We use $t(v_0 \dots v_n)$ to denote a term t whose variables form a subset of $\{v_0, \dots, v_n\}$. Similarly, we use $\varphi(v_0 \dots v_n)$ to denote a formula φ whose free variables form a subset of $\{v_0, \dots, v_n\}$.

Note that we do not require that all of the variables $v_0, \dots, v_n$ be free variables of $\varphi(v_0 \dots v_n)$. In fact, $\varphi(v_0 \dots v_n)$ could even have no free variables. Also, we make no restriction on the bound variables. For example, each of the following formulas is of the form $\varphi(v_0 v_1 v_2)$:

$$(\forall v_1)(\exists v_3)R(v_0 v_1 v_3), \qquad R(v_0 v_1 v_2), \qquad S(v_0 v_2), \qquad (\forall v_4)S(v_4 v_4).$$

A *sentence* is a formula with no free variables.

Note that even if $\mathscr{L}$ has no symbols, there are still formulas of $\mathscr{L}$. These formulas are built up entirely from the identity symbol $\equiv$ and the other logical symbols listed. Such formulas are called *identity formulas* and they *occur in every language*. The following proposition is simple but important.

PROPOSITION 1.3.4. *The cardinal of the set of all formulas of $\mathscr{L}$ is $\|\mathscr{L}\|$.*

To make all the above syntactical notions into a *formal system* we need *logical axioms* and *rules of inference*. The logical axioms for $\mathscr{L}$ are divided into three groups.

1.3.5. *Sentential Axioms*: Every formula φ of $\mathscr{L}$ which can be obtained from a tautology ψ of $\mathscr{S}$ by (simultaneously and uniformly) substituting formulas of $\mathscr{L}$ for the sentence symbols of ψ is a logical axiom for $\mathscr{L}$. From now on we call such a formula φ a *tautology* of $\mathscr{L}$.

1.3.6. *Quantifier Axioms*:

(i). If φ, ψ are formulas of $\mathscr{L}$ and v is a variable not free in φ, then the formula

$$(\forall v)(\varphi \to \psi) \to (\varphi \to (\forall v)\psi)$$

is a logical axiom.

(ii). If φ, ψ are formulas and ψ is obtained from φ by freely substituting each free occurrence of v in φ by the term t (i.e., no variable x in t shall occur bound in ψ at the place where it is introduced), then the formula

$$(\forall v)\varphi \to \psi$$

is a logical axiom.

1.3.7. *Identity Axioms*: Suppose x, y are variables, $t(v_0 \dots v_n)$ is a term and $\varphi(v_0 \dots v_n)$ is an atomic formula. Then the formulas

$$x \equiv x,$$
$$x \equiv y \to t(v_0 \dots v_{i-1} x v_{i+1} \dots v_n) \equiv t(v_0 \dots v_{i-1} y v_{i+1} \dots v_n),$$
$$x \equiv y \to (\varphi(v_0 \dots v_{i-1} x v_{i+1} \dots v_n) \to \varphi(v_0 \dots v_{i-1} y v_{i+1} \dots v_n)),$$

are logical axioms.

There are two rules of inference.

1.3.8. *Rule of Detachment* (or *Modus Ponens*): From φ and $\varphi \to \psi$ infer ψ.

1.3.9. *Rule of Generalization*: From φ infer $(\forall x)\varphi$.

Given the axioms and the rules of inference, we assume that the resulting notions of *proof*, *length of proof*, *theorem* are already familiar to the reader. As we are dealing with the usual first-order logic with identity, we shall assume as known and make free use of all of the basic theorems and metatheorems of such formal systems.

Following standard usage, $\vdash \varphi$ means that φ is a theorem of $\mathscr{L}$. If Σ is a set of sentences of $\mathscr{L}$, then $\Sigma \vdash \varphi$ means that there is a proof of φ from the logical axioms and Σ. If $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ is finite, we write $\sigma_1 \dots \sigma_n \vdash \varphi$. As the logical axioms are always assumed, we say that *there is a proof of φ from Σ*, or *φ is deducible from Σ*, whenever $\Sigma \vdash \varphi$. *Σ is inconsistent* iff every formula of $\mathscr{L}$ can be deduced from Σ. Otherwise Σ is *consistent*. A sentence σ is consistent iff $\{\sigma\}$ is. Σ is *maximal consistent* (*in $\mathscr{L}$*) iff Σ is consistent and no set of sentences (of $\mathscr{L}$) properly containing Σ is consistent. We list in the proposition below some useful, though simple, properties of consistent and maximal consistent sets of sentences. (Many of these properties are found also in Proposition 1.2.8.)

