CHAPTER I

ON THE ORIGIN AND SIGNIFICANCE OF THE AXIOMS OF GEOMETRY

(*Vorträge und Reden*, 5th ed., vol. II, pp. 1–31)

The fact that a science like geometry can exist, and can be built up in the way it is, has necessarily demanded the closest attention of anyone who ever felt an interest in the fundamental questions of epistemology. There is no other branch of human knowledge which resembles it in having seemingly sprung forth ready-made, like a fully armed Minerva from the head of Jupiter, none before whose devastating aegis dispute and doubt so little dared to lift their eyes. In this it wholly escapes the troublesome and tedious task of gathering empirical facts, as the natural sciences in the narrower sense are obliged to, but instead the form of its scientific procedure is exclusively deduction[1]. Conclusion is developed from conclusion, and yet nobody in his right mind ultimately doubts that these geometrical theorems must have their very practical application to the actuality surrounding us. In surveying and architecture, in mechanical engineering and mathematical physics, we all constantly calculate the most varied kinds of spatial relationships in accordance with geometrical theorems. We expect the issue of their constructions and experiments to be subject to these calculations, and no case is yet known in which we were deceived in this expectation, provided we calculated correctly and with sufficient data.

Thus in the conflict over that issue which forms, as it were, the focus of all the oppositions between philosophical systems, the fact that geometry exists and achieves such things has always been used to prove, as an impressive example, that knowledge of propositions of real content is possible without recourse to a corresponding basis taken from experience. In answering especially Kant's famous question "How are synthetic *a priori* propositions possible?", the axioms of geometry probably constitute the examples which seem to show most evidently, that synthetic propositions *a priori* are in general possible[2]. The circumstance that such propositions exist, and necessarily force our assent, is moreover for him a proof that space is a form, given *a priori*, of all outer intuition[3]. By that he seems to mean not merely that this form given *a priori* has the character of a purely formal scheme, in itself devoid of any content[4], and into which any arbitrary content of experience would fit. Rather, he seems also to include certain details in the schema whose effect is precisely that only a content restricted in a certain lawlike way can enter it, and become intuitable for us*.

It is precisely this epistemological interest of geometry which gives me the courage to speak of geometrical matters in a gathering of whose members only the smallest proportion have penetrated more deeply into mathematical studies than their school days required. Fortunately, the knowledge of geometry which is normally taught in high school will suffice to convey to you the sense, at least, of the propositions to be discussed in what follows.

My intention is namely to give you an account of a recent series of interconnected mathematical studies, which concern the axioms of geometry, their relations to experience, and the logical possibility of replacing them with others. The relevant original studies of the mathematicians are, in the first instance, intended only to provide proofs for the experts in a field which demands a higher capacity for abstraction than any other, and are fairly inaccessible to the non-mathematician. So I shall try to give even such a person an intuitive conception of what is involved. I hardly need remark that my discussion is not meant to give any proof of the correctness of the new insights. Whoever seeks this must simply take the trouble to study the originals.

If a person once enters geometry, in other words the mathematical theory of space[5], by the gates of its first elementary propositions, he faces on his further journey the unbroken chain of conclusions of which I spoke already. Through them, successively more numerous and complicated spatial forms receive their laws. But in those first elements are

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* In his book *Über die Grenzen der Philosophie* ["On the limits of philosophy"], W. Tobias claims that some previous utterances of mine to a similar effect were a misunderstanding of Kant's opinion. But Kant specifically quotes the propositions that the straight line is the shortest (*Kritik der reinen Vernunft* ["Critique of pure reason"], 2nd ed., p. 16), that space has three dimensions (p. 41) and that between two points only one straight line is possible (p. 204), as ones which "express the conditions of sensible intuition *a priori*". Whether these propositions are given originally in spatial intuition, or whether this only gives the starting points from which the faculty of understanding can develop such propositions *a priori* – this being what my critic attaches importance to – is quite irrelevant here.

laid down a few propositions concerning which even geometry declares it cannot prove them, but can only count on the recognition of their correctness by whoever understands their sense. These are the so-called axioms of geometry.

If we call the shortest line that can be drawn between two points a *straight line*, then there is firstly the axiom that there can only be one straight line between two points, and not two different ones. It is a further axiom that a *plane* can be placed through any three points of space that do not lie in a straight line, namely a surface which wholly includes every straight line connecting any two of its points. Another and a much-discussed axiom asserts that through a point lying outside a straight line only one single straight line can be drawn parallel to the first, and not two different ones†. *Parallel* is what one calls two lines lying in the same plane which never meet, however far they are extended. Besides this, the axioms of geometry also assert propositions which specify the number of dimensions of both space and its surfaces, lines and points, and which elucidate the concept of the continuity of these formations. Such are the propositions that the limit of a body is a surface, the limit of a surface is a line, the limit of a line is a point, and a point is indivisible. And also the propositions that a line is described by the motion of a point, a line or a surface by the motion of a line, a surface or a body by the motion of a surface, but only another body by the motion of a body[6].

Where do such propositions come from, being unprovable and yet indubitably correct in the domain of a science in which everything else has submitted to the authority of inference[7]? Are they an inheritance from the divine source of our reason, as the idealist philosophers[8] believe, or have earlier generations of mathematicians merely not yet had sufficient ingenuity to find the proof? Every new disciple of geometry, coming to this science with fresh zeal, naturally tries to be the fortunate one who outdoes all his predecessors. It is also quite proper that everyone should have a try at it afresh, since in the situation to date one could only convince oneself by one's own fruitless attempts that the proof was impossible[9]. Unfortunately, there still arises the occasional obsessive searcher who entangles himself for so long and so

† [This is strictly speaking not Euclid's postulate (axiom) of parallels, but a common substitute for it.]

deeply in intricate reasonings that he can no longer discover his mistakes, and believes he has found the solution. The axiom of parallels especially has evoked a large number of apparent proofs.

The greatest difficulty in these investigations has been, and still is, that so long as the only method of geometry was the intuitive method taught by Euclid[10], it was all too easy to intermix results of everyday experience, as apparent necessities of thought, with the logical development of concepts. Proceeding in this way, it is in particular extraordinarily difficult to ascertain that one is nowhere involuntarily and unknowingly helped, in the steps which one successively prescribes for the demonstration, by certain very general results of experience, which have already taught us practically that certain prescribed parts of the procedure can be carried out.

In drawing any subsidiary line in any proof, the well-trained geometer always asks whether it will indeed always be possible to draw a line of the required kind. Constructional tasks, as is well known, play an essential role in the system of geometry. Viewed superficially, these seem to be practical applications which have been inserted as an exercise for pupils. But in truth they ascertain the existence of certain structures. They show that points, straight lines or circles such as those whose construction is required as a task, are either possible under all conditions, or characterise the exceptional cases that may be present.

It is essentially a point of this kind, about which the investigations to be discussed in what follows turn. In the Euclidean method, the basis of all proofs is demonstration of the congruence of the relevant lines, angles, plane figures, bodies, and so on. In order to make the congruence intuitive, one imagines the relevant geometrical structures moved up to each other[11], naturally without changing their form and dimensions. That this is in fact possible, and can be carried out, is something we have all experienced from earliest youth onwards. But if we want to erect necessities of thought upon this assumption, that fixed† spatial structures can be moved freely without distortion to any location in space, then we must raise the question of whether this assumption involves any presupposition which has not been logically proved. We shall soon see below that it does in fact involve one, and indeed one of far-reaching

† [Helmholtz' use of 'fest' ('fixed') instead of 'starr' ('rigid') is alluded to in note 31 below.]

implications. But if it does, then every congruence proof is supported by a fact drawn only from experience[12].

I bring up these considerations here in order, in the first place, only to make clear what difficulties we stumble upon, when we analyse fully all of the presuppositions made by us in using the intuitive method. We escape them, if in our investigation of basic principles we employ the analytic method[13] developed in modern calculative geometry. The calculation is wholly carried out as a purely logical operation[14]. It can yield no relationship between the quantities subjected to the calculation, which is not already contained in the equations forming the starting point of the calculation. For this reason, the mentioned recent investigations have been pursued almost exclusively by means of the purely abstract method of analytic geometry.

Yet it is possible besides this to give to some extent an intuitive conception of the points at issue, now that they have been made known by the abstract method. This is best done if we descend into a narrower domain than that of our own spatial world. Let us think of intelligent beings, having only two dimensions, who live and move in the surface of one of our solid bodies – in this there is no logical impossibility. We assume that although they are not capable of perceiving anything outside this surface, they are able to have perceptions, similar to our own, within the expanse of the surface in which they move. When such beings develop their geometry, they will naturally ascribe only two dimensions to their space. They will ascertain that a moving point describes a line and a moving line a plane, this being for them the most complete spatial structure of their acquaintance. But they will as little be able to have any imagination of a further spatial structure that would arise if a surface moved out of their surface-like space, as we are of a structure that would arise if a body moved out of the space known to us.

By the much misused expression 'to imagine' † or 'to be able to think of how something happens', I understand[15] that one could depict the series of sense impressions which one would have if such a thing happened in a particular case. I do not see how one could understand anything else by it without abandoning the whole sense of the expression. But suppose no sense impression whatsoever is known that would

† [See Translator's Note.]

relate to such a never observed process as for us a motion into a fourth dimension of space, or for the surface beings a motion into the third dimension known to us. Then no such 'imagining' is possible, just as little as someone absolutely blind from youth will be able to 'imagine' colours, even if he could be given a conceptual description of them.

The surface beings would besides also be able to draw shortest lines in their surface space. These would not necessarily be straight lines in our sense, but what in geometrical terminology we would call *geodetic* lines of the surface on which they live, ones which will be described by a taut thread applied to the surface and able to slide freely upon it. In what follows, I shall permit myself to term such lines the *straightest* lines[16] of the relevant surface (or of a given space), in order to emphasise the analogy between them and the straight line in a plane. I hope this intuitive expression will make the concept more accessible to my non-mathematical listeners, but without causing confusions.

If moreover beings of this kind lived in an infinite plane, they would lay down precisely our planimetric geometry. They would maintain that only *one* straight line is possible between two points, that through a third point lying outside it only one line parallel to the first can be drawn, that furthermore straight lines can be extended infinitely without their ends meeting again, and so on. Their space might be infinitely extended. But even if they encountered limits to their motion and perception, they would be able to imagine intuitively a continuation beyond those limits. In imagining this, their space would seem to them to be infinitely extended just as ours does to us, although we too cannot leave our earth with our bodies, and our sight only reaches as far as fixed stars are available.

