Math Preliminaries

2023-2024 AIMPN

Computational Neuroscience

Emmanouil D. Protonotarios

Special Teaching Staff

Department of History & Philosophy of Science

National & Kapodistrian University of Athens

(based on material by V. Cutsuridis, I. Dellis)

Outline

- **Linear algebra**
  - **Matrices**
  - **Vectors**
- **Calculus**
  - **Limits**
  - **Derivatives**
  - **Sums**
  - **Integrals**
- **Differential equations**
  - **Numerical analysis**

Outline

- **Linear algebra**
  - Matrices
  - Vectors
- **Calculus**
  - Limits
  - Derivatives
  - Sums
  - Integrals
- **Differential equations**
  - Numerical analysis

Matrices

A matrix is a rectangular array of numbers. Example:

Column 1

↓

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Matrices

A matrix is a rectangular array of numbers. Example:

Column 2 ↓

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Matrices

A matrix is a rectangular array of numbers. Example:

Column n ↓

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Matrices

A matrix is a rectangular array of numbers. Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \longleftarrow \text{Row 1}$$

Matrices

A matrix is a rectangular array of numbers. Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

← Row 2

Matrices

A matrix is a rectangular array of numbers. Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

← Row m

Matrices - Presentation

A matrix is a rectangular array of numbers. Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Elements or entries of the matrix: a_{ij} , A_{ij} , A[i , j].

Note: Rows first, then columns.

Dimensions of the matrix: number of rows m, number of columns n.

Rows: horizontal lines of the matrix.

Columns: vertical lines of the matrix.

Matrices – Simple example

Let's see a simple example of a matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

A 4 × 3 (four by three) matrix – a matrix with 4 rows and 3 columns

a_{31} = ?, a_{13} = ?, a_{42} = ?

a_{31} = 7

a_{13} = 3

a_{42} = 11

Matrices – Special matrices

Row matrix (there is only one row)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

1 × *n* matrix

Matrices – Special matrices

Row matrix (there is only one row)

example:

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

1 × 3 matrix

Matrices – Special matrices

Column matrix (there is only one column)

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix}$$

m **× 1 matrix**

Matrices – Special matrices

Column matrix (there is only one column)

example:

$$A = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

3 × 1 matrix

Matrices – Special matrices

Square matrix (same number of rows and columns)

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$

$m \times m$ matrix

Matrices – Special matrices

Square matrix (same number of rows and columns)

example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

3 × 3 matrix

main diagonal of a square matrix:

top-left corner to bottom-right corner

Matrices – Special matrices

Diagonal matrix (all elements apart from the main diagonal are equal to zero)

example 1:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

3 × 3 matrix

Matrices – Special matrices

Diagonal matrix (all elements apart from the main diagonal are equal to zero)

example 2:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 × 3 matrix

Matrices – Special matrices

Identity matrix (a square diagonal matrix with all elements in main diagonal equal to 1)

example 1:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

example 2:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The $n \times n$ identity matrix is denoted by I_n

Matrices – Trace

Trace of a matrix: the sum of all elements on the main diagonal of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$

$$Tr(A) = a_{11} + a_{22} + \dots + a_{mm}$$

example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$Tr(A) = 1 + 5 + 9 = 15$$

Matrices – Operation between scalar and matrix

Matrix scalar multiplication (scalar: βαθμωτό μέγεθος)

Given a m × n matrix $A = (a_{ij})_{m\times n}$ and a scalar λ, the scalar multiplication λA is:

$$\lambda A = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda a_{m1} & \lambda a_{m2} & \dots & \lambda a_{mn} \end{bmatrix}$$

example:

$$2\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$$

Matrices – Operations between matrices

Matrix addition

Given two m × n matrices A = $(a_{ij})_{m\times n}$ and B = $(b_{ij})_{m\times n}$, their sum A + B is also an *m × n* matrix computed by adding the corresponding elements.

$$A + B = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$$

In full form:

$$A+B=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

If $A = (a_{ij})_{m \times k}$ and $B = (b_{ij})_{k \times n}$ then AB is an $m \times n$ matrix with elements given by:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & \\ & \\ & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ & \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

If $A = (a_{ij})_{m\times k}$ and $B = (b_{ij})_{k\times n}$ then AB is an $m \times n$ matrix with elements given by:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

If $A = (a_{ij})_{m \times k}$ and $B = (b_{ij})_{k \times n}$ then AB is an $m \times n$ matrix with elements given by:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

If $A = (a_{ij})_{m\times k}$ and $B = (b_{ij})_{k\times n}$ then AB is an $m \times n$ matrix with elements given by:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

If $A = (a_{ij})_{m\times k}$ and $B = (b_{ij})_{k\times n}$ then AB is an $m \times n$ matrix with elements given by:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

If $A = (a_{ij})_{m \times k}$ and $B = (b_{ij})_{k \times n}$ then AB is an $m \times n$ matrix with elements given by:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

If $A = (a_{ij})_{m \times k}$ and $B = (b_{ij})_{k \times n}$ then AB is an $m \times n$ matrix with elements given by:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

If $A = (a_{ij})_{m\times k}$ and $B = (b_{ij})_{k\times n}$ then AB is an $m \times n$ matrix with elements given by:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

If $A = (a_{ij})_{m \times k}$ and $B = (b_{ij})_{k \times n}$ then AB is an $m \times n$ matrix with elements given by:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 3 + 3\cdot 5 & 1\cdot 2 + 2\cdot 4 + 3\cdot 6 \\ 4\cdot 1 + 5\cdot 3 + 6\cdot 5 & 4\cdot 2 + 5\cdot 4 + 6\cdot 6 \\ 7\cdot 1 + 8\cdot 3 + 9\cdot 5 & 7\cdot 2 + 8\cdot 4 + 9\cdot 6 \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+2\cdot 3+3\cdot 5 & 1\cdot 2+2\cdot 4+3\cdot 6 \\ 4\cdot 1+5\cdot 3+6\cdot 5 & 4\cdot 2+5\cdot 4+6\cdot 6 \\ 7\cdot 1+8\cdot 3+9\cdot 5 & 7\cdot 2+8\cdot 4+9\cdot 6 \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+2\cdot 3+3\cdot 5 & 1\cdot 2+2\cdot 4+3\cdot 6 \\ 4\cdot 1+5\cdot 3+6\cdot 5 & 4\cdot 2+5\cdot 4+6\cdot 6 \\ 7\cdot 1+8\cdot 3+9\cdot 5 & 7\cdot 2+8\cdot 4+9\cdot 6 \end{bmatrix}$$

Matrices – Operations between matrices

Matrix multiplication

example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 & 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 6 \\ 4 \cdot 1 + 5 \cdot 3 + 6 \cdot 5 & 4 \cdot 2 + 5 \cdot 4 + 6 \cdot 6 \\ 7 \cdot 1 + 8 \cdot 3 + 9 \cdot 5 & 7 \cdot 2 + 8 \cdot 4 + 9 \cdot 6 \end{bmatrix} = \begin{bmatrix} 22 & 28 \\ 49 & 64 \\ 76 & 100 \end{bmatrix}$$

Matrices – Transpose Matrix

Transpose Matrix: rows become columns and columns become rows

notation: the transpose of *A* is denoted as A^T

it is the "reflection" of *A* by its main diagonal

dimensions: if *A* is an $m \times n$ matrix, then A^T is an $n \times m$ matrix

elements:

$$A^T_{ji} = A_{ij} \quad \text{for } 1 \leq i \leq n,\ 1 \leq j \leq m$$

example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

Matrices – Transpose Matrix

Properties:

$$(A^{\mathsf{T}})^{\mathsf{T}} = A$$

$$(A + B)^{\mathsf{T}} = A^{\mathsf{T}} + B^{\mathsf{T}}$$

$$(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$$

$$(cA)^{\mathsf{T}} = cA^{\mathsf{T}}$$

Matrices – Symmetric Matrix

Symmetric Matrix: a matrix which is equal to its transpose

$$A^{\mathsf{T}} = A$$

For a symmetric matrix $a_{ij} = a_{ji}$

example

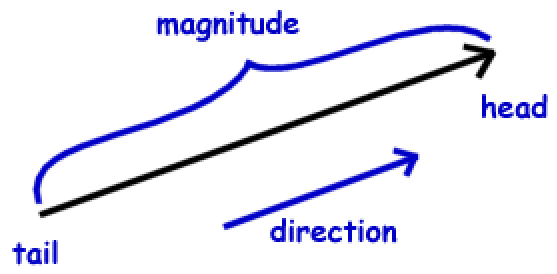
$$\begin{bmatrix} 1 & 10 & 20 \\ 10 & 2 & 10 \\ 20 & 10 & 3 \end{bmatrix}$$

Vectors

Vector: A vector is a matrix that one of its dimensions is equal to one.