PROPOSITION 1.3.10.

(i). *Σ is consistent if and only if every finite subset of Σ is consistent.*

(ii). *Let σ be a sentence. $\Sigma \cup \{\sigma\}$ is inconsistent if and only if $\Sigma \vdash \neg\sigma$. Whence $\Sigma \cup \{\sigma\}$ is consistent if and only if $\neg\sigma$ is not deducible from Σ.*

(iii). *If Σ is maximal consistent, then for any sentences σ, τ*

$$\Sigma \vdash \sigma \text{ if and only if } \sigma \in \Sigma;$$
$$\sigma \notin \Sigma \text{ if and only if } \neg\sigma \in \Sigma;$$
$$\sigma \wedge \tau \in \Sigma \text{ if and only if } \sigma \text{ and } \tau \text{ belong to } \Sigma.$$

(iv) (Deduction Theorem). *$\Sigma \cup \{\sigma\} \vdash \tau$ if and only if $\Sigma \vdash \sigma \to \tau$.* (*Here σ is a sentence, although τ need not be one.*)

The next proposition duplicates Lemma 1.2.9. There is no change in the proof.

Proposition 1.3.11 (Lindenbaum's Theorem). *Any consistent set of sentences of $\mathscr{L}$ can be extended to a maximal consistent set of sentences of $\mathscr{L}$.*

We now come to the key definition of this section. In fact, the following definition of satisfaction is the cornerstone of model theory. We first give the motivation for the definition in a few remarks. If we compare the models of Section 1.2 and the models discussed here, we see that with the former we were only concerned with whether a statement is true or false in it, while here the situation is more complicated because the sentences of $\mathscr{L}$ say something about the individual elements of the model. The whole question of the (first-order) truths or falsities of a possible world (i.e., model) is just not a simple problem. For instance, there is no way to decide whether a given sentence of $\mathscr{L} = \{+, \cdot, \mathrm{S}, 0\}$ is true or false in the standard model $\langle N, +, \cdot, \mathrm{S}, 0\rangle$ of arithmetic (where S is the successor function). Whereas we have already seen in Section 1.2 that there is such a decision procedure for every model for $\mathscr{S}$ and for every sentence of $\mathscr{S}$. To define the notion

the sentence σ is true in the model $\mathfrak{A}$,

we have first to break up σ into smaller parts and to examine each part. If σ is $\neg\varphi$ or if σ is $\varphi \wedge \psi$, then we see that the truth or falsity of σ in $\mathfrak{A}$ follows once we know the truth or falsity of φ and ψ in $\mathfrak{A}$. If, on the other hand, σ is $(\forall x)\varphi$, then the same method for deciding the truth of σ breaks down as φ may not be a sentence and it would be meaningless to ask if φ is true or false in $\mathfrak{A}$.

The free variable x in φ is supposed to range over the elements of A. For each particular a in A it is meaningful to ask whether

the formula φ is true in $\mathfrak{A}$ if φ is talking about a.

If for each a in A the answer to this question is yes, then we can say that σ is true in $\mathfrak{A}$. If there exists an a in A so that the answer is no, then we say that σ is false in $\mathfrak{A}$. But in order to answer the above question, even for a fixed element of A, we shall run into the same difficulty if φ happens to be $(\forall y)\psi$. Then we are led naturally to ask whether

ψ is true in $\mathfrak{A}$ if ψ is talking about a pair of elements a and b in A.

It takes but a very small step before we see that the crucial question is the following:

Given a formula $\varphi(v_0 \dots v_p)$ and a sequence $x_0, \dots, x_p$ in A, what does it mean to say that φ is true in $\mathfrak{A}$ if the variables $v_0, \dots, v_p$ are taken to be $x_0, \dots, x_p$?