But intelligent beings of this kind could also live in the surface of a sphere. For them, the shortest or straightest line between two points would then be an arc of the great circle through the points in question. Every great circle through two given points is divided thereby into two parts. When their two lengths are not equal, the shorter part is certainly the unique shortest line on the sphere between these two points. But the other and greater arc of the same great circle is also a geodetic or straightest line, meaning that each of its smaller parts is a shortest line between its two endpoints. Because of this circumstance, we cannot simply identify the concept of a geodetic or straightest line with that of a shortest line. If moreover the two given points are endpoints of the same diameter of the sphere, then any plane through this diameter intersects the surface of the sphere in semicircles, all of which are shortest lines between the two endpoints. So in such a case there are infinitely many shortest lines, all equal to each other, between the two given points. Accordingly, the axiom that only one shortest line exists between two points would not be valid, for the sphere dwellers, without a certain exception.

Parallel lines would be quite unknown to the inhabitants of the sphere. They would maintain that two arbitrary straightest lines, suitably extended, must eventually intersect not in just one point, but in two. The sum of the angles in a triangle would always be greater than two right angles, and would increase with the area of the triangle. For just that reason, they would also lack the concept of geometrical similarity of form between greater and smaller figures of the same kind, since a greater triangle would necessarily have different angles from a smaller one[17]. Their space would be found to be unbounded, yet finitely extended, or at least would have to be imagined to be such.

It is clear that the beings on the sphere, though having the same logical capabilities, would have to lay down a quite different system of geometrical axioms from what the beings in the plane would, and from what we ourselves do in our space of three dimensions. These examples already show us that beings, whose intellectual powers could correspond entirely to our own, would have to lay down different geometrical axioms according to the kind of space in which they lived.

But let us go further, and think of intelligent beings existing in the surface of an egg-shaped body. Between any three points of such a surface one could draw shortest lines, and so construct a triangle. But if one tried to construct congruent triangles at different locations in this surface, it would be found that the angles of two triangles having equally long sides would not turn out to be equal. A triangle drawn at the pointed end of the egg would have angles whose sum differed more from two right angles, than would a triangle with the same sides drawn at the blunt end. It emerges from this, that even such a simple spatial structure as a triangle, in such a surface, could not be moved from one location to another without distortion. It would be found equally, that if circles of equal radii (the length of the radii being always measured by shortest lines along the surface) were constructed at different locations in such a surface, their periphery at the blunt end would turn out to be greater than at the pointed end.

It also follows from this, that it is a special geometrical property of a surface to be such that the figures lying in it can be displaced freely without altering any of their angles or lines as measured along the surface, and that this will not be the case on every kind of surface. The condition for a surface's having this important property was already established by Gauss in his famous essay on the curvature of surfaces. The condition is that what he called the 'measure of curvature', namely the reciprocal of the product of the two principal radii of curvature, should have the same magnitude throughout the whole expanse of the surface.

At the same time, Gauss showed that if the surface is bent without being stretched or shrunk in any part, this measure of curvature is not altered. Thus we can roll up a flat sheet of paper into a cylinder or cone without any alteration in the measurements of figures on the sheet, if the measurements are taken along its surface. And equally we can roll together the hemispherical closed half of a pig's bladder into the form of a spindle, without altering the measurements in this surface itself. The geometry on a plane will therefore also be the same as that in a cylindrical surface. We must only imagine in the latter case that many layers of this surface lie without limit on top of one another, like the layers of a rolled-up sheet of paper, and that with every complete circuit round the circumference of the cylinder one enters another layer, different from the one in which one was before.

These observations are necessary, in order to give you an idea of a kind of surface whose geometry is on the whole similar to that of the plane, but for which the axiom of parallels does not hold. It is a kind of curved surface which behaves in geometrical terms like the contrary of a sphere. For this reason the outstanding Italian mathematician E. Beltrami*, who investigated its properties, called it the *pseudospherical surface*. It is a saddle-shaped surface, of which only bounded pieces or strips can be displayed as a connected whole in our space. However, one can think of its being continued infinitely in all directions,

* *Saggio di Interpretazione della Geometria Non-Euclidea*, ["An essay at interpreting non-Euclidean geometry"], Naples, 1848. "Teoria Fondamentale degli Spazii di Curvatura Costante", ["The fundamental theory of spaces of constant curvature"], *Annali di Matematica*, 2nd series, II, 232–255.

since one can think of taking each surface portion lying at the boundary of the constructed part of the surface, pushing it back towards its middle, and then continuing it[18]. A surface portion displaced in this process must change its flexure but not its dimensions, just as one can push a sheet of paper back and forth on a cone which has arisen by rolling a plane up together. Such a sheet will everywhere adapt itself to the surface of the cone, but it must be more strongly bent nearer the tip, and it cannot be pushed on past the tip in such a way as to remain adapted to both the existing cone and its ideal continuation beyond the tip.

Pseudospherical surfaces are of constant curvature like the plane and the sphere, so that every piece of them fits exactly when laid upon any other part of the surface. Therefore, any figure constructed at one place in the surface can be transferred to any other place with a form completely congruent, and with complete equality of all dimensions lying in the surface itself. The measure of curvature laid down by Gauss, which is positive for the sphere and equal to zero for the plane, will have a constant negative value for pseudospherical surfaces, because the two principal curvatures of a saddle-shaped surface have their concavities on opposite sides[19].

It is, for example, possible to display a strip of a pseudo-spherical surface furled up as the surface of a ring. Think of a surface such as *aabb* (Figure 1), rotated about its axis of symmetry *AB*, in which case the two arcs *ab* would describe such a pseudospherical ringlike surface. The two edges of the surface, above at *aa* and below at *bb*, would bend more and more sharply outwards until the surface was perpendicular to the axis. It would end there at the rim with infinite curvature. One could also furl up half of a pseudospherical surface into a calyx-shaped champagne glass with an infinitely prolonged and tapering stem, as Figure 2. But the surface is necessarily always limited on one side by an abrupt edge beyond which a continuous extension is not immediately realisable. Only if one thinks of each individual portion of the edge being cut loose and pushed along the surface of the ring or calyx-shaped glass, can one bring such portions of the surface to locations of different flexure where they can be extended further.

In this way, the straightest lines of the pseudospherical surface can also be prolonged infinitely. They do not return upon themselves, like those of the sphere. There can instead, as on the plane, be only one

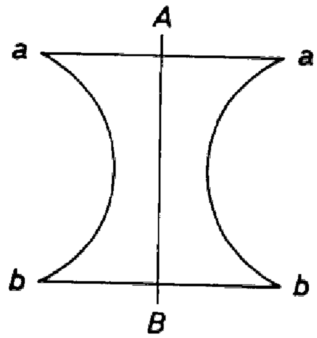

Fig. 1

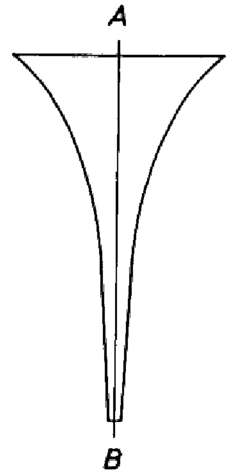

Fig. 2

single shortest line between any two given points. But the axiom of parallels does not hold. Given a straightest line on the surface and a point lying outside it, one can find a whole pencil of straightest lines through the point, none of which intersects the given line even when prolonged infinitely. They are namely all lines lying between a certain pair of straightest lines which bound the pencil. One of these meets the given line when infinitely prolonged in one direction, and the other in the other direction.

Such a geometry, which drops the axiom of parallels, was moreover completely elaborated as long ago as 1829 by the mathematician N. J. Lobatchewsky* in Kazan, using Euclid's synthetic method[20]. It was found that this geometrical system could be implemented in a manner as deductive and as free of contradiction as can that of Euclid. This geometry accords completely with that of pseudospherical surfaces, more recently developed by Beltrami.

From all this we see that the assumption, in two-dimensional geometry, that any figure can be moved in any direction without at all altering its dimensions in the surface, characterises the surface concerned as being a plane, a sphere or a pseudospherical surface. The axiom that there exists only one shortest line between any two points separates the plane and the pseudospherical surface from the sphere, and the axiom of parallels divides off the plane from the pseudosphere. These three axioms are therefore necessary and sufficient, in order to characterise

* *Prinzipien der Geometrie* ['Principles of geometry'], Kazan, 1829–30.

the surface to which Euclidean planimetric geometry refers as a plane, as opposed to all other two-dimensional spatial structures.

The distinction between geometry in a plane and on a spherical surface has been clear and intuitive for a long time. But the sense of the axiom of parallels could only be understood after Gauss had developed the concept of surfaces which can be bent without stretching, and therefore of the possibility of extending pseudospherical surfaces infinitely. We of course, being inhabitants of a three-dimensional space and endowed with sense organs for perceiving all of these dimensions, can intuitively imagine the various cases in which surface-like beings would have to develop their spatial intuition, because to this end we only have to restrict our own intuitions to a narrower domain. It is easy to think away intuitions which one has. But to imagine with our senses intuitions for which one has never had any analogue, is very difficult. So when we go over to three-dimensional space, we are impeded in our powers of imagination by the structure of our organs and the experiences thereby acquired, which are only appropriate to the space in which we live.

However, we have also another way of dealing with geometry scientifically. All spatial relationships known to us are namely measurable, meaning that they can be reduced to the specification of magnitudes (lengths of lines, angles, areas, volumes). For just this reason, the problems of geometry can also be solved by looking for the methods of calculation whereby the unknown spatial magnitudes have to be derived from the known ones. This occurs in *analytic geometry*, where all of the structures of space are only treated as magnitudes and specified by means of other magnitudes. Our axioms themselves also speak of spatial magnitudes[21]. The straight line is defined as the *shortest* between two points – in this one is specifying a magnitude. The axiom of parallels declares that if two straight lines in the same plane do not intersect (are parallel), the corresponding angles, or the alternate angles, made by a third line intersecting them are pairwise equal[†]. Or it is postulated, in place of this, that the sum of the angles any triangle is equal to two right angles. These too are cases of specifying magnitudes.

