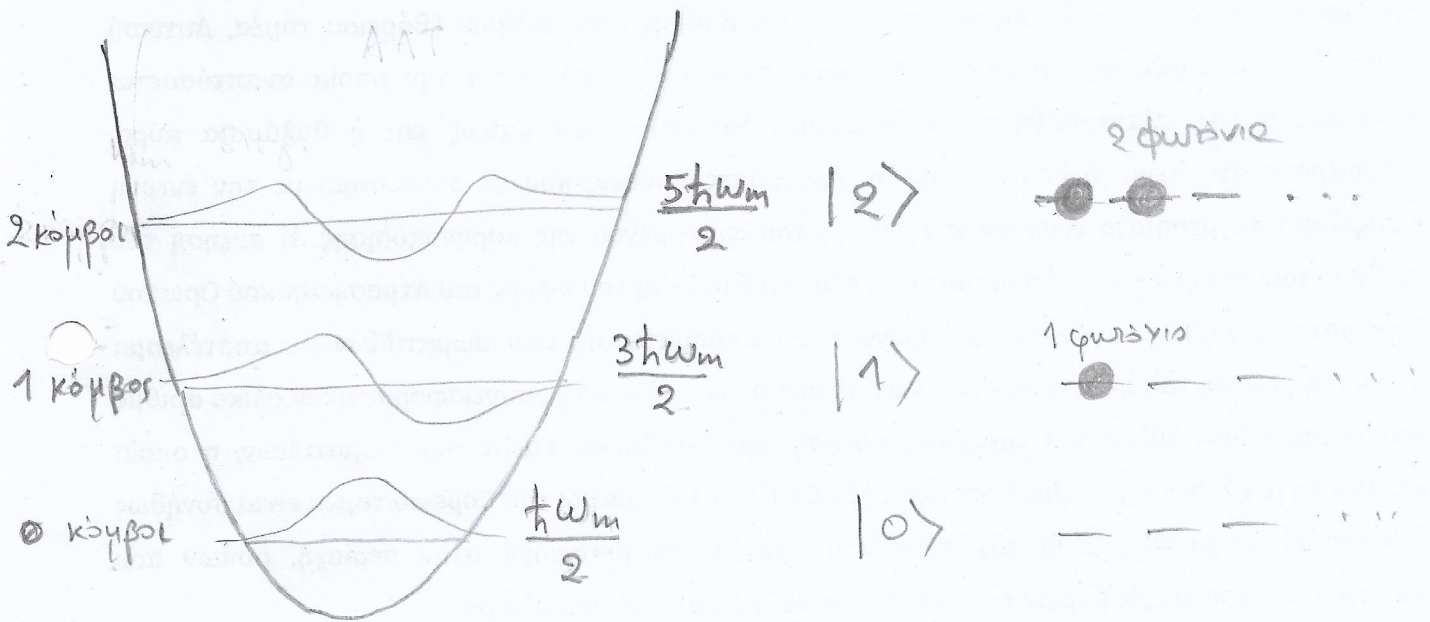
$$\hat{a}_m |0\rangle = 0$$

$$\langle n_m | l_m \rangle = \delta_{nl}$$

Η θεμελιώδης κατάσταση με ιδιοenergeia $\frac{\hbar\omega_m}{2}$ αντιστοιχεί στο κενό κανένα φωτόνιο (φωτόνιο)

Η 1η διεγερμένη κατάσταση με ιδιοenergeia $\frac{3\hbar\omega_m}{2}$ αντιστοιχεί σε 1 φωτόνιο (φωτόνιο)

Η 2η διεγερμένη κατάσταση με ιδιοenergeia $\frac{5\hbar\omega_m}{2}$ αντιστοιχεί σε 2 φωτόνια (φωτόνια)



φωτονίων = # κόμβων της ιδιοσυναρτήσεως του AAT

Με τη βοήθεια των τελεστών καταστροφής και δημιουργίας
μπορούμε να γράψουμε

$$\hat{E}_x^m(z,t) = \left(\frac{\hbar \omega_m}{\epsilon V} \right)^{1/2} \sin\left(\frac{m\pi z}{L}\right) (\hat{a}_m^\dagger + \hat{a}_m)$$

$$\hat{B}_y^m(z,t) = \frac{i}{c} \left(\frac{\hbar \omega_m}{\epsilon V} \right)^{1/2} \cos\left(\frac{m\pi z}{L}\right) (\hat{a}_m^\dagger - \hat{a}_m)$$

ΣΧΕΣΕΙΣ ΜΕΤΑΘΕΣΕΩΣ ΜΠΟΣΟΝΙΩΝ

$$[\hat{a}_m, \hat{a}_e] = 0$$

$$[\hat{a}_m^\dagger, \hat{a}_e^\dagger] = 0$$

$$[\hat{a}_m, \hat{a}_e^\dagger] = \delta_{me}$$