Vectors are commonly used to represent entities with a <u>magnitude</u> and a <u>direction</u> that can be added to each other and multiplied with scalars.

Such entities are for example forces, velocity and acceleration.



Vectors

Vector: A vector is a matrix that one of its dimensions equals to one.

column vector: an $m \times 1$ matrix (one column and m rows) **(same as column matrix)**

row vector: an $1 \times n$ matrix (one row and n columns) **(same as row matrix)**

example 1: a 3-element row vector, or an 1×3 matrix

$$\mathbf{a} = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \qquad \mathbf{a} = [1, 3, 4]$$

example 2: a 3-element column vector, or a 3×1 matrix

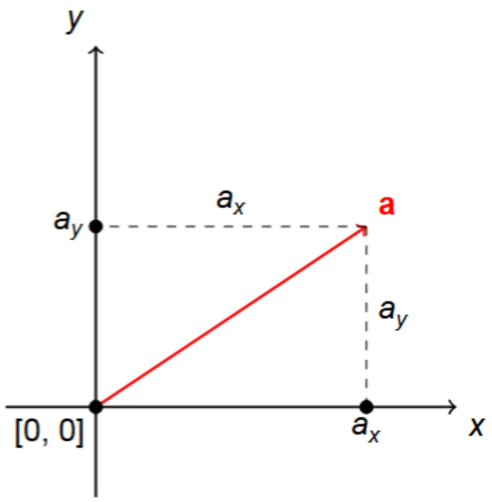
$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Vectors – Geometric representation

Vector: An *n*-dimensional vector (a vector with *n* entries) vector can be represented as a directed line in a *n*-dimensional space.

example 1: a 2-element vector (2 dimensions)

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

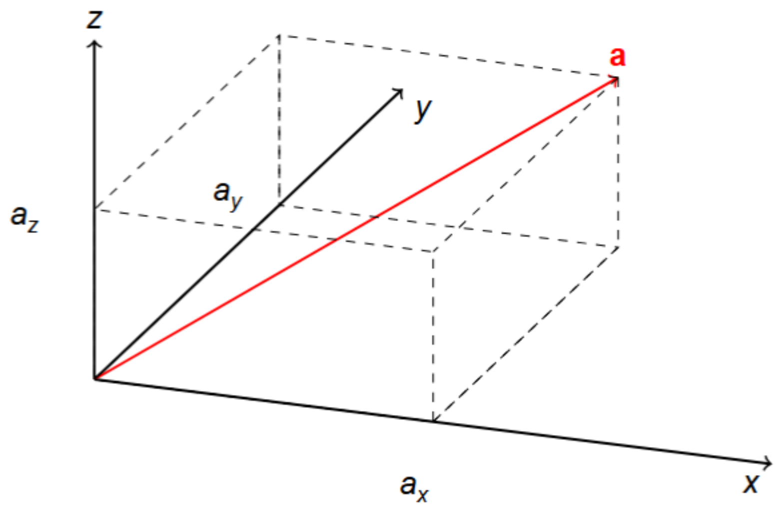


Vectors – Geometric representation

Vector: An *n*-dimensional vector (a vector with *n* entries) vector can be represented as a directed line in a *n*-dimensional space.

example 2: a 3-element vector (3 dimensions)

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$



Vectors – Magnitude (norm)

Vector magnitude: The length of the vector (L^2 norm, Euclidean norm)

general expression:

$$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \sqrt{\sum_{i=1}^{n} a_i^2}$$ (summation notation)

notation: $||\mathbf{a}|| = ||\mathbf{a}||_2 = |\mathbf{a}|$

norm 1: $||\mathbf{a}||_1 = |a_1| + |a_2| + \ldots + |a_n|$

norm 3: $||\mathbf{a}||_3 = \sqrt[3]{a_1^3 + a_2^3 + \cdots + a_n^3} = \sqrt[3]{\sum_{i=1}^{n} a_i^3}$

norm (general): $||\mathbf{a}||_p = \sqrt[p]{a_1^p + a_2^p + \cdots + a_n^p} = \sqrt[p]{\sum_{i=1}^{n} a_i^p}, p \in \mathbb{N}_+$

Vectors – Magnitude

Vector magnitude: The length of the vector (norm L2, Euclidean norm)

general expression:

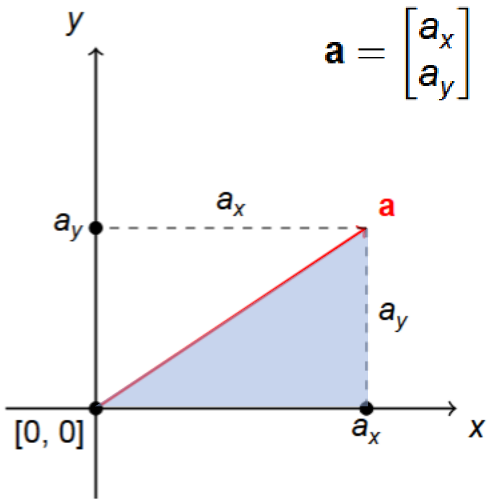
$$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \sqrt{\sum_{i=1}^{n} a_i^2}$$

(summation notation)

example 1: a 2-element vector (2 dimensions)

$$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2} = \sqrt{a_x^2 + a_y^2}$$

Pythagorean theorem



$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

Vectors – Magnitude

Vector magnitude: The length of the vector.

general expression:

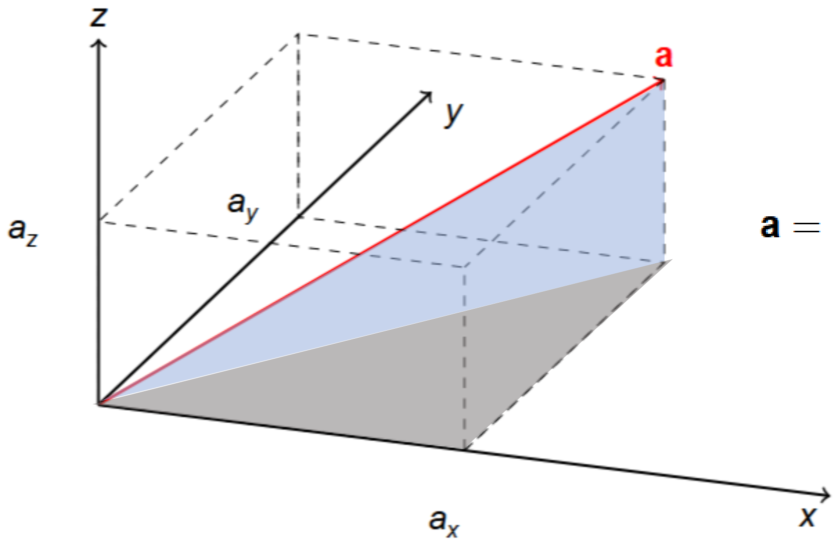
$$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \sqrt{\sum_{i=1}^{n} a_i^2}$$

(summation notation)

example 2: a 3-element vector (3 dimensions)

$$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Pythagorean theorem

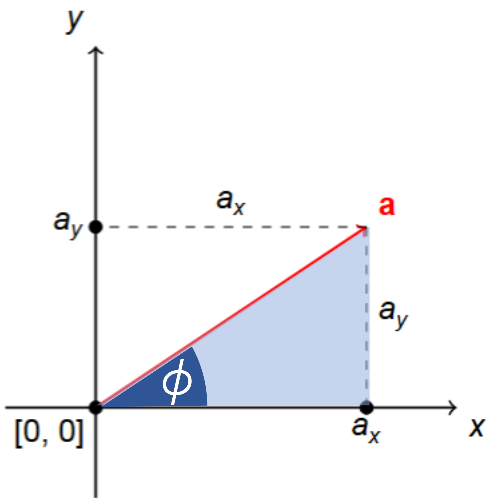


$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

Vectors – Description

Vector description: We need to define the magnitude and the direction

example in two dimensions:



$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

magnitude:

$$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2} = \sqrt{a_x^2 + a_y^2}$$

direction:

determined by the angle ϕ between the vector and the horizontal axis *x*

$$\phi = \arctan \frac{a_y}{a_x}$$

Vectors – Multiplication by scalar

Vector multiplication by scalar: **all elements are multiplied by the same scalar**

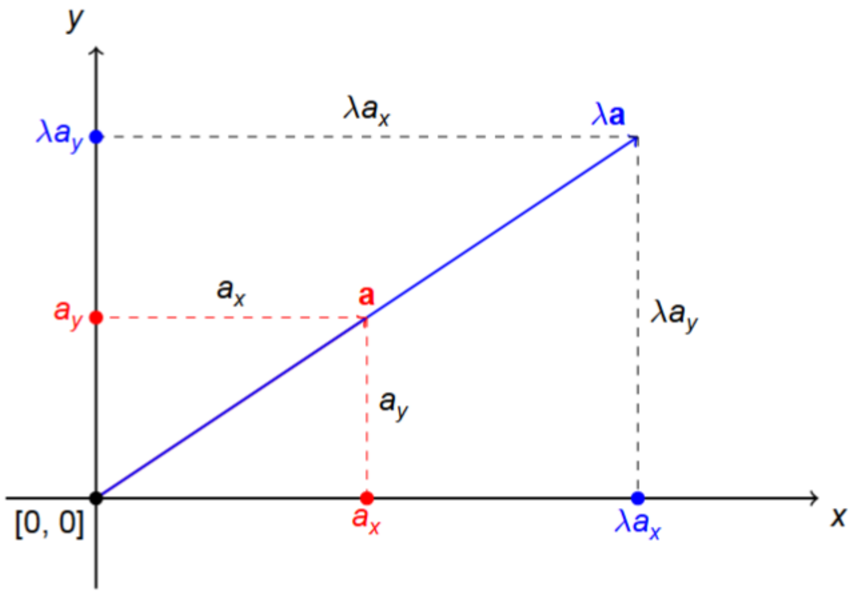
magnitude gets scaled, and direction remains the same (or reverses if λ < 0)

$$\lambda[a_1, a_2, ..., a_n] = [\lambda a_1, \lambda a_2, ..., \lambda a_n]$$

example in 2 dimensions:

$$\lambda[a_x, a_y] = [\lambda a_x, \lambda a_y]$$

$$||\lambda \mathbf{a}|| = \sqrt{(\lambda a_x)^2 + (\lambda a_y)^2} = |\lambda| \cdot ||\mathbf{a}||$$



Vectors – Multiplication by scalar

Vector multiplication by scalar: **all elements are multiplied by the same scalar**

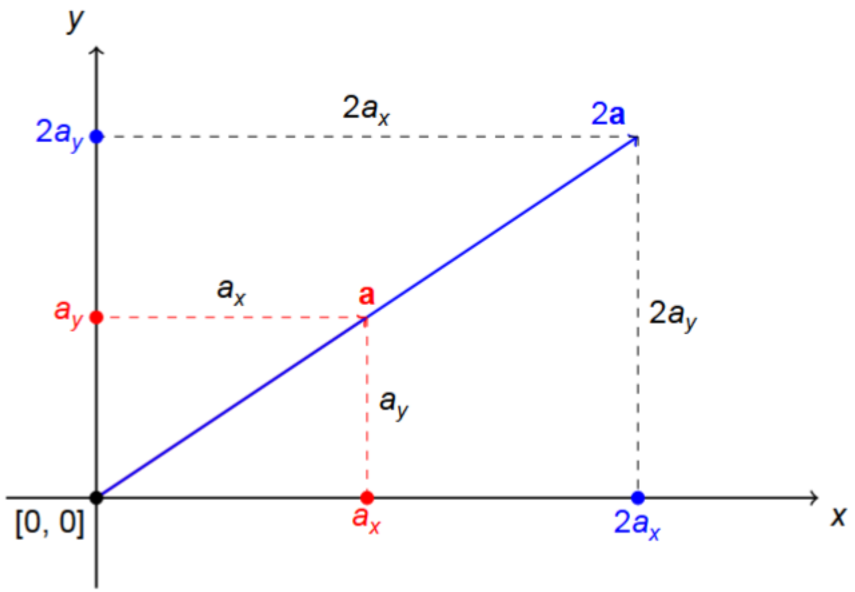
magnitude gets scaled, and direction remains the same (or reverses if *λ* < 0)

$$\lambda[a_1, a_2, ..., a_n] = [\lambda a_1, \lambda a_2, ..., \lambda a_n]$$

example in 2 dimensions:

$$2[a_x, a_y] = [2a_x, 2a_y]$$

$$||2\mathbf{a}|| = \sqrt{(2a_x)^2 + (2a_y)^2} = 2||\mathbf{a}||$$



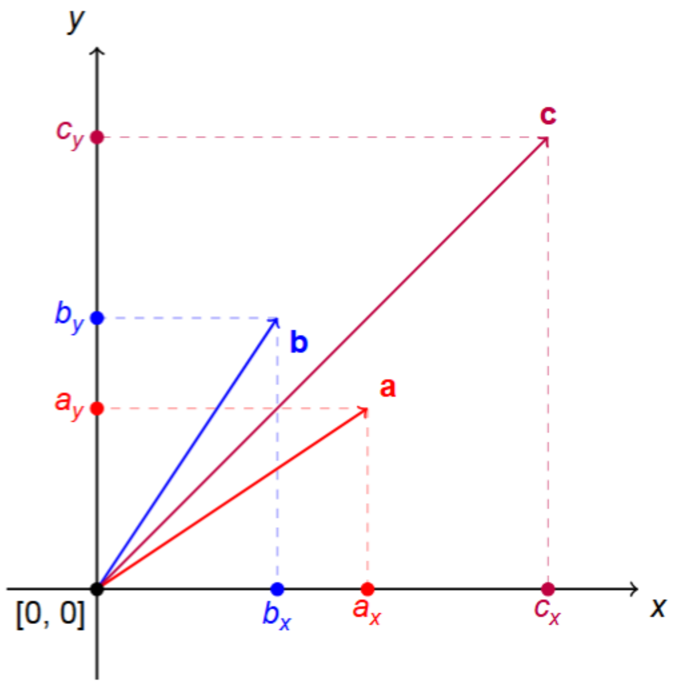
Vectors – Addition

Vector addition: a new vector with elements the sum of the corresponding elements

$$\mathbf{a} + \mathbf{b} = [a_1, a_2, ..., a_n] + [b_1, b_2, ..., b_n] = [a_1 + b_1, a_2 + b_2, ..., a_n + b_n]$$

example: in two dimensions

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = [a_x, a_y] + [b_x, b_y] = [a_x + b_x, a_y + b_y]$$



Vectors – Dot product (εσωτερικό γινόμενο)

Vector dot product: Definition (the result is a scalar)

(summation notation)

$$\mathbf{a} \cdot \mathbf{b} = [a_1, a_2, ..., a_n] \cdot [b_1, b_2, ..., b_n] = a_1 \cdot b_1 + a_2 \cdot b_2 + ... + a_n \cdot b_n = \sum_{i=1}^{n} a_i b_i$$

Matrix notation:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathsf{T}}\mathbf{b} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 \cdot b_1 + a_2 \cdot b_2 + ... + a_n \cdot b_n$$

Vectors – Dot product

example: calculate the dot product of the vectors [1   3   5] and [2   4   3]

$$\begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 & 3 \end{bmatrix} = 1 \cdot 2 + 3 \cdot 4 + 5 \cdot 3 = 2 + 12 + 15 = 29$$

in matrix notation:

$$\begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = 1 \cdot 2 + 3 \cdot 4 + 5 \cdot 3 = 2 + 12 + 15 = 29$$