So one can also start from this aspect of the concept of space, according to which the situation of any point can be specified by measuring certain

† [A theorem of Euclid closely related to his postulate of parallels.]

magnitudes with respect to some spatial structure which is regarded as fixed (coordinate system). One can then look and see what particular specifications accrue to our space as it manifests itself in the actual measurements to be executed, and whether there are such by which it is distinguished from magnitudes of extension of similar manifold. This approach was first initiated in Göttingen by Riemann*, who was sadly lost too early to science. It has the peculiar advantage, that all of its operations consist purely in specifying magnitudes by calculation, whereby the danger is completely obviated that familiar, intuitive facts might insinuate themselves as necessities of thought.

The number of measurements needed for determining the position of a point is equal to the total number of dimensions of the space concerned. In a line the distance from one fixed point, thus one magnitude, suffices. In a surface one must already give the distances from two fixed points in order to fix the position of the point. In space one must give those from three, so that we need, as on the earth, longitude, latitude and height above sea-level, or, as is usual in analytic geometry, the distances from three coordinate planes. Riemann terms a system of differences in which the individual can be specified by n measurements, a *n-fold extended manifold*, or a *manifold of n dimensions*. Thus the space known to us, in which we live, is in these terms a threefold extended manifold of points. A surface is a twofold one, a line a onefold one, and time equally a onefold one. The colour system too consists of a threefold manifold, inasmuch as every colour, according to Thomas Young's and Maxwell's** investigations, can be described as a mixture of three basic colours, of each of which a definite amount is to be used. Using the coloured spinning top, one can actually carry out such mixtures and measurements.

We could equally regard the realm of simple tones*** as a manifold of two dimensions, if we just distinguished them by pitch and intensity and disregarded differences of timbre[23]. This generalisation of the concept is very suitable for bringing out what distinguishes space from other

* 'Über die Hypothesen, welche der Geometrie zu Grunde liegen' ['On the hypotheses underlying geometry'], inaugural dissertation of 10 June 1854. Published in vol. XIII of the *Abhandlungen der königlichen Gesellschaft zu Göttingen*[22].

** See [Helmholtz'] *Vorträge und Reden*, vol. I, p. 307.

*** *Vorträge und Reden*, vol. I, p. 141.

manifolds of three dimensions[24]. In space, as you all know from everyday experience, we can compare the distance between two points lying one above the other with the horizontal distance between two points on the floor, because we can apply one measuring rod alternately to the one pair and to the other. But we cannot compare the distance between two tones of equal pitch and different intensity with that between two tones of equal intensity and different pitch.

By considerations of this kind, Riemann showed that the essential basis of any geometry is the expression giving the separation between two points for any arbitrary direction of separation, and to be exact that in the first place between two infinitesimally separated points. From analytic geometry, he obtained the most general form* which this expression takes for a completely arbitrary choice of the kind of measurements yielding the place of each point. He then showed that the kind of freedom of motion without distortion which characterises bodies in our space, can only exist if certain magnitudes** emerging from the calculation have the same value everywhere. In respect of the relationships in surfaces, these magnitudes reduce to the Gaussian measure of curvature. For this very reason, when these calculated magnitudes have the same value in all directions from a given location, Riemann terms them the measure of curvature, at that location, of the space concerned.

To prevent misunderstandings, I just wish also to emphasize here that this so-called measure of curvature of space is a calculated magnitude obtained in a purely analytic way, and that its introduction in no way rests upon insinuating relationships which would only have a sense as ones intuited by the senses. The name, as a brief expression for denoting a complicated relationship, has simply been adopted from the one case where the denoted magnitude corresponds to something intuited by the senses[25].

If moreover this measure of curvature of space has everywhere the value zero, such a space corresponds everywhere to the axioms of Euclid. In this case we can term it a *flat* space, in contrast with other

* Namely, for the square of the distance between two infinitely close points a homogeneous second degree function of the differentials of their coordinates.

** It is an algebraic expression which is constructed out of the coefficients of the individual terms in the expression for the square of the separation between two neighbouring points and their derivatives.

analytically constructible spaces which one might term curved, because their measure of curvature has a value different from zero. At the same time, the analytic geometry of spaces of the latter kind can be worked out as completely and self-consistently, as the customary geometry of our in fact existing[26] flat space. For a positive measure of curvature we obtain *spherical* space, in which straightest lines return upon themselves and in which there are no parallel lines. Such a space would, like the surface of a sphere, be unbounded but not infinitely large. On the other hand, a constant negative measure of curvature gives *pseudospherical* space, in which straightest lines go on to infinity, and where in any flattest surface one can find a pencil of straightest lines through any point which do not intersect some other given straightest line of that surface[27].

This latter situation has been made intuitively accessible by Beltrami*, through his showing how one can form an image of the points, lines and surfaces of a three-dimensional pseudospherical space in the interior of a sphere of Euclidean space, and in such a way that every straightest line of the pseudospherical space is represented in the sphere by a straight line, and every flattest surface of the former by a plane in the latter. The surface itself of the sphere corresponds here to the infinitely distant points of the pseudospherical space. In their image in the sphere, the various parts of that space become progressively smaller as they approach the surface of the sphere, and indeed more strongly in the direction of its radii than in directions perpendicular to them. Straight lines in the sphere which intersect only outside its surface, correspond to straightest lines of the pseudospherical space which never intersect.

It thus emerged that space, considered as a domain of measurable magnitudes, by no means corresponds to the most general concept of a three-dimensional manifold. Instead, it contains further particular specifications, which are occasioned by the completely free mobility without distortion of fixed bodies to every place and with every possible change of direction. And occasioned too by the particular value of the measure of curvature, which is to be set equal to zero for the in fact existing space, or whose value is at least not noticeably distinct from zero. This

* 'Teoria Fondamentale degli Spazii di Curvatura Costante' (*op. cit.*).

last stipulation is given in the axioms of straight lines and of parallels.

Riemann entered this new field by starting from the most general basic questions of analytic geometry. I myself had meanwhile arrived at similar considerations, partly through investigations into portraying the colour system spatially, thus through comparing one threefold extended manifold with another, and partly through investigations into the origin of our visual estimation for measurements in the visual field. Riemann started out from the above mentioned algebraic expression, which describes in the most general form the separation between two infinitely close points, as his basic assumption, and he derived from it the theorem on the mobility of fixed spatial structures, whereas I started out from the observational fact[28], that in our space the motion of fixed spatial structures is possible with that degree of freedom with which we are acquainted, and I derived from this fact the necessity of the algebraic expression which Riemann set down as an axiom. The following were the assumptions on which I had to base the calculation[29].

It must *firstly* be presupposed – in order to make treatment by calculation possible at all – that the situation of any point A can be specified by measuring certain spatial magnitudes, whether lines or angles between lines or angles between surfaces or whatever else, with respect to certain spatial structures which are regarded as unchangeable and fixed. As is well known, one terms the measurements needed for specifying the situation of the point A its *coordinates*. The total number of coordinates needed in general for specifying completely the situation of any point whatever, specifies the total number of dimensions of the space concerned. It is further presupposed that the spatial magnitudes used as coordinates change continuously as the point A moves.

One must *secondly* give such a definition of a fixed body, or fixed point system, as is needed to enable comparison of spatial magnitudes by congruence. As we should not yet presuppose here any special methods of measuring[30] spatial magnitudes, the definition of a fixed body can now only be given by the following characteristic: there must exist, between the coordinates of any two points belonging to a fixed body, an equation which expresses an unchanged spatial relationship between the two points (which finally turns out to be their separation) for any motion of the body, and one which is the same for congruent point pairs, while point pairs are congruent, if they can successively coincide with the same point pair fixed in space[31].

This definition has extremely far-reaching implications, although its formulation seems so incompletely specific, because with an increasing number of points the total number of equations grow much faster than the number of the coordinates (of these points) that they determine. Five points, *A*, *B*, *C*, *D* and *E*, give ten different point pairs:

$$\begin{array}{rrrr} AB, & AC, & AD, & AE, \\ & BC, & BD, & BE, \\ & & CD, & CE, \\ & & & DE, \end{array}$$

and so ten equations, which in three-dimensional space contain fifteen variable coordinates. Of these, however, six must remain freely disposable if the system of five points is to be freely movable and rotatable. Thus the ten equations are to determine only nine coordinates, as dependent upon the six variable ones. With six points we get fifteen equations for twelve variable magnitudes, with seven points twenty-one equations for fifteen magnitudes, and so on.

However, from n mutually independent equations we can determine n magnitudes appearing in them. If we have more than n equations, the surplus ones must themselves be derivable from the first n of them. From this it follows that the equations which exist between the coordinates of each point pair of a fixed body must be of a particular kind, namely such that if they are satisfied in three-dimensional space for nine of the point pairs formed from any five points, the equation for the tenth pair must follow from them identically. It is in virtue of this circumstance, that the stated assumption for the definition of fixity is indeed sufficient in order to specify the kind of equations which exist between the coordinates of two fixedly interconnected points.

It resulted *thirdly* that the calculation had to be based on the fact of yet one more particular peculiarity of the motion of fixed bodies, a peculiarity so familiar to us, that without this investigation we might never have chanced to regard it as something that might also not be so. If we namely hold fixed two points of a fixed body in our three-dimensional space, it can then only rotate about the straight line connecting them as axis of rotation. If we rotate it once right round, it comes back exactly to the situation in which it was initially. Now it has to be specifically stated, that rotation of any fixed body without reversal always brings it back to its initial situation [32]. There might be a geometry where this was not so. This is to be seen most simply for the geometry of the plane. If one thinks of every plane figure increasing its linear dimensions under any rotation in proportion to the angle of rotation, then after a whole rotation through 360 degrees the figure would no longer be congruent with its initial state. Yet of course any other figure which was congruent with it in its initial situation could also be made congruent with it in its second situation, if the second figure too were rotated through 360 degrees. A self-consistent system of geometry, not falling within Riemann's scheme, would be possible under this assumption too.

I have shown, on the other hand, that the enumerated three assumptions are together sufficient in order to establish the starting point assumed by Riemann in the investigation. With it are also established all further results of his work which concern the distinction of the various spaces according to measure of curvature.