Our plan is to give an answer to this question first for every atomic formula $\psi(v_0 \dots v_p)$ and all elements $x_0, \dots, x_p$. Then, by an inductive procedure based on our inductive definition of formula (1.3.1–1.3.3), we shall give an answer for all formulas $\varphi(v_0 \dots v_p)$ and elements $x_0, \dots, x_p$.

There is still one difficulty with our plan: if all the free variables of a formula φ are among $v_0, \dots, v_p$, it does not follow that all the free variables of every subformula of φ are among $v_0, \dots, v_p$. For a quantifier makes a free variable bound. This will cause trouble in the induction part of our plan. To overcome this difficulty we observe that the following is true. If all the variables, free or bound, of a formula φ are among $v_0, \dots, v_q$, then all the variables of every subformula of φ are also among $v_0, \dots, v_q$. So we shall modify our plan thus: First, we answer the question for all atomic formulas $\psi(v_0 \dots v_q)$ and all elements $x_0, \dots, x_q$. Then by an inductive procedure we answer the question for all formulas φ such that all its *free and bound* variables are among $v_0, \dots, v_q$, and all elements $x_0, \dots, x_q$. Finally, we *prove* that the answer to the question for a formula $\varphi(v_0 \dots v_p)$ and elements $x_0, \dots, x_q$, $p \leqslant q$, depends only on the elements $x_0, \dots, x_p$ corresponding to *free* variables of φ, so that the values of $x_{p+1}, \dots, x_q$ are irrelevant.

We are now ready for the formal definition. The crucial notion to be defined is the following: Let φ be any formula of $\mathscr{L}$, all of whose free and bound variables are among $v_0, \dots, v_q$, and let $x_0, \dots, x_q$ be any sequence of elements of A. We define the predicate

1.3.12. *φ is satisfied by the sequence $x_0, \dots, x_q$ in $\mathfrak{A}$*, or *$x_0, \dots, x_q$ satisfies φ in $\mathfrak{A}$.*

The definition proceeds in three stages (compare with 1.3.1–1.3.3).

Let $\mathfrak{A}$ be a fixed model for $\mathscr{L}$.

1.3.13. The *value of a term $t(v_0 \dots v_q)$ at $x_0, \dots, x_q$* is defined as follows (we let $t[x_0 \dots x_q]$ denote this value):

(i). If $t = v_i$, then $t[x_0 \dots x_q] = x_i$.

(ii). If t is a constant symbol c, then $t[x_0 \dots x_q]$ is the interpretation of c in $\mathfrak{A}$.

(iii). If $t = F(t_1 \dots t_m)$, where F is an m-placed function symbol, then

$$t[x_0 \dots x_q] = G(t_1[x_0 \dots x_q] \dots t_m[x_0 \dots x_q]),$$

where G is the interpretation of F in $\mathfrak{A}$.

1.3.14.

(i). Suppose $\varphi(v_0 \dots v_q)$ is the atomic formula $t_1 \equiv t_2$, where $t_1(v_0 \dots v_q)$ and $t_2(v_0 \dots v_q)$ are terms. Then $x_0, \dots, x_q$ *satisfies* φ if and only if

$$t_1[x_0 \dots x_q] = t_2[x_0 \dots x_q].$$

(ii). Suppose $\varphi(v_0 \dots v_q)$ is the atomic formula $P(t_1 \dots t_n)$, where P is an n-placed relation symbol and $t_1(v_0 \dots v_q), \dots, t_n(v_0 \dots v_q)$ are terms. Then $x_0, \dots, x_q$ *satisfies* φ if and only if

$$R(t_1[x_0 \dots x_q] \dots t_n[x_0 \dots x_q]),$$

where R is the interpretation of P in $\mathfrak{A}$.

For brevity, we write
$\mathfrak{A} \models \varphi[x_0 \dots x_q]$ for: $x_0, \dots, x_q$ satisfies φ in $\mathfrak{A}$.
Thus 1.3.14 can also be formulated as:
(i). $\mathfrak{A} \models (t_1 \equiv t_2)[x_0 \dots x_q]$ if and only if $t_1[x_0 \dots x_q] = t_2[x_0 \dots x_q]$.
(ii). $\mathfrak{A} \models P(t_1 \dots t_n)[x_0 \dots x_q]$ if and only if $R(t_1[x_0 \dots x_q] \dots t_n[x_0 \dots x_q])$.