Vectors – Dot product – Geometric representation

example: dot product of two vectors in 2D Euclidean space

$$\mathbf{a} \cdot \mathbf{b} = a_x \cdot b_x + a_y \cdot b_y = ||\mathbf{a}|| \cdot \underbrace{||\mathbf{b}|| \cdot \cos\theta}_{\text{(projection of } \mathbf{b} \text{ on } \mathbf{a}\text{)}}$$

(projection of **b** on **a**)

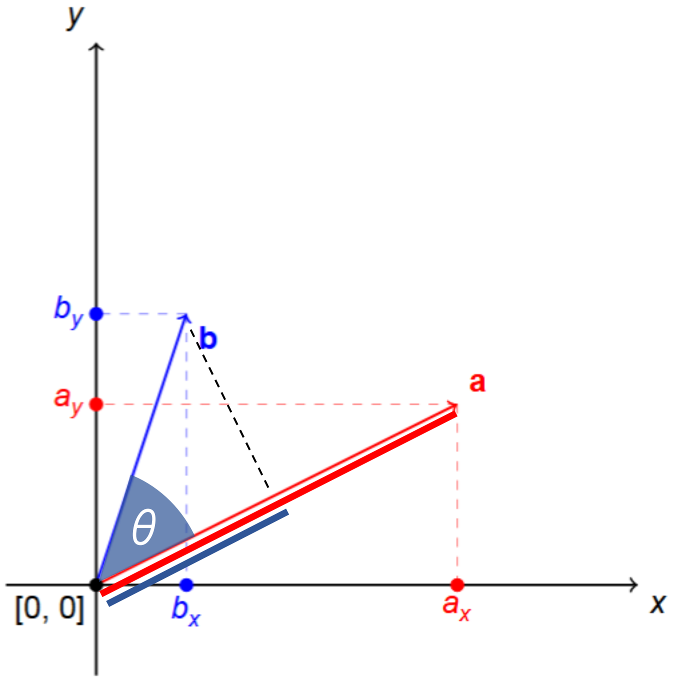
It depends on:

a) the magnitudes of the two vectors

b) the angle between the two vectors

when we vary the angle between the two vectors keeping their magnitude constant, the dot product becomes maximum when they are perfectly aligned

parallel and co-directional (ομόρροπα)



Vectors – Dot product – Geometric representation

Commutative property (αντιμεταθετική ιδιότητα) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

If vectors a and b are perpendicular to each other, their dot product is zero.

If angle ϑ = 0, cos(ϑ) = 1 (co-directional). **Dot product equals magnitudes' product.**

If angle ϑ = 90^o, cos(ϑ) = 0. **Dot product is zero.**

If angle ϑ = 180^o, cos(ϑ) = -1 (antiparallel). **Dot product equals negative magnitudes' product.**

The dot product of a vector with itself equals to the square of its magnitude:

$$\mathbf{a} \cdot \mathbf{a} = a_x \cdot a_x + a_y \cdot a_y = ||\mathbf{a}|| \cdot ||\mathbf{a}|| \cdot 1 = ||\mathbf{a}||^2$$

Given two vectors, the angle between them can be determined by:

$$\theta = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| \cdot ||\mathbf{b}||}\right)$$

Outline

→ **Linear algebra**
- **Matrices**
- **Vectors**

Calculus
- **Limits**
- **Derivatives**
- **Sums**
- **Integrals**

Differential equations
- **Numerical analysis**

Outline

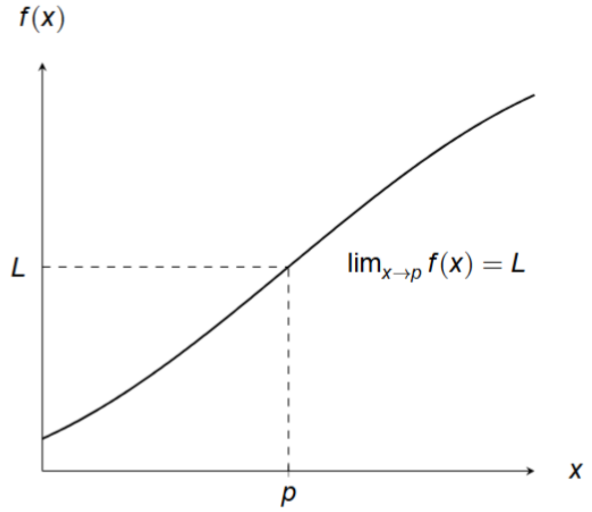
Linear algebra
- Matrices
- Vectors

→ Calculus
- Limits
- Derivatives
- Sums
- Integrals

Differential equations
- Numerical analysis

Limits

Limit definition: A function *f (x)* has a limit *L* at a point *p* if the value of *f (x)* can be made as close to *L* as desired, by making *x* close enough to *p*.



We say that the limit of *f*, as *x* approaches *p*, is *L*, **and we write:** $\lim_{x \to p} f(x) = L$

Limits

Definition (Limit of a Function): Let $f : D \to \mathbb{R}$ be a function defined on a domain $D \subseteq \mathbb{R}$, and let $L \in \mathbb{R}$ and x_0 be an accumulation point of D. We say that the limit of f as x approaches x_0 is L, written as

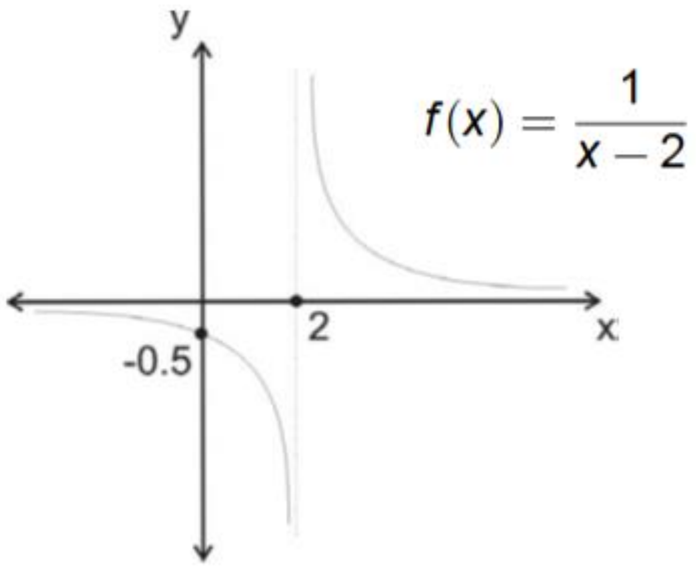
$$\lim_{x \to x_0} f(x) = L,$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$ with $0 < |x - x_0| < \delta$, it holds that

$$|f(x) - L| < \epsilon.$$

Limits

Limit example: We will find the limit of function *f*(*x*) as *x* approaches $x_0 = 0$.

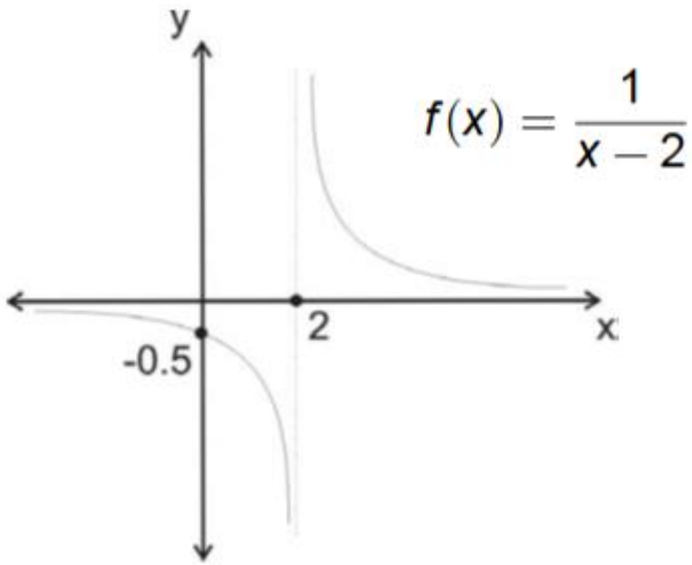


$$\lim_{x \to x_0} f(x) = \lim_{x \to 0} \frac{1}{x-2} = -\frac{1}{2} = -0.5$$

Limits

Limit example: We will find the limit of function *f*(*x*) as *x* approaches $x_0 = 2$.

Observation: approaching 2 from the left side leads to a different limit value than approaching 2 from the right side.