One might yet ask whether the laws of motion, and of its dependence on motive forces, can also be carried over without contradiction to spherical or pseudospherical space. This investigation has been carried out by Herr Lipschitz* in Bonn. One can actually carry over the comprehensive expression of all laws of dynamics, namely Hamilton's Principle[33], directly to spaces whose measures of curvature are not equal to zero. Thus the deviating systems of geometry are not subject to contradiction in this respect either.

We shall now have to go on and ask for the origin of these particular specifications [which characterise our space as a flat space]†, since it has emerged that they are not included in the general concept of an extended three-dimensional magnitude and of the free mobility of the bounded structures contained in it. They are not *necessities of thought*[34], deriving from the concept of such a manifold and its measurability, or from the most general concept of a fixed structure contained in it and of the freest mobility of the latter.

We now wish to investigate the contrary assumption that may be

* 'Untersuchungen über die ganzen homogenen Funktionen von *n* Differentialen' ['Investigations on total homogeneous functions of *n* differentials'], *Borchardts Journal für Mathematik* **70**, 71 and **72**, 1. 'Untersuchung eines Problems der Variationsrechnung' ['Investigation of a problem in the calculus of variations'], *ibid.*, **74**.

† [Hertz and Schlick appear to have omitted this clause by mistake.]

made about their origin, namely the question of whether their origin is *empirical*, whether they are to be derived or made evident from facts of experience, or correspondingly to be tested and perhaps even refuted by these[35]. This latter eventuality would then include too, that we should be able to imagine series of observable facts of experience by which another value of the measure of curvature would be indicated, than that of Euclid's flat space. If indeed spaces of another kind are imaginable in the sense stated, this will also refute the claim that the axioms of geometry are in Kant's sense necessary consequences of a transcendental form, given *a priori*, of our intuitions[36].

As noted above, the distinction between Euclidean, spherical and pseudospherical geometry rests upon the value of a certain constant, which Riemann terms the measure of curvature of the space concerned, and whose value must be equal to zero if Euclid's axioms hold. If it is not equal to zero, triangles of large area will necessarily have a different sum of angles from small ones, the former having a larger sum in spherical and a smaller one in pseudospherical space. Moreover, geometrical similarity between larger and smaller bodies or figures is only possible in Euclidean space. All systems of practically executed geometrical measurements in which the three angles of large rectilinear triangles have been measured individually, and so in particular all systems of astronomical measurements which yield zero for the parallax[37] of immeasurably distant fixed stars (in pseudospherical space even the infinitely distant points must have positive parallax), confirm the axiom of parallels empirically[38]. They show that in our space and using our methods of measurement, the measure of curvature of space appears to be indistinguishable from zero. One must of course raise, with Riemann, the question of whether this perhaps might not be different if instead of our limited base lines, of which the greatest is the major axis of the earth's orbit, we could use greater base lines.

But we should not forget here, that all geometrical measurements rest ultimately on the principle of congruence[39]. We measure separations between points by moving a pair of dividers or measuring rod or measuring chain up to them. We measure angles by bringing a protractor or theodolite up to the vertex of the angle. We determine straight lines, additionally, by the path of light rays too, which in our experience is rectilinear; but it could also be carried over equally to spaces of other measures of curvature that light, as long as it stays in an unchanging refracting medium, is propagated along shortest lines. So all our geometrical measurements rest upon the presupposition that the measuring instruments which we take to be fixed, actually[40] are bodies of unchanging form. Or that they at least undergo no kinds of distortion other than those which we know, such as those of temperature change, or the small extensions which ensue from the different effect of gravity in a changed location.

When we measure, we are only doing, with the best and most reliable auxiliary means known to us, the same thing as what we otherwise ordinarily ascertain through observation by visual estimation and touch, or through pacing something off. In the latter cases, our own body with its organs is the measuring instrument which we carry around in space. At different moments our hand or our legs are our dividers, or our eye turning in all directions is the theodolite with which we measure arcs or plane angles in the visual field.

Thus any comparative estimation of magnitudes, or measurement of spatial relationships, starts from a presupposition about the physical behaviour of certain bodies, whether of our own body or of applied measuring instruments. This presupposition may incidentally have the highest degree of probability[41] and be in the best agreement with all physical circumstances otherwise known to us, but it still goes beyond the domain of pure spatial intuitions.

One indeed can instance a certain behaviour[42] of the bodies that appear fixed to us, such that measurements in Euclidean space would turn out as if they had been performed in pseudospherical or spherical space. To see this[43], I will point out first that if the whole of the linear dimensions of the bodies surrounding us, and with them those of our own body, were all decreased in the same proportion – say by half, or say they were all increased to double their size – we would be quite unable to notice such a change with our means of spatial intuition. The same would moreover also be the case if the extension or contraction were different in different directions, provided that our own body changed in the same way, and provided further that a rotating body adopted, without undergoing or exercising mechanical resistance, at every moment the degree of extension of its various dimensions that corresponded to its momentary situation[44].

Think of the image of the world in a convex mirror. The familiar silvered globes that are commonly set up in gardens display the essential phenomena of such an image, if indeed distorted by some optical irregularities. A well-made convex mirror of not too great an aperture displays, in a definite situation and at a definite distance behind its surface, the apparently corporeal image of any object lying in front of it. But the images of the distant horizon and of the sun in the sky lie behind the mirror at a limited distance, equal to its focal length. The images of all other objects lying in front of it are contained between those images and its surface, but such that the images are progressively more diminished and flattened with increasing distance of their objects from the mirror. The flattening, in other words the diminution in the dimension of depth, is relatively more pronounced than the diminution in the areal dimensions. Nonetheless, every straight line of the outer world is portrayed by a straight line in the image, and every plane by a plane. The image of a man measuring, with a measuring rod, a straight line stretching away from the mirror, would progressively shrivel up as its original moved away. But the man in the image would count up, with his equally shrivelled measuring rod, exactly the same number of centimeters as the man in actuality. In general, all geometrical measurements of lines or angles carried out with the regularly changing mirror images of the actual instruments, would yield exactly the same results as those in the outer world. All congruences would match in the images, if the bodies concerned were actually laid against each other, just as in the outer world. All lines of sight in the outer world would be replaced by straight lines of sight in the mirror.

In short, I do not see how the men in the mirror could bring it out that their bodies were not fixed bodies and their experiences [not] good examples of the correctness of Euclid's axioms. But if they could look out into our world, as we look into theirs, without being able to cross the boundary, then they would have to pronounce our world to be the image of a convex mirror, and speak of us just as we of them. And as far as I can see, if the men of the two worlds could converse together, then neither would be able to convince the other that he had the true and the other the distorted situation. I cannot even recognise that such a question has at all any sense[45], as long as we introduce no mechanical considerations[46].

Now Beltrami's mapping[47] of pseudospherical space onto a whole sphere of Euclidean space is of a quite similar kind, except that the background surface is not a plane, as with the convex mirror, but the surface of a sphere, and that the proportion in which the images contract, as they approach the surface of the sphere, has a different mathematical expression. Suppose one therefore thinks conversely of bodies which move in the sphere in whose interior Euclid's axioms hold, and which always contract, like the images in the convex mirror, when moving away from the centre, and which moreover contract in such a way that their images constructed in pseudospherical space preserve unchanged dimensions. Then observers whose own bodies regularly underwent this change would obtain results from geometrical measurements, made as they could make them, as if they themselves lived in pseudospherical space.

Starting from here we can go yet one step further. We can deduce how the objects of a pseudospherical world would appear to an observer, whose visual estimation and spatial experiences had, exactly as ours, been developed in flat space, if he could enter such a world. Such an observer would continue to see the lines of light rays, or the lines of sight of his eyes, as straight lines like those existing in flat space, and like they actually are in the spherical image of pseudospherical space. The visual image of the objects in pseudospherical space would therefore give him the same impression as if he were at the centre of Beltrami's spherical image[48]. He would believe he could see all round himself the most distant objects of this space at a finite distance*, let us for example assume at a distance of 100 feet. But if he approached these distant objects they would expand in front of him, and indeed more in depth than in area, while behind him they would contract. He would discern that he had judged by visual estimation falsely.

Suppose he saw two straight lines which, in his estimation, stretched away parallel for this whole distance of 100 feet, to where the world seemed to end for him. On pursuing them he would then discern that, together with this expansion of the objects he approached, these lines would spread apart the more he advanced along them. Behind him, on the other hand, the distance between them would seem to dwindle, so that as he advanced they would seem to be progressively more divergent

* The negative inverse square of this distance would be the measure of curvature of the pseudospherical space.

and separated from each other. But two straight lines which seemed, from the first viewpoint, to converge to one and the same point in the background at a distance of 100 feet, would always do this however far he went, and he would never reach their point of intersection.

Now we can obtain quite similar pictures of our actual world, if we put before our eyes a large convex lens of corresponding negative focal length[†], or indeed only two convex eye glasses, which would have to be ground slightly prismatically as if they were parts of a larger, connected lens. These, just like the convex mirror mentioned above, show us distant objects brought up closer, the most distant ones up to the distance of the focal point of the lens. If we move with such a lens before our eyes, there take place quite similar expansions of the objects we approach to those I have described for pseudospherical space. If, moreover, someone puts such a lens before his eyes, and not even a lens of focal length 100 feet, but a much stronger one of focal length only 60 inches, then perhaps for the first moment he notices that he sees objects brought up closer. But after a little going back and forth the illusion fades, and he judges distances correctly despite the false images. We all have grounds to surmise that it would soon enough be the same for us in pseudospherical space, as it is after just a few hours for someone who begins to wear glasses. In short, pseudospherical space would seem to us to be relatively not very strange at all. Only at first would we find ourselves subject to illusions, in measuring the magnitude and distance of more distant objects by their visual impression.

A three-dimensional spherical space would be accompanied by the opposite illusions, if we entered it with the visual estimation acquired in Euclidean space. We would take more distant objects to be further off and larger than they were. On approaching them, we would find ourselves reaching them more quickly than we should assume from their visual image. However, we would also see before us objects which we could only fixate with diverging lines of sight: this would be the case with all those more distant from us than a quadrant of a great circle. This kind of view would hardly strike us as very unusual, since we can produce it for earthly objects too, by putting before one eye a weakly prismatic glass whose thicker side is turned towards the nose. Then too

[†] [i.e. a concave lens.]

we have to set our eyes divergently in order to fixate distant objects. It produces in the eyes a certain feeling of unaccustomed strain, but does not notably change the visual features of objects seen thus, whereas the strangest visual feature of the spherical world would consist of the back of our own head, upon which all of our lines of sight would reconverge, to the extent that they could pass freely between other objects, and which would have to fill up the furthest background of the whole perspective image.