1.3.15. Suppose that φ is a formula of $\mathscr{L}$ and all free and bound variables of φ are among $v_0, \dots, v_q$.
(i). If φ is $\theta_1 \wedge \theta_2$, then
$\mathfrak{A} \models \varphi[x_0 \dots x_q]$ if and only if both $\mathfrak{A} \models \theta_1[x_0 \dots x_q]$ and $\mathfrak{A} \models \theta_2[x_0 \dots x_q]$.
(ii). If φ is $\neg\,\theta$, then

$$\mathfrak{A} \models \varphi[x_0 \dots x_q] \text{ if and only if not } \mathfrak{A} \models \theta[x_0 \dots x_q].$$

(iii). If φ is $(\forall v_i)\psi$, where $i \leqslant q$, then
$\mathfrak{A} \models \varphi[x_0 \dots x_q]$ if and only if for every $x \in A$, $\mathfrak{A} \models \psi[x_0 \dots x_{i-1} x x_{i+1} \dots x_q]$.

Our definition of 1.3.12 is now completed. As simple exercises, the reader should check that the abbreviations $\vee, \rightarrow, \leftrightarrow, \exists$ have their usual meanings. In particular, if φ is $(\exists v_i)\psi$, where $i \leqslant q$, then

$\mathfrak{A} \models \varphi[x_0 \dots x_q]$ if and only if there exists $x \in A$ such that

$$\mathfrak{A} \models \psi[x_0 \dots x_{i-1} x x_{i+1} \dots x_q].$$

More important, the reader should realize that we can formulate a precise definition of $t[x_0 \dots x_q]$ and $\mathfrak{A} \models \varphi[x_0 \dots x_q]$ in set theory, based upon 1.3.13–1.3.15.

Having finished our definition, our first task is to prove the proposition that the relation

$$\mathfrak{A} \models \varphi(v_0 \dots v_p)[x_0 \dots x_q]$$

depends only on $x_0, \dots, x_p$, where $p < q$. This is the last part of the plan we have outlined.

Proposition 1.3.16

(i). *Let* $t(v_0 \dots v_p)$ *be a term and let* $x_0, \dots, x_q$ *and* $y_0, \dots, y_r$ *be two sequences such that* $p \leqslant q$, $p \leqslant r$, *and* $x_i = y_i$ *whenever* v_i *is a variable of* t. *Then*

$$t[x_0 \dots x_q] = t[y_0 \dots y_r].$$

(ii). *Let* φ *be a formula all of whose free and bound variables are among* $v_0, \dots, v_p$, *and let* $x_0, \dots, x_q$ *and* $y_0, \dots, y_r$ *be two sequences such that* $p \leqslant q, p \leqslant r$, *and* $x_i = y_i$ *whenever* v_i *is a free variable of* φ. *Then*

$$\mathfrak{A} \models \varphi[x_0 \dots x_q] \textit{ if and only if } \mathfrak{A} \models \varphi[y_0 \dots y_r].$$

Remark. Proposition 1.3.16 shows that the value of a term t at $x_0, \dots, x_q$ and whether a formula φ is satisfied or not by a sequence $x_0, \dots, x_q$ *depend only* on those values of x_i for which v_i is a free variable, and are *independent* of the other values of the sequence as well as the length of the sequence. The length q of the sequence must be high enough to cover all the free and bound variables of t and φ in order for the expressions $t[x_0 \dots x_q]$, $\mathfrak{A} \models \varphi[x_0 \dots x_q]$ to be defined at all. We can now immediately infer that if σ is a sentence, then $\mathfrak{A} \models \sigma[x_0 \dots x_q]$ is entirely independent of the sequence $x_0, \dots, x_q$. The importance of the above proposition is that it allows us to make the following definition.

1.3.17. Let $\varphi(v_0 \dots v_p)$ be a formula all of whose free and bound variables are among $v_0, \dots, v_q$, $p \leqslant q$. Let $x_0, \dots, x_p$ be a sequence of elements of A. We say that φ *is satisfied