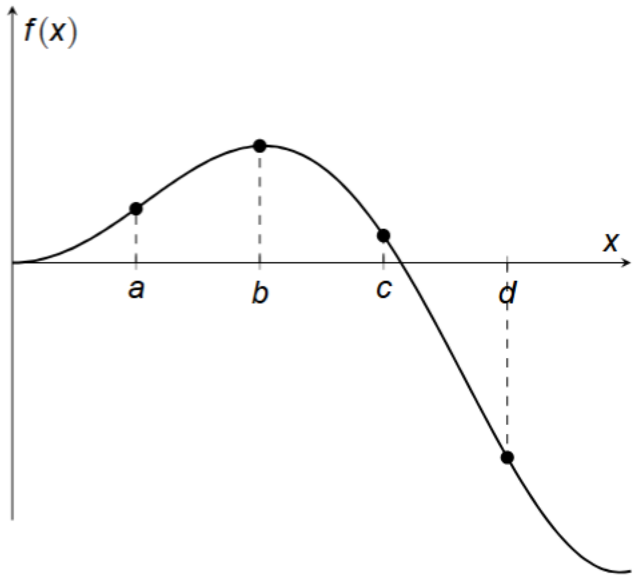
$$\lim_{x \to 2^+} \frac{1}{x-2} = \infty$$

$$\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$$

Derivatives - Differentiation

Differentiation: It measures the rate of change at any given point on a curve.

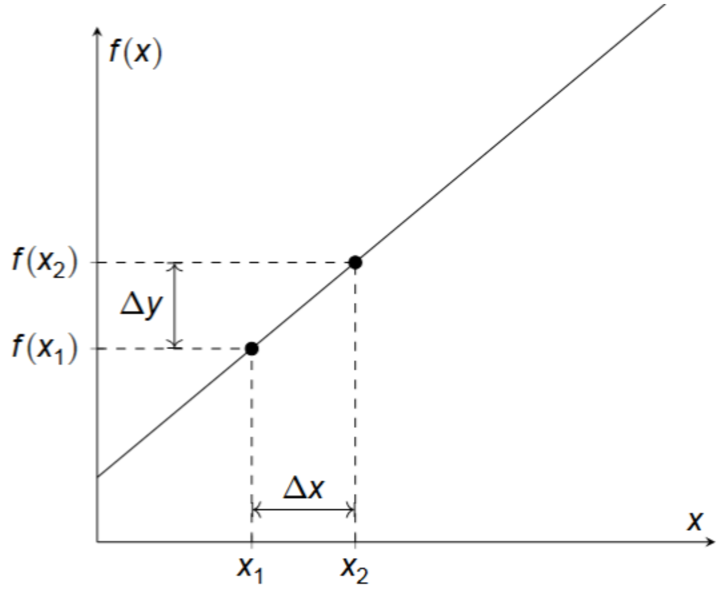
This rate of change is called the derivative.



Derivatives – Straight line

Differentiation: For a straight line, the slope expresses the derivative. It is constant at any given x_0.

The slope is defined as the difference in *y*s divided by the corresponding difference in *x*s.



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

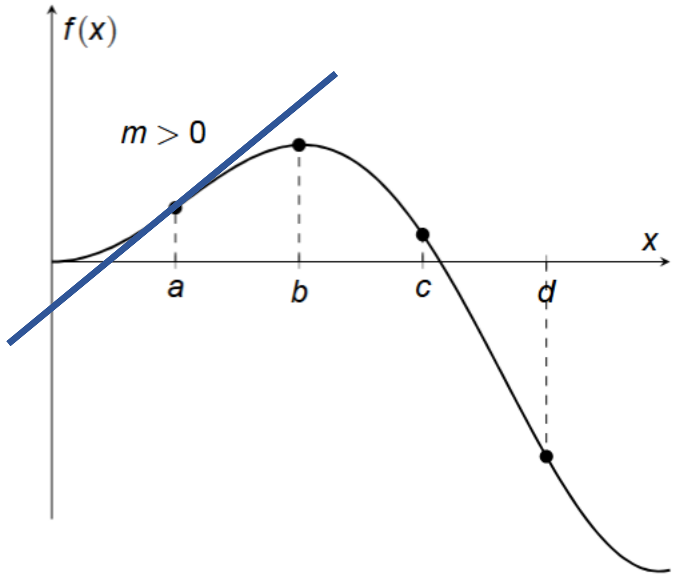
Derivatives – Arbitrary function

Differentiation: For arbitrary functions the derivative may change across *x*.

The slope is considered only at the neighborhood of the point.

The tangential straight line has a positive slope

$$m = \frac{dy}{dx}$$



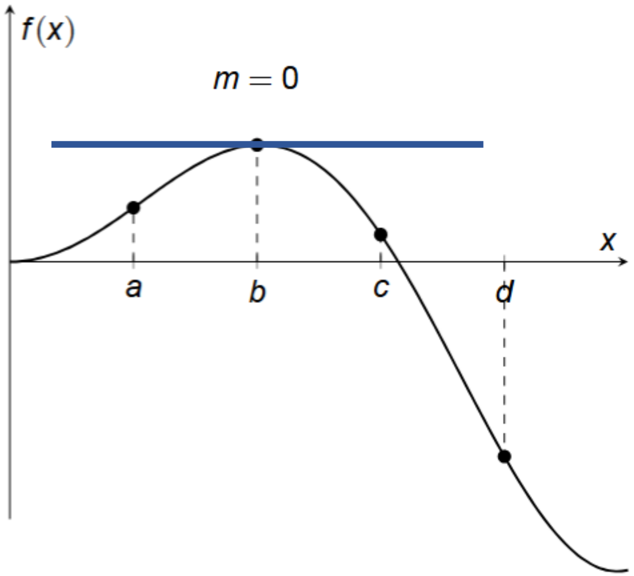
Derivatives – Arbitrary function

Differentiation: For arbitrary functions the derivative may change across *x*.

The slope is considered only at the neighborhood of the point.

The tangential straight line is horizontal, zero slope.

$$m = \frac{dy}{dx}$$



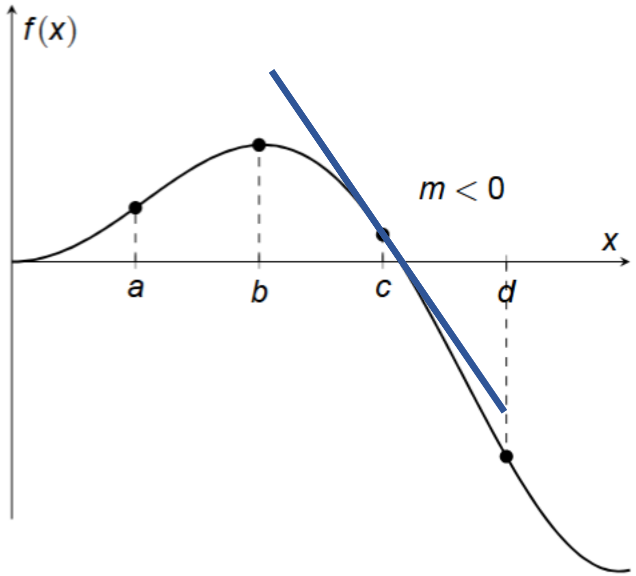
Derivatives – Arbitrary function

Differentiation: For arbitrary functions the derivative may change across *x*.

The slope is considered only at the neighborhood of the point.

The tangential straight line has a negative slope.