At the same time, the following is of course also to be noted. Just as a small flat elastic plate, say a small flat rubber disk, can only be accommodated to a weakly curved spherical surface with a relative contraction of its edge and stretching of its middle, so also our own body, which has grown up in Euclidean flat space, could not go over into a curved space without undergoing similar stretchings and compressions of its parts. Naturally, the interconnexion of these parts could only be maintained as long as their elasticity allowed yielding without tearing and breaking. The kind of stretching would have to be the same as if we thought of a small body at the centre of Beltrami's sphere, and then transposed from it to its pseudospherical or spherical image. For such a transposition to appear possible, it must always be presupposed that the body transposed is sufficiently elastic and small, in comparison with the real or imaginary radius of curvature of the curved space into which it is to be transposed.

This will suffice to show how one can deduce from the known laws of our sense perceptions, continuing in the way begun here, the series of sense impressions which a spherical or pseudospherical world would give us if it existed. In this respect too we nowhere meet an impossibility or deductive fault, just as little as in the calculative treatment of the metrical relationships. We can just as well depict the view in all directions, in a pseudospherical world, as we can develop its concept. For this reason, we also cannot admit that the axioms of our geometry are based upon the given form of our faculty of intuition, or are connected with such a form in any way[49].

It is otherwise with the three dimensions of space. All our means of intuition by the senses only stretch to a space of three dimensions, and the fourth dimension would not be a mere modification of what exists, but something completely new. Thus if only on account of our bodily makeup, we find ourselves absolutely unable to imagine a way of intuitively conceiving a fourth dimension[50].

I wish finally to stress further, that the axioms of geometry are certainly not propositions belonging to the pure theory of space alone[51]. As I have already mentioned, they speak of magnitudes. One can only talk of magnitudes if one knows and intends some procedure, whereby one can compare these magnitudes, split them up into parts and measure them. Thus all spatial measurement, and therefore in general all magnitude concepts applied to space, presuppose the possibility of the motion of spatial structures whose form and magnitude one may take to be unchanging despite the motion. In geometry, such spatial forms are indeed by custom referred to only as geometrical bodies, surfaces, angles and lines, because one abstracts from all other distinctions of a physical and chemical kind manifested by natural bodies. But one retains nonetheless one of their physical properties, namely fixity. Now we have no criterion for the fixity of bodies and spatial structures other than that when applied to one another at any time, in any place and after any rotation, they always show again the same congruences as before. But we certainly cannot decide in a purely geometrical way, without bringing in mechanical considerations[52], whether the bodies applied to each other have not themselves both changed in the same manner.

If we ever found it useful for some purpose, we could, in a completely logical way, regard the space in which we live as the apparent space behind a convex mirror with its shortened and contracted background. Or we could regard a bounded sphere of our space, beyond whose bounds we perceive nothing more, as infinite pseudospherical space. We would then only have to ascribe the corresponding stretchings and shortenings to the bodies which appear to us to be fixed, and equally to our own body at the same time. We would of course at the same time have to change our system of mechanical principles completely. For even the proposition, that every moving point upon which no force acts continues to move in a straight line with unchanged velocity, no longer applies to the image of the world in the convex mirror. The path would indeed still be straight, but the velocity dependent upon place[53].

Thus the axioms of geometry certainly do not speak of spatial relationships alone, but also, at the same time, of the mechanical behaviour of our most fixed bodies during motions[54]. One could admittedly also take the concept of fixed geometrical spatial structures to be a transcendental[55] concept, which is formed independent of actual experiences and to which these need not necessarily correspond, as in fact our natural bodies are already not even in wholly pure and undistorted correspondence to those concepts which we have abstracted from them by way of induction. By adopting such a concept of fixity, conceived only as an ideal, a strict Kantian certainly could then regard the axioms of geometry as propositions given *a priori* through transcendental intuition, ones which could be neither confirmed nor refuted by any experience, because one would have to decide according to them alone whether any particular natural bodies were to be regarded as fixed bodies. But we would then have to maintain that according to this conception, the axioms of geometry would certainly not be synthetic propositions in Kant's sense. For they would then only assert something which followed analytically from the concept of the fixed geometrical structures necessary for measurement, since only structures satisfying those axioms could be acknowledged to be fixed ones.

But suppose we further add propositions concerning the mechanical properties of natural bodies to the axioms of geometry, if only the law of inertia, or the proposition that the mechanical and physical properties of bodies cannot, under otherwise constant influences, depend upon the place where they are. Then such a system of propositions is given an actual content, which can be confirmed or refuted by experience, but which for just that reason can also be obtained by experience[56].

Incidentally, I naturally do not intend to maintain that humankind only obtained intuitions of space, corresponding to Euclid's axioms, by carefully executed systems of exact geometrical measurements. A series of everyday experiences, and especially the intuition of the geometrical similarity of larger and smaller bodies – which is only possible in flat space – must rather have led to the rejection as impossible of every geometrical intuition opposed to this fact. For this no knowledge was needed of the conceptual connexion between the observed fact of geometrical similarity and the axioms, but only an intuitive acquaintance with the typical behaviour of spatial relationships, obtained by observing them frequently and exactly, namely the kind of intuition of objects to be portrayed which the artist has, and by means of which he decides, with assurance and refinement, whether or not an attempted new combination corresponds to the nature of the object to be portrayed. We indeed know of no other name in our language to refer to this but 'intuition', but it is an empirical acquaintance, obtained by the accumulation and reinforcement in our memory of impressions which recur in the same manner. It is no transcendental form of intuition given before all experience.

That this sort of intuition of a typical lawlike occurrence, gained empirically and not yet developed to the clarity of a definitely expressed concept, has often enough imposed itself upon metaphysicians as a proposition given *a priori*, is a matter which I need not discuss further here.

NOTES AND COMMENTS

1 In logic one understands by 'deduction' the derivation of a judgment from more comprehensive judgments, i.e. from ones having more general validity. It is the only procedure giving a completely *rigorous* foundation for a truth by means of other truths. The most usual form of deduction is the syllogism (compare note IV.43)†. Deduction stands in opposition to logical "induction", which endeavours to infer more generally valid truths from particular and individual ones, and which, to this end, must submit to the "troublesome and tedious task" of gathering empirical facts, of which Helmholtz speaks in this same sentence. Inductive inference lacks absolute certainty, for inasmuch as it extends, by generalization, a proposition extracted from a number of individual cases to cases that have not yet been observed††, it goes beyond the actual content of the presuppositions. Thus propositions obtained by induction can only claim to hold with *probability* (though with frequently an extremely high one).

2 In this question Kant summarizes the problem of his *Kritik der reinen Vernunft* [Critique of Pure Reason].

By a 'synthetic judgment' he understands a proposition which attributes to an object a predicate that does not belong to the object already in virtue of its definition. As against this, an 'analytic judgment' only asserts of a subject something that is already contained in this subject by definition. If, for instance, by a 'body' I always understand something extended, then the proposition 'all bodies are extended' is evidently analytic, whereas we would have a synthetic judgment if we said 'all bodies are subject on the surface of the earth to an acceleration of about 981 cm/sec^2'. The first judgment follows simply from the definition, from the concept of a body. The second cannot be derived from this, but only experience can teach us whether, and with what acceleration, the objects characterised by the concept 'body' fall towards the earth.

A judgment whose validity is assured wholly independent of any experience is termed by Kant '*a priori*'; if its validity rests only upon experience, it is called '*a posteriori*'. It follows from what has been said that every analytic judgment is *a priori*, for indeed one needs no experience to perceive its validity, but only an analysis of the concept of its

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† [In modern logic the syllogism no longer has a position of any prominence; its significance is now historical as the first fragment of logic to have been analyzed with full logical rigour.]

†† [This is not the only kind of inductive inference: see R. Carnap, *Logical Foundations of Probability*, 2nd ed., 1962, §44B. The whole subject is still controversial.]

subject, and this concept is completely given by its definition. It also follows that most synthetic judgments, both in everyday life and in science, are *a posteriori*, because they express the results of experience.

Are there also synthetic *a priori* judgments, thus in other words propositions which assert *more* about an object than already exists in its concept, yet without drawing this 'more' from experience? One easily recognizes the enormous importance of the question. For only in synthetic judgments is there a real advance of knowledge; only they extend our knowledge, while analytic ones merely elucidate what we have put into our concepts by definition. But if all synthetic judgments are only *a posteriori*, then no advance of knowledge occurs with absolutely certain validity, since a validity resting upon items of experience reaches no further than these very items themselves and may be overthrown by new observations.

We are only certain of the universal, necessary validity of a proposition if it is valid *a priori*. Necessity, and validity without exception, are therefore the characteristic by which the *a priori* is recognized. According to Kant, this characteristic attaches to the axioms of geometry, and he also, as noted by Helmholtz in the text, does not doubt that they are synthetic. Thus he believed that synthetic *a priori* judgments, this highest form of knowledge, indeed exist, and the only question for him was: how are they possible? How can I with certainty assert of an object something which is neither deduced from its definition nor drawn from experience? Helmholtz' investigation, as will emerge, examines the question as to whether the axioms of geometry actually should, as Kant presupposed, be regarded as synthetic *a priori* judgments.

3 In taking space to be a 'form of intuition', and hence a lawlike feature characteristic of our intuiting consciousness, Kant precisely wanted to explain the possibility of synthetic *a priori* judgments about space. Everything that we can perceive and intuitively imagine must necessarily be subject to the laws governing the manner in which we intuit. Thus these laws, according to Kant, must be valid *a priori* for everything that can be experienced.

4 "...schema devoid of any content" – this will later need comment in greater detail (compare note IV.33).

5 The definition of geometry as 'the theory of space' suffices at this point, although it raises several questions – especially whether one can state more precisely what is to be understood here by 'space'. It could, for instance, be that the mathematician, the physicist and the psychologist do not at all mean the same thing when speaking of 'space'. Helmholtz' investigation will also lead of itself to this question.

6 This list of 'axioms' is clearly to be taken as only a collection of examples, intended to indicate what is involved. Helmholtz clearly does not want to say here that precisely these propositions, and in exactly this formulation, must appear as axioms in a self-contained system of geometry.