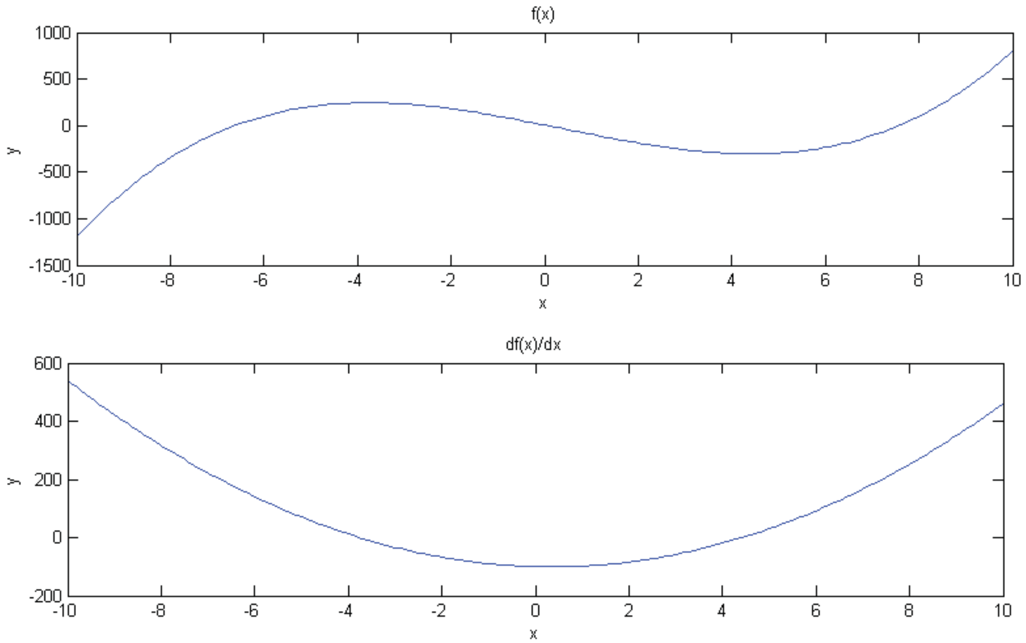
$$m = \frac{dy}{dx}$$



Derivatives – Straight line

Differentiation: For arbitrary functions the derivative may change across *x*.

example: graph of f(*x*) (top) and graph of the derivative of *f(x)* (bottom)



The derivatives can be considered as composite functions (functions acting on other functions)

Derivatives – Definition for arbitrary function

Differentiation: Suppose a function *f*(*x*) is defined in an interval containing point *a*.

The derivative of *f* with respect to *x* at point *a* is given by the limit:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If *f'*(*a*) exists, then the derivative of the function exists.

If the limit does not exist, then the derivative of the function does not exist.

If *f'*(*a*) exists, then *f* is said to be differentiable at *a*.

If *f* is differentiable at every point in its domain, then it's called a differentiable function.

Derivatives – Notation

Common notation for the derivative of a function *y = f (x)* is:

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}f(x)$$

The notation *dy/dx* (Leibniz notation) is read "the derivative of *y* with respect to *x*".

This notation suggests that the derivative of *f* is the rate of change of *f* with respect to *x*.

Similarly, the derivative of function *f* (*x*) at a specific point *a* is also notated as:

$$f'(a) = f'(x)\Big|_{x=a} = y'\Big|_{x=a} = \frac{dy}{dx}\Big|_{x=a} = \frac{d}{dx}f(x)\Big|_{x=a}$$

Derivatives – Rules

1. **Constant Rule:**

$$\frac{d}{dx}(c) = 0$$

2. **Constant Multiple Rule:**

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

3. **Power Rule:**

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

4. **Derivative of sin:**

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

5. **Derivative of cos:**

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

6. **Sum Rule:**

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

7. **Difference Rule:**

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

8. **Product Rule:**

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

9. **Quotient Rule:**

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

10. **Chain Rule:**

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

Derivatives – Simple Applications

$$\frac{d}{dx}(7) = 0$$

$$\frac{d}{dx}[3x^2] = 3 \cdot 2x = 6x$$

$$\frac{d}{dx}(x^5) = 5x^4$$

$$\frac{d}{dx}[x^2 + 4x] = 2x + 4$$

$$\frac{d}{dx}[x \cdot \sin(x)] = x \cdot \cos(x) + \sin(x) \cdot 1$$

$$\frac{d}{dx}[4\sin(3x)] = 4\cos(3x) \cdot 3 = 12\cos(3x)$$

Explanation:

Define u as a function of x:

$$u = 3x \quad f(x) = 4\sin(u)$$

$$\frac{du}{dx} = 3 \qquad \frac{df}{du} = 4\cos(u)$$

Chain rule:

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

$$\frac{df}{dx} = 4\cos(u) \cdot 3 = 12\cos(3x)$$

Summary:

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = 4\cos(3x) \cdot 3 = 12\cos(3x)$$

Derivatives – Partial derivative

Partial derivative: It concerns functions of two or more variables.

The partial derivative of *f* is its derivative with respect to one of those variables while the others are held fixed during differentiation.

$$\frac{\partial}{\partial x_i} f(a) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}$$

<u>Common notation for partial derivative</u>

First-order partial derivative: $\frac{\partial f}{\partial x} = f_x = \partial_x f$

Second-order partial derivative: $\frac{\partial^2 f}{\partial x^2} = f_{xx} = \partial_{xx} f$

Derivatives – Partial derivative

Mixed derivative: Partial derivative that involves more than one variable.

Example:

$$\partial_{xy} f = \partial_{xy} f = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

$$\partial_{yx} f = \partial_{yx} f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

For most functions (continuous) the mixed partial derivatives are equal irrespectively of the order:

$$f_{xy} = f_{yx}$$

$$f_{xyz} = f_{zxy} = f_{yzx} = f_{zyx} = f_{xzy} = f_{yxz}$$

Sums - Description

Summation is the addition of a set of numbers. The result of summation is called *sum*.

It is possible to add an infinite number of elements. An infinite sum is called a *series*.

For a concise representation of a sum, we use the summation symbol, capital sigma.

Example:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

***i* is the index of summation running from 1 to *n* (*i* takes integer values)**

Additional examples:

$$\sum_{i=1}^{n} (i+1) = 2 + 3 + 4 + \cdots + (n+1)$$

$$\sum_{i=m}^{n} x_i = x_m + x_{m+1} + x_{m+2} + \cdots + x_n$$

Sums - Application

Application: Write the following sum using the summation notation.

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

Solution:

$$S = \sum_{k=1}^{n} \frac{1}{k}$$

Sums - Description

Some additional common types of sums

The sum of *f* (*k*) over all (integer) *k* in the specified range:

$$\sum_{0 \le k < 100} f(k)$$

The sum of *f* (*x*) over all elements *x* in the set *S* (*S* should be countable) :

$$\sum_{x \in S} f(x)$$

Sums – Multiple indices

We can have sums with multiple indices (independent or not)

$$\sum_{1 \leq i \leq m,\, 1 \leq j \leq n} a_{ij} = \sum_{\substack{i=1 \\ j=1}}^{m \quad n} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

Example 1: $\sum_{i=1}^{4} \sum_{j=1}^{3} a_{ij} = a_{11} + a_{12} + a_{13} + a_{21} + a_{22} + a_{23} + a_{31} + a_{32} + a_{33} + a_{41} + a_{42} + a_{43}$

$$\begin{aligned} \sum_{i=1}^{4} \sum_{j=1}^{3} a_{ij} = a_{11} \quad &+ \quad a_{12} \quad + \quad a_{13} \quad + \quad \cdots \\ a_{21} \quad &+ \quad a_{22} \quad + \quad a_{23} \quad + \quad \cdots \\ a_{31} \quad &+ \quad a_{32} \quad + \quad a_{33} \quad + \quad \cdots \\ a_{41} \quad &+ \quad a_{42} \quad + \quad a_{43} \end{aligned}$$

Sums – Multiple indices

We can have sums with multiple indices (independent or not)

$$\sum_{1 \le i \le m,\, 1 \le j \le n} a_{ij} = \sum_{\substack{i=1 \\ j=1}}^{m \quad n} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

Example 2:

$$\sum_{i=1}^{4} \sum_{j>i}^{4} a_{ij} = a_{12} + a_{13} + a_{14} + a_{23} + a_{24} + a_{34}$$

$$\begin{aligned} \sum_{i=1}^{4} \sum_{j>1}^{4} a_{ij} = \quad & a_{12} + a_{13} + a_{14} + \cdots \\ & \qquad\quad\; a_{23} + a_{24} + \cdots \\ & \qquad\qquad\qquad\; a_{34} \end{aligned}$$

Integrals – Description

The integral of a function is an extension of the concept of a sum.

The process of finding integrals is called *integration*.