7 The great successes of the method of logical inference in mathematics explain the wish to give a proof for everything, and at the same time the belief in the success of such an undertaking (an endeavour to which, in the field of philosophy, the systems of Fichte and Reinhold owe their origin; both tried to derive a whole system from *one* single proposition). However, the simple consideration that every inference needs premises shows the

impossibility of proving everything, and the necessity of making presuppositions. Such starting points of logical thought are the axioms, whose origin is precisely to be investigated here.

[8] The prototype of the idealist philosopher is Plato. He explains the self-evidence of the axioms of geometry by the hypothesis of '*anamnesis*' or 'recollection'. In earlier stages of its existence, before its earthly birth, the human soul supposedly became acquainted with the truths of mathematics through immediate contemplation, and it now knows about them without earthly experience, by mere recollection.

It perhaps deserves to be mentioned that in this Platonic doctrine the explanation is, properly speaking, still based on experience, even though not on that of the senses which belongs to earthly life, but on an entirely different kind of experience during a mythical pre-existence.

[9] This of course does not yet provide a rigorous proof of the fruitlessness of the attempts. The impossibility of deriving a proposition from certain axioms (i.e. the "independence" of this proposition from those axioms, which requires its being placed with the other axioms as a new one) is in general proved by dropping the proposition in question – or replacing it with another – while retaining all the other axioms, and testing the system of axioms which results for freedom from contradiction. Thus, for instance, the independence of the axiom of parallels from the other Euclidean axioms follows from the fact, that by giving it up we variously arrive at the geometrical system of Bolyai and Lobatchewsky or at that of Riemann. Now one can prove that these two non-Euclidean geometries are free from internal contradiction, *provided* that Euclidean geometry is. The freedom of the latter from contradiction (which was probably never in doubt) can be proved if one presupposes that the structure of number theory – arithmetic – is free from contradiction. To date nobody seems to have succeeded in rigorously demonstrating the correctness of this presupposition[†].

[10] The Greek mathematician Euclid compiled, around 300 B.C., the mathematical knowledge of his time in a text-book of such excellence that it occasionally (e.g. in English schools) serves as the basis for teaching geometry even to this day.

[11] In talking, as Helmholtz does here, of 'movement' as something that can in fact be experienced, as an actual process, one must by 'geometrical structures' understand here rigid corporeal models, and not purely mathematical lines or surfaces as mere structures of abstraction. There arises the important question of to what extent a distinction can be made at all between 'geometrical' and 'corporeal' structures, thus the problem of the relation of geometry to physics, on which Helmholtz, towards the end of the lecture, quite specifically adopts a position.

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† A number of proofs have been given for the consistency of the arithmetic of the natural numbers, starting with G. Gentzen, 'Die Wiederspruchsfreiheit der reinen Zahlentheorie' ['The freedom from contradiction of pure number theory'], *Math. Annalen*, CXII (1936), 493–565. However, all of these proofs have the methodological drawback that they can be formulated only in a mathematical form of language (e.g. mathematical and even transfinite induction are used). There are moreover theoretical reasons for supposing that this drawback is insuperable. See Elliott Mendelson, *Introduction to Mathematical Logic*, Van Nostrand, Princeton, 1964, Appendix and pp. 148–9.

12 It should be noted that the conclusion drawn by Helmholtz in the last sentence is only admissible, if it is true that every presupposition must either have a logical foundation or have originated in experience, and thus cannot have come from a third source.

13 'Analytic' geometry reduces all proofs to calculations through denoting each point of space by three numbers (coordinates), for instance by giving its distances from three mutually perpendicular planes (compare p. 13 of the text). In this way it becomes possible to treat, in a purely calculative manner, the relations between spatial structures as numerical relations (namely between the coordinates of their points).

14 'purely logical operation', i.e. deduction (compare note 1).

15 As it is appropriate to restrict the term 'to think' to the logical operations of judgment and inference, the first of the two formulations, 'to imagine', is doubtless to be preferred. The sense of the explanation given by Helmholtz in this sentence can hardly be misunderstood in its simplicity. 'Imaginable' means everything that is intuitively reproducible in the sense of psychology. Although a certain subjectivity or relativity is inherent in this definition, due to individual differences of imaginative capacity, it should not affect the *basic* meaning of this delimitation of the concept 'imaginable' (compare note IV.42).

16 The concept of the 'straightest' path also plays a decisive role in the mechanics of Heinrich Hertz.

17 Thus J. Wallis (1614–1703) substituted for the Euclidean axiom of parallels the proposition: for any figure there exists a similar one of arbitrary magnitude. See Bonola-Liebmann, *Die nichteuclidische Geometrie* ['Non-Euclidean geometry'], Leipzig, 1908, pp. 14f.

18 D. Hilbert (*Grundlagen der Geometrie* ['Foundations of geometry'], 3rd ed., Leipzig and Berlin, 1909, p. 251) has proved that "there does not exist an analytic surface of constant negative curvature which is without singularities and everywhere regular. Thus in particular, the question of whether one can realize in Beltrami's manner the whole of the Lobatchewskian plane by means of a regular analytic surface in space, must also be answered in the negative." But this mathematical result, to which Riehl also refers (*Helmholtz in seinem Verhältnis zu Kant* ['Helmholtz in his relation to Kant'], 1904, p. 37), is not of fundamental significance for the basic epistemological thought behind Helmholtz' lecture.

19 ...and the radii of curvature are therefore taken to have opposite signs.

20 'Euclid's synthetic method' is the same as what Helmholtz on p. 4 called the 'intuitive method'.

21 Namely the axioms of our ordinary metric geometry. One can also think of a geometric system containing no concepts of magnitude at all. Such is *analysis situs*, which deals exclusively with those properties of spatial structures which contain no *metric* concepts. For this reason Poincaré terms *analysis situs* the true "*qualitative* geometry" (*Der Wert der Wissenschaft* ['The value of science'], 2nd ed., 1910, p. 49).

22 Riemann's dissertation has recently been published with detailed comments by H. Weyl, Berlin, 1919, 2nd ed., 1921.

[23] Thus in the manifold of colours, the three amounts of the basic colours needed for mixing a specific colour would have to be regarded as the three 'coordinates' of the latter (see notes II.8, III.29); in the manifold of tones, the intensity and pitch of each tone would be its two coordinates.

[24] The distinguishing characteristic stressed by Helmholtz in the following is not sufficient, though indeed necessary, to specify what is proper to 'space' as against other three-dimensional manifolds. To what extent can one at all speak of a 'definition' of space? What spatial extension is in the psychological-intuitive sense, can only become known through perceptual experiences; it is just as little definable, as it is possible to give a person born blind an idea of what 'red' is. Only those properties of space are to be regarded as definable which are accessible to mathematical analysis. But these are the sole properties that physics has to make use of, and from this one can draw important conclusions.

[25] One can hardly draw attention emphatically enough to what Helmholtz is stressing here: the expression 'measure of curvature' in its application to space should only ever be understood in a metaphorical sense, it is not intended to denote anything that we can in any way intuitively perceive or imagine as a curvature. A typical example of this widespread misunderstanding is found in the polemic *Kant und Helmholtz* ['Kant and Helmholtz'] by Albrecht Krause, Lahr, 1878, where it is said on p. 84: "Lines, surfaces, axes of bodies in space have an orientation and thus also a measure of curvature, but space as such has no orientation, precisely because everything oriented is in space, and it therefore has *no* measure of curvature; but this is something other than having a measure of curvature *equal to zero*."

Another misunderstanding, frequently met, is connected with this. Since we are only able to imagine a curved surface in a three-dimensional space, it is namely inferred that a 'curved' three-dimensional manifold presupposes the existence of a four-dimensional one. But this inference from analogy is false. Were it correct, then 'flat' Euclidean space would also have to be imaginable only as embedded in a four-dimensional space: it is namely evident that we also cannot imagine a flat surface otherwise than as in space, since certainly for our imagination it always remains possible to leave the flat surface, just as a curved one, by stepping out on either side into space. The layman too easily forgets that 'curvature' in the Gaussian sense is a wholly *internal* property of the surface, and that a curved surface is just as much a merely two-dimensional structure as a flat one.

Helmholtz says correctly that in the case of a surface an intuition of the senses corresponds to the 'curvature'. But not every surface which is curved for intuition is thus also in the mathematical sense. Notably, the surfaces of e.g. a cone or a cylinder have the Gaussian measure of curvature zero.

[26] Helmholtz speaks of our "in fact existing" space; he never doubted that physical space is not merely a partially arbitrary mental construction, but something actual, whose properties can be ascertained by observation. In what sense this presupposition is founded and justified we shall see from Helmholtz' own statements. Here he describes the in fact existing space as 'flat'; on p. 18 he says further that besides flatness a tiny measure of curvature, not noticeably different from zero, is also compatible with the data of experience.

[27] Further types of space again are elliptical space and Klein-Clifford space; compare note II.30.

[28] Helmholtz terms it an 'observational fact' that motion of fixed spatial structures, i.e. change of place of rigid bodies, is possible. To what extent this should actually count as an observational fact will be discussed later on (note 40).

[29] The mathematical developments whose results are briefly stated here by Helmholtz, will be found in the paper 'On the Facts Underlying Geometry'. For more detailed criticism of Helmholtz' presuppositions – which however is of no importance for the epistemological outcome – we may refer to the comments on that paper.

[30] The characteristic of a fixed body (the unchanging separation between two points in the body) cited by Helmholtz in what follows, gives the basis on which every physical measurement must rest, for in the last analysis a measurement always occurs through repeatedly applying a measuring rod. But a measuring rod is a body on which two (or more) points are marked whose separation is regarded as constant. To 'apply' the measuring rod to an object to be measured means: to bring those two points into coincidence with specific points in the object.

[31] This definition reduces congruence (the equality of two extents*) to the coincidence of point pairs in rigid bodies "with the same point pair fixed in space", and thus presupposes that 'points in space' can be distinguished and held fixed. This presupposition was explicitly made by Helmholtz in the preceding paragraph, but for this he had to presuppose in turn the existence of "certain spatial structures which are regarded as unchangeable and fixed" (p. 15). Unchangeability and fixity (the term 'rigidity' is more usual nowadays) cannot for its own part again be specified with the help of that definition of congruence, for one would otherwise clearly go round in a circle. For this reason the definition seems not to be logically satisfactory.

One escapes the circle only by stipulating by convention that certain bodies are to be regarded as rigid, and one chooses these bodies such that the choice leads to a simplest possible system of describing nature (Poincaré, *Der Wert der Wissenschaft* [*op. cit.*], pp. 44f.) It is easy to find bodies which (if temperature effects and other influences are excluded) fulfil this ideal sufficiently closely in practice. Then congruence can be defined unobjectionably (as by Einstein in *Geometrie und Erfahrung* ['Geometry and Experience'], p. 9) as follows: "We want to term an extent what is embodied by two marks made on a body which is in practice rigid. We think of two practically rigid bodies with an extent marked on each. These two extents shall be called 'equal to each other' if the marks on the one can constantly be brought into coincidence with the marks on the other."

[32] This is the so-called monodromic principle. Compare on this point the second paper in this volume, and in particular the comments upon it.

[33] 'Hamilton's principle' expresses the laws of nature in such a general and comprehensive form that it has also held good for all modern extensions of physical conceptions. According to it, the course of all physical processes is such that a certain function of the magnitudes that determine the momentary state takes a minimum value (in exceptional cases a maximum one) for the transition from the initial to the final state.

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* ['Strecken', i.e. 'stretches' or 'linear extensions'; the same word is used by Einstein in the quotation below.]

[34] 'Necessities of thought'. The results of our thought should only be called *necessary* if they have been obtained by deduction (compare note 1). But since the particular specifications of our in fact existing space cannot be deduced from the mere concept of a three-dimensional manifold, Helmholtz rightly denies that they are necessary for thought. In Kant's terminology this conclusion would have to be formulated by saying: the propositions which assert those particular properties of our space – i.e. the axioms of geometry – are not analytic. For indeed those judgments were called analytic which only assert of an object what is already contained in its concept, thus what can be deduced from it.

[35] The 'contrary assumption' would, in the first instance, be only that the axioms of geometry constitute *synthetic* propositions, for this is the contrary of analytic, and the latter coincides, according to the preceding note, with 'necessary for thought'. But Helmholtz already goes one step further and raises the question of whether the axioms are empirical propositions, i.e. synthetic *a posteriori* judgments. He therefore seems to have in view only these two possibilities for the axioms: either analytic, or synthetic *a posteriori*. (Thus already above: compare note 12). He seems from the outset not to allow for the very class of judgments on which everything depends here, namely the synthetic *a priori* ones.

Empirical propositions, which Helmholtz opposes to judgments necessary *for thought*, thus to analytic judgments, are in fact the contrary of *necessary* judgments as such; as against this, Kant's synthetic *a priori* judgments, if there were such, would indeed not be necessary *for thought*, but they would constitute *intuitive* necessities. Even according to Kant's doctrine, non-Euclidean axiom systems would be wholly *thinkable*, i.e. they could be set up without contradiction. But they would not be *imaginable*, not realizable in products of intuitive imagination, and they could find no application in physical actuality, for this, as being intuitively perceptible, would be subject to the laws governing the manner in which we intuit. If the Euclidean axioms were amongst these laws, then we would be unable to perceive and imagine the corporeal world other than as ordered in Euclidean space.

Kantians have objected against Helmholtz that he did not distinguish between intuitive necessity and necessity for thought, and indeed he omitted to mention this important distinction here. But he was fully aware, in fact, that the resolution of his problem depends precisely upon whether some other geometry is *intuitively imaginable* besides Euclidean geometry. This already emerges very clearly from the next sentence of the text, for there Helmholtz uses the word 'imagine', whose sense he explicitly defined above (compare note 15). He rightly attaches great importance to that definition, for he comes back to it several times (p. 23 of this lecture, also in "The Facts in Perception", below p. 130; likewise in his *Abhandlungen*, vol. II, p. 644).

[36] This sentence formulates the problem precisely and is completely correct. If "spaces of another kind are imaginable in the sense stated" (compare the previous note) then Euclidean space has no intuitive necessity for us; and since Euclid's axioms, as already shown by Helmholtz, are not necessary *for thought* either, proof will have been given that they are not necessary *at all*, and consequently (compare note 2) not valid *a priori*. They will then indeed have to have originated in experience and be *a posteriori*. Helmholtz quickly applies himself in the text to the demonstration that non-Euclidean spaces in fact are imaginable. The following paragraphs start with preparatory considerations.

[37] By the parallax of a fixed star is understood the angle which the major axis of the

earth's orbit subtends as viewed from the star. The fixed star and the two extremities of the axis of the earth's orbit form together the terminal points of a very elongated triangle; the angle at the tip is the parallax. As the sum of angles of a Euclidean triangle equals two right angles, one obtains the value of the parallax by substracting from two right angles the sum of the two angles at the base of the triangle. For immeasurably distant stars, the two long sides of the triangle become markedly parallel, the sum of the base angles becomes equal to two right angles and the parallax zero. As the sum of angles of a pseudospherical triangle is smaller than two right angles, the sum of the base angles cannot reach this total. Thus if one substracts it from two right angles, the result will always have to be positive even for the most distant stars, as Helmholtz mentions in parentheses. In spherical space, on the other hand, one will even get negative values for the angle, since the sum of the two base angles could exceed two right angles.

[38] It holds only with certain reservations that the axiom of parallels, and consequently the Euclidean structure of space, can be "empirically confirmed" by astronomical observations. Above all, it only holds under the presupposition that light rays are to be regarded as straight lines. If the determinations of parallax e.g. yielded negative values, this result could always be explained just as well by the assumption of curvilinear propagation of light as by the hypothesis of positive curvature of space. This has been pointed out with particular emphasis by H. Poincaré, who says with regard to this case (*Wissenschaft und Hypothese* ['Science and hypothesis'], 2nd ed., 1906, p. 74): "one would thus have the choice between two conclusions: we could renounce Euclidean geometry, or alter the laws of optics and allow that light is not propagated exactly in a straight line." It is unquestionably a fault in Helmholtz' account that he does not sufficiently emphasize the second possibility and sharply contrast the two alternatives with each other. He thereby appeared to offer an easy target for the attacks of the Kantians. We shall return to this later, and then see that he perceived the true state of affairs with complete clarity.

Poincaré is thoroughly wrong in continuing at that point: "It is needless to add that everybody would consider the latter solution to be the more advantageous." The successes of Einstein's general theory of relativity, which sacrifices the validity of the Euclidean axioms, prove the error of Poincaré's assertion, and it may be said with certainty that he would today gladly withdraw it in the face of those successes.

He believed that Euclidean geometry would always have to retain its preferential status in physics because it was the 'simplest'. Yet it is not the simplicity of an individual branch or an auxiliary means of science which is decisive, but it ultimately comes down to the simplicity of the *system* of science, a simplicity which is identical with the *unity* of natural knowledge. This maximal simplicity is nowadays attained more perfectly by dropping Euclidean geometry than by retaining it. The observation that the planets do not travel around the sun precisely in simple Keplerian ellipses, but describe extremely complicated orbits, nonetheless made the world picture simpler, because it enabled one to ascribe to Newton's law of gravitation a more precise validity. Just as Poincaré (*ibid.*, p. 152) recognized that "the simplicity of Kepler's laws is only apparent", he would equally say today: the simplicity of Euclidean geometry in its application to nature is only apparent. Helmholtz, attaching himself to Riemann, advocated the admissibility of this standpoint clearly and resolutely; what he contributed as well to its philosophical justification will shortly be considered.

[39] 'Congruence' is established by observing the coincidence of material points. All physical measurements can be reduced to this same principle, since any reading of any of our

instruments is brought about with the help of coincidences of movable parts with points on a scale, etc. Helmholtz' proposition can therefore be extended to the truth that no occurrences whatsoever can be ascertained physically other than meetings of points, and from this Einstein has logically drawn the conclusion that all physical laws should contain basically only statements about such coincidences. The following paragraphs of Helmholtz' lecture contain statements going in the same direction.

[40] In the little word 'actually' there lurks the most essential philosophical problem of the whole lecture. What kind of sense is there in saying of a body that it is *actually* rigid? According to Helmholtz' definition of a fixed body (p. 15), this would presuppose that one could speak of the distance between points 'of space' without having regard to bodies; but it is beyond doubt that without such bodies one cannot ascertain and measure the distance in any way. Thus one gets into the difficulties already described in note 31. If the content of the concept 'actually' is to be such that it can be empirically tested and ascertained, then there remains only the expedient already mentioned in that note: to declare those bodies to be 'rigid' which, when used as measuring rods, lead to the *simplest* physics. Those are precisely the bodies which satisfy the condition adduced by Einstein (compare note 31). Thus what has to count as 'actually' rigid is then not determined by a logical necessity of thought or by intuition, but by a convention, a definition.

[41] Here it seems as if Helmholtz did, after all, regard the concept of a rigid measuring rod as something absolute, continuing to exist independent of our conventions, for one can speak of the probability or truth of an assertion only if its correctness can in principle be tested; the concept of probability is not applicable to definitions. But we obviously have to see in this formulation only a concession to the reader's capacity of comprehension, and Helmholtz only gradually introduces him to the rigorous consistency of the more radical thoughts.

[42] This behaviour is in fact not specified immediately, but only after three purely preparatory paragraphs, in the one which follows them.

[43] With regard to what follows, compare the presentation of the same thoughts in Delboeuf, *Prolégomènes Philosophiques de la Géométrie* ['Philosophical prolegomena to geometry'], 1860; Mongré, *Das Chaos in kosmischer Auslese* ['Chaos in cosmic selection'], 1898, ch. 5; Poincaré, *Der Wert der Wissenschaft*, [*op. cit.*], pp. 46f.; *Science et Méthode* ['Science and method'], 1908, pp. 96ff.; *Dernières Pensées* ['Last thoughts'], 1913, pp. 37ff.; Schlick, *Raum und Zeit* ['Space and time'], 3rd ed., 1920, ch. 3.

[44] The hypothetical change described here by Helmholtz can be expressed in mathematical language thus: if one thinks of the whole world deformed such that its new shape arises from its old one by a one-to-one and continuous, but otherwise quite arbitrary point transformation, then the new world is in actuality completely indistinguishable from the old. The mirror example which follows in the text corresponds to a special case of such a transformation. (Compare the literature cited in note 43.)