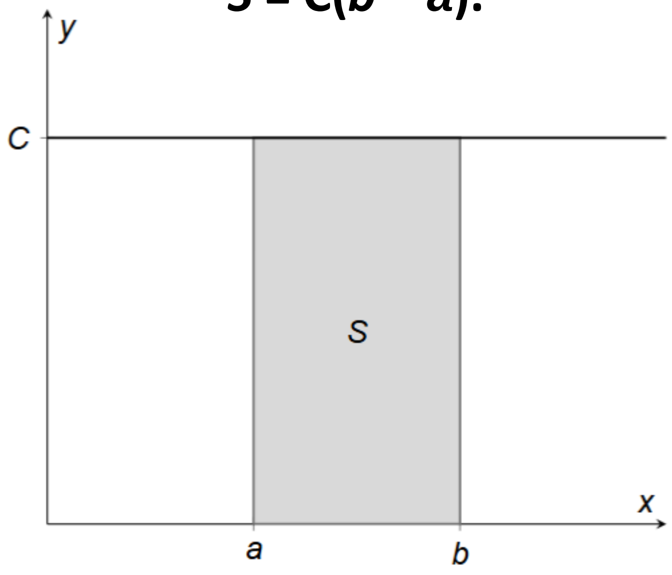
Integration is usually used to compute areas, volumes, mass, displacements etc., when their distribution or rate of change with respect to some other quantity (position, time, etc.) is specified.

Integrals – Description

The integral of *f* (*x*) on the interval [*a*, *b*] is equal to the signed area bounded by the lines *x* = *a*, *x* = *b*, the *x*-axis, and the curve defined by the graph of *f*.

The area is the width of the rectangle times its height, so the value of the integral is

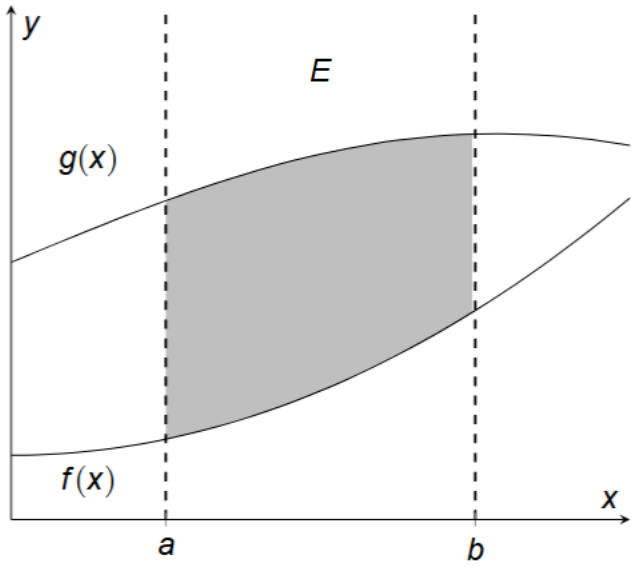
$$S = C(b - a).$$



The same result (as we will see) can be found by integrating *f*(*x*)

Integrals – Description

To find the area between the two curves and between two limits *a* and *b*, would be to evaluate the integral of the function representing the difference in the value of the two functions between those limits.



Integrals – Definite integral

Leibniz introduced the standard "long *S*" notation for the integral:

$$\int_a^b f(x)\,dx$$

The $\int$ sign represents integration

The *a* and *b* are the endpoints of the interval

***f*(*x*) is the function we are integrating known as the integrand**

***x* is the variable of integration**

***dx* is a notation for the variable of integration.**

Integrals – Definite integral

If a function has an integral, it is said to be integrable.

The function for which the integral is calculated is called the integrand.

Definite integrals result in a number, not another function.

If the domain of the function to be integrated is the real numbers, and

If the region of integration is an interval, then

the greatest lower bound of the interval is called the lower limit of integration,

and the least upper bound is called the upper limit of integration.

$$\int_a^b f(x)\,dx$$

Integrals – Indefinite integral

The notation of an "indefinite integral" is

$$\int f(x)\,dx$$

this is read "integral of f (x) dx".

note: There are no limits of integration in this notation.

an indefinite integral, results in a function, not a number.

Integrals– Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus states that:

differentiation and integration are inverse processes.

Mathematically, this is expressed as:

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$$

$$\int \frac{d}{dx} f(x)\,dx = f(x) + C$$

Integrals– Fundamental Theorem of Calculus

Examples

$$F(x) = x^3 - 5x^2 + 3x$$

$$f(x) = \frac{d}{dx}(x^3 - 5x^2 + 3x) = 3x^2 - 10x + 3$$

$$\int (3x^2 - 10x + 3)\, dx = x^3 - 5x^2 + 3x + C$$

$$F(x) = \sin(3x)$$

$$f(x) = \frac{d}{dx}(\sin(3x)) = 3\cos(3x)$$

$$\int 3\cos(3x)\, dx = \sin(3x) + C$$

Integrals– Fundamental Theorem of Calculus

Examples

$$F(x) = x^3 - 5x^2 + 3x$$

$$f(x) = \frac{d}{dx}(x^3 - 5x^2 + 3x) = 3x^2 - 10x + 3$$

$$\int (3x^2 - 10x + 3)\, dx = x^3 - 5x^2 + 3x + C$$

$$F(x) = \sin(3x)$$

$$f(x) = \frac{d}{dx}(\sin(3x)) = 3\cos(3x)$$

$$\int 3\cos(3x)\, dx = \sin(3x) + C$$

Important example

$$F(x) = \ln(x)$$

$$x > 0$$

$$f(x) = \frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\int \frac{1}{x}\, dx = \ln(x) + C$$

Outline

Linear algebra
- Matrices
- Vectors

→ Calculus
- Limits
- Derivatives
- Sums
- Integrals

Differential equations
- Numerical analysis

Outline

Linear algebra
- Matrices
- Vectors

Calculus
- Limits
- Derivatives
- Sums
- Integrals

→ **Differential equations**
- Numerical analysis

Differential equations – Ordinary differential equations

All living things are dynamical systems, that is they change over time.

Differential equations describe dynamical systems.

Every system must pass through a continuous path of intervening states to get from one state to another.

Example: The position of an object a tiny amount of time in the future can be calculated from its current position and current velocity. Differential equations quantify this procedure.

Ordinary differential equations (ODEs) describe how individual properties of a system change with time, or along a single spatial dimension, but not with both together.

Differential equations – Ordinary differential equations

Example

The rate of growth $y'= dy/dt$ of a population y is proportional to the population itself.

$$y' = ky \quad \Rightarrow \quad y' - ky = 0$$

If we know the population $y(t_0)$ at time $t = t_0$, the above differential equation tells us how the population changes from that point on.

The population $y(t_1)$ at time $t = t_1$ will only depend on the difference $\Delta t = t_1 - t_0$, rather than the specific value of time t_1.

If we assume that the population at time instant $t = 0$ is y_0, i.e. $y(0) = y_0$. This is an initial value problem.

We would like to find a function $y(t)$ which satisfies both the differential equation and the initial condition.

Differential equations – Ordinary differential equations

Example **(solution)**

Given the differential equation, we are interested in predicting the evolution of the population across time.

To reach an explicit mathematical expression for the function y(t), we will need to do some mathematical manipulation.

$$y' = ky \quad \Rightarrow \frac{1}{y}\frac{dy}{dt} = k$$

Differential equations – Ordinary differential equations

Example (solution)

We will need to isolate the terms which contain variables *y* and *t*.