[45] The realization that the question is meaningless as to which of the two worlds described is the 'actual' and which the deformed one, is of the highest importance for the whole problem. There exists no 'actual' difference at all between the two worlds, but only one of description, in that a different kind of coordinate system is assumed in each case (com-

pare again the literature cited above in note 43). In other words: in both cases we are dealing with the same objective actuality, which is portrayed by two different systems of symbols.

[46] If Helmholtz wanted to say, in the last clause of this sentence, that the men of the two worlds would ascertain differing mechanical laws, then this would obviously be an error, so long as he does not drop the presupposition that in the world of the mirror all measuring rods participate in the distortions of the bodies. For otherwise all measurements there, all readings of instruments, must lead to precisely the same numbers as in the original world; all point coincidences remain constant (compare note 39). But a law is only a summary expression for the results of measurements; consequently the physical laws in the two worlds are not different. Yet perhaps Helmholtz is thinking in terms of the observer in the deformed world somehow getting offered measuring rods which do not participate in the deformation otherwise prevailing there, and with which he can continually compare his bodies. In this case he would ascertain peculiar laws of motion, and he could infer that it would be more reasonable to regard the other world as undistorted, because there (with respect to those measuring rods) a much simpler mechanics prevails.

[47] F. Klein has introduced in a purely projective way the specification of the non-Euclidean metric in the interior of a second degree surface (*Ges. Abhandlungen*, vol. I, pp. 287f.).

[48] Wherever the observer may be situated, there always exists a mapping into the Euclidean sphere such that his situation corresponds to the centre of the sphere. Let P be a point in the non-Euclidean space and Q its image in the space of the sphere. If now AA' are the eyes of the observer in the non-Euclidean space and BB' are their images, which are taken to be very close to each other, then the non-Euclidean angles PAA' and $PA'A$ must, apart from infinitely small quantities of higher order, be equal to the angles QBB' and $QB'B$. Thus someone who has Euclidean habits will see Q at the very distance which it in fact has in Beltrami's spherical image.

[49] In this paragraph Helmholtz has formulated the chief epistemological result of his investigation. Having shown, in the preceding paragraphs, that the sense impressions which one would receive in a non-Euclidean world can very well be imagined intuitively, he infers that Euclidean space is *not* an inescapable form of our faculty of intuition, but a product of experience. Is this inference really cogent?

This has been particularly contested on the part of Kantian philosophy (compare for instance A. Riehl, *Helmholtz in seinem Verhältnis zu Kant* [*op. cit.*], pp. 35ff.), with the objection that the various series of perceptions described by Helmholtz need not necessarily be interpreted as processes in a non-Euclidean space, but that one can equally well, as it clearly emerges precisely from Helmholtz' own account, conceive of them as peculiar changes of place and shape in Euclidean space. And this latter kind of interpretation – thus the *a priori* school goes on to assert – is the one which our intuition inevitably forces upon us. It obliges us, if any strange behaviour of bodies is observed, to make their *physical* constitutions and laws alone responsible for this, and forbids us to seek instead the ground for that behaviour in a constitution of 'space' which deviates from the Euclidean one.

We have already emphasized more than once that actually both possibilities of interpretation exist, one beside the other. But Helmholtz was so firmly convinced that we would choose the *geometrical* interpretation for certain experiences of perception and measurement, that he did not explicitly refute here the other possibility. He was himself obviously so capable of freeing himself from rooted habits of intuition, that self-observation seemed to show him immediately the absence of the alleged pure intuition. Yet, probably not everybody can struggle through so easily to this attitude; and thus a gap is to be found here in the reasoning. One should have refuted the *a priori* view according to which we have available, for the interpretation of actuality, only physical hypotheses and not purely geometrical ones, because the latter, though possible in thought, are excluded by an intuitive constraint.

The gap in the proof may be variously filled in. Thus Poincaré makes it extremely plausible, in his books on epistemology, that for instance those lines which we call *straight* are certainly not distinguished over and above other lines by immediately given intuitive properties, but solely by the role which, according to experience, they play in physical actuality; and this result is then in fact decisive. Furthermore, and perhaps above all, it might be pointed out that we do not possess a *purely* geometrical intuition at all, in that for instance surfaces without any thickness, colour, etc. are not imaginable at all. (Compare also F. Klein's lecture of 14 October 1905 to the Philosophical Society of Vienna University; M. Pasch, *Vorlesungen über neuere Geometrie* ['Lectures on modern geometry']. Also already J. S. Mill, and before him Hume in the *Treatise*, Book I, part II, section IV.) Then only physical-corporeal structures are accessible at all to our sensible-intuitive imagination, and it follows that the physical and the geometrical are already inseparably fused together in our intuitive geometry, so that it therefore coincides with 'practical geometry'. This was undoubtedly Helmholtz' opinion. Thus even the gap found here in the proof is, in this sense, closed by him in Appendix III to the address on 'The Facts in Perception'; see below, p. 153. Compare on the question of the existence of a pure intuition also Schlick, *Allgemeine Erkenntnislehre* ['General Epistemology'], 1918, § 37.

[50] On this point a number of mathematicians go beyond Helmholtz, and are of the opinion that we cannot be denied off-hand the capacity to imagine a world of four spatial dimensions. Thus, for example, Poincaré. He points out (*Wissenschaft und Hypothese* [*op. cit.*], p. 70) that the three-dimensionality of our visual space arises, *inter alia*, from the fact that the simultaneous perceptions of one eye, which are basically only two-dimensional, follow each other according to quite specific laws – namely the laws of perspective. Following this he thinks that we would automatically imagine the world to have four spatial dimensions, if observation revealed a certain transformation of three-dimensional corporeal forms, taking place according to quite specific laws. These forms would then have to appear to us as three-dimensional perspective views of four-dimensional structures, which would therefore be intuitively imaginable in the same sense as the third dimension of visual space is for a one-eyed person.

If this is already true for visual space, to which Helmholtz of course restricts his considerations, then for the other ideas of the senses – such as those of the sense of touch and the muscular sense – it seems much less certain still that they must necessarily appear arranged in a three-dimensional manifold. At any rate, the question of the imaginability of a fourth spatial dimension must at least be regarded as a problem deserving serious consideration.

[51] This sentence contains the fundamental insight – which Riemann had already attained and which has recently become truly fruitful in the general theory of relativity – that geometry is to be regarded as a part of physics, and thus not as a science of purely ideal structures, such as say arithmetic, the theory of numbers. The sentences of the text which follow give an irreproachable foundation for this insight.

The sentence is perhaps not quite happily formulated, inasmuch as Helmholtz seems in it to contrast geometry with a 'pure theory of space', yet without stating how the concept of a pure theory of space is to be delimited. Today we would rather say that precisely the 'theory of space' is simply an empirical science, and namely, to use Einstein's way of putting it, that part of physics which deals with the 'situabilities' * of bodies. Even Newton already thought the same, as is proved by the quotation which Helmholtz gives elsewhere (p. 163 of this volume). The purely mathematical discipline of geometry, as opposed to practical or physical geometry, is nothing but a structure of theorems, which are derived in a purely logico-formal way from a series of axioms, without regard to whether there exist any objects (e.g. spatial structures) for which those axioms hold (compare Schlick, *Allgemeine Erkenntnislehre* [*op. cit.*], §7). This rigorous 'geometry' therefore, properly speaking, bears its name without justification, and can raise no claim to be a 'theory of space'. That there indeed exists no 'pure' theory of space was also Helmholtz' opinion, as can easily be seen from his remarks on pp. 155 ff.

[52] Compare note 46.

[53] In contrast to less careful statements (compare notes 31, 40, 41), this paragraph shows how clear Helmholtz was about the interdependence between geometry and physics: he rightly declares that we can conceive the in fact existing space of our world to be an arbitrary non-Euclidean one, provided only we introduce, at the same time, entirely new laws of a physical kind. Why don't we do this? Why do we in practice, e.g. in technology, always employ Euclidean geometry? Kant would have answered: because our *a priori* spatial intuition does not let it be otherwise! Poincaré would say: because geometry thereby becomes the science of the situabilities of our fixed bodies, and our physical formulae thereupon assume their simplest form. This is naturally also Helmholtz's opinion; for him too, as the first line of this paragraph shows, what decides the conception to be chosen is scientific *usefulness*.

[54] This sentence says it once again: the theory of space is not a purely mathematical discipline, but a theory of the situabilities of rigid bodies, or, as Einstein also terms it, 'practical geometry'. Helmholtz also calls it *physical* geometry (see below, p. 154).

[55] Helmholtz does not use the word 'transcendental' correctly in its Kantian sense; in particular he often confuses it, as also in this passage, with '*a priori*'.

The sense of the sentences which follow should be further elucidated with a few words in view of the importance of their content. Having rejected the existence of an *a priori* intuition of 'space', Helmholtz examines the question of whether we perhaps possess an *a priori* intuition of the 'fixed body'. What, he asks, would be the consequence? The propositions of the then existing intuitive geometry, though indeed *a priori*, would be *analytic*, for they would be deducible from the properties of those ideal rigid bodies with whose situabilities this geometry would deal. But as to what are the situabilities of any *actual* bodies – on this question of practical geometry it could indeed teach us nothing; the propositions of the latter would continue to be synthetic *a posteriori* judgments.

Let us assume for a moment that this supposed *a priori* geometry were Euclidean, then the only consequence would be that we would be allowed to *call* rigid only those bodies

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* ["Lagerungsmöglichkeiten"].

whose situational laws obey Euclidean rules. However, it is a purely empirical fact if our most convenient mesuring rods actually are rigid in this sense. This *a priori* intuition would therefore be superfluous, it would contribute nothing to the knowledge of actuality and thus would not fulfil the very task assigned to it by Kant. It would indeed even obstruct and be harmful for our knowledge, as soon as it turned out (as is evidently the case nowadays) that we can only maintain the assumption of strictly Euclidean situabilities if we sacrifice for this the beauty and perfection of our system of physics. In short, the assumption of an *a priori* intuition of the rigid body would be scientifically mistaken in every respect. Helmholtz also follows similar trains of thought below on pp. 155 ff.

[56] This paragraph repeats once again that the geometry applied to actuality – which arises from the formally abstract discipline as soon as one brings it into coincidence with actuality at any point whatsoever, and gives its empty concepts a real content – that this practical geometry is an *empirical science* throughout.