$$\int_0^{t_1} \frac{1}{y}\frac{dy}{dt}\, dt = \int_0^{t_1} k\ dt \quad \Rightarrow \quad \int_{y_0}^{y(t_1)} \frac{1}{y}\, dy = kt_1$$

$$\Rightarrow \quad [\ln|y|]_{y_0}^{y(t_1)} = kt_1$$

$$\Rightarrow \quad \ln\left|\frac{y(t_1)}{y_0}\right| = kt_1$$

$$\Rightarrow \quad \frac{y(t_1)}{y_0} = e^{kt_1}$$

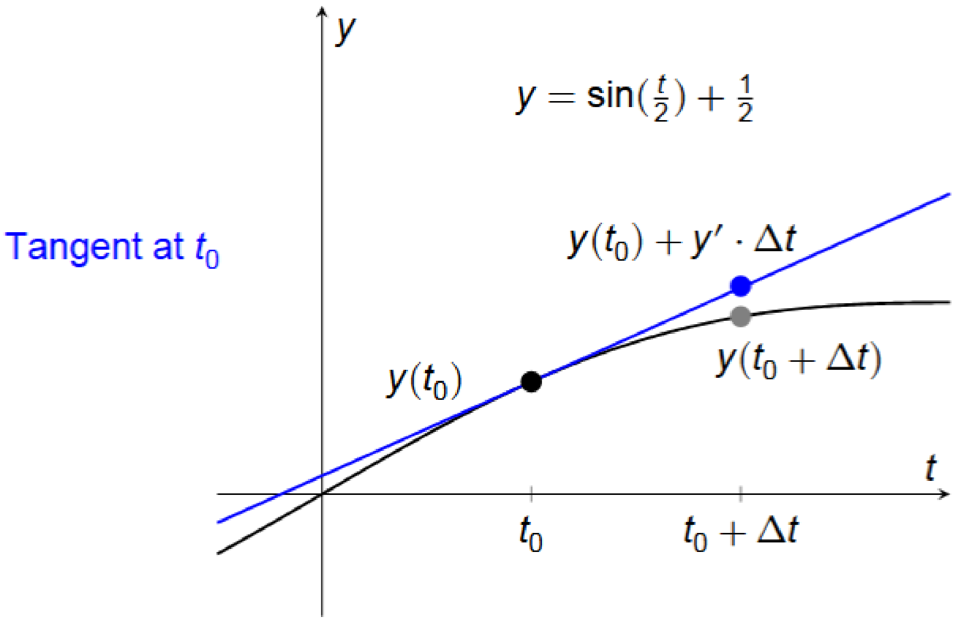
$$\Rightarrow \quad y(t_1) = y_0 e^{kt_1}$$

$$\int \frac{1}{x}\, dx = \ln(x) + C$$

Differential equations – Solution Methods

<u>Basic Idea</u>: Given a known $y(t_0)$ for a specific t_0, we can approximately predict the future value of y at a close enough timepoint $t_0 + \Delta t$, $y(t_0 + \Delta t)$, if we know the derivative of y' at timepoint t_0. The approximation is better the smaller the Δt we use.

The predicted value of y is: $\hat{y} = y(t_0) + y' \cdot \Delta t$



Differential equations – Solution Methods

Euler's Method

Most differential equations cannot be solved exactly (i.e. closed form solutions); hence they must be solved numerically.

A numerical solution for function $y(t)$ consists of a set of approximated function values y_1, y_2, y_3,… for specific values at specific points t_1, t_2, t_3, … of the independent variable t.

Euler's method: use the first two terms of a Taylor series to generate the solution of a first order differential equation with an initial condition ($y(0) = y_0$) .

$$y' = U(t, y)$$

Taylor series: $$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(t_0)}{2!}(t - t_0)^2 + \frac{y'''(t_0)}{3!}(t - t_0)^3 + \cdots$$

Differential equations – Solution Methods

Euler's Method

The problem of solving these equations amounts to determining y as a function of t on some interval, say $t \in [a,b]$. To solve a first order diff. equation numerically on [0, 1] with the initial condition $y(0) = y_0$, we proceed as follows:

Let $t_0, t_1, \ldots, t_n$ be a set of equally spaced points on [0, 1] with

$$t_0 = 0,\ t_n = 1,\ t_{i+1} - t_i = h = 1/n,\ (0 \le i < n)$$

Euler's method computes the solution on these points according to the following formula:

$$y' = U(t, y) \qquad\qquad y_{i+1} = y_i + U(t_i, y_i)h$$

Differential equations – Solution Methods

<u>Euler's Method (example)</u>

Given the initial value problem:

$$y' = y, \qquad y(0) = 1$$

we would like to use the Euler method to approximate $y(4)$.

The Euler method is:

$$y_{n+1} = y_\mathrm{n} + h\, f(t_n, y_\mathrm{n})$$

so first we must compute $f(t_0, y_0)$. In this simple differential equation, the function f is defined by $f(t, y) = y$.

We have $f(t_0, y_0) = f(0, 1) = 1$. By doing the above step, we have found the slope of the line that is tangent to the solution curve at the point (0, 1).

Differential equations – Solution Methods

Euler’s Method (example): The next step is to multiply the above value by the step size h, which we take equal to 1 here:

$$h \; f(y_0) = 1 \cdot 1 = 1$$

Since the step size is the change in t, when we multiply the step size and the slope of the tangent, we get a change in y value. This value is then added to the initial y value to obtain the next value to be used for computations.

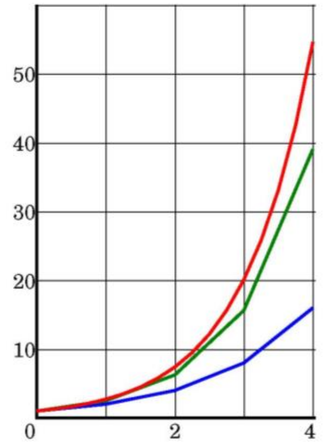
$$y_0 + h\,f(y_0) = y_1 = 1 + 1 \cdot 1 = 2$$

The above steps should be repeated to find y_2 , y_3 and y_4.

$$y_2 = y_1 + h\,f(y_1) = 2 + 1 \cdot 2 = 4$$

$$y_3 = y_2 + h\,f(y_2) = 4 + 1 \cdot 4 = 8$$

$$y_4 = y_3 + h\,f(y_3) = 8 + 1 \cdot 8 = 16$$



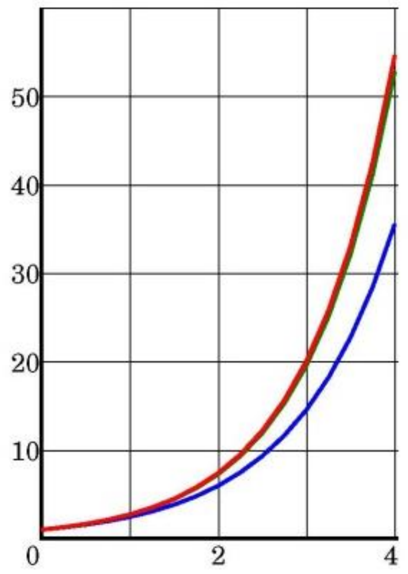
The conclusion of this computation is that $y_4 = 16$. The exact solution of the differential equation is $y(t) = e^t$, so $y(4) = e^4 \sim 54.598$. Thus, the approximation of the Euler method is not very good in this case. However, as the figure shows, its behavior is qualitatively right.

Differential equations – Solution Methods

Euler’s Method (example): by using smaller step size, we get smaller errors.

The error recorded in the last column of the table is the difference between the exact solution and the Euler approximation. In the bottom of the table, the step size is half the step size in the previous row, and the error is also approximately half the error in the previous row. This suggests that the error is roughly proportional to the step size, at least for fairly small values of the step size.

step size	Euler's method prediction	error
1	16	38.598
0.25	35.53	19.07
0.1	45.26	9.34
0.05	49.56	5.04
0.025	51.98	2.62
0.0125	53.26	1.34



Differential equations – Solution Methods

Runge-Kutta Method

Let an initial value problem be spaced as follows.

$$y' = f(t, y);\; y(t_0) = y_0$$

Then, the Runge-Kutta method for this problem is given by the following equation:

$$y_{n+1} = y_n + h\,(k_1 + 2k_2 + 2k_3 + k_4) / 6$$

Differential equations – Solution Methods

Runge-Kutta Method

$$k_1 = f(t_n; y_n)$$

$$k_2 = f(t_n + h/2; y_n + k_1 h/2)$$

$$k_3 = f(t_n + h/2; y_n + k_2 h/2)$$

$$k_4 = f(t_n + h; y_n + k_3 h)$$

Thus, the next value (y_{n+1}) is determined by the present value (y_n) plus the product of the size of the interval (h) and an estimated slope. The slope is a weighted average of slopes:

- **k_1 is the slope at the beginning of the interval**
- **k_2 is the slope at the midpoint of the interval, using slope k_1 to determine the value of y at the point $t_n + h/2$ using Euler's method**
- **k_3 is again the slope at the midpoint, but now using the slope k_2 to determine the y-value**
- **k_4 is the slope at the end of the interval, with its y-value determined using k_3.**

The End

Thank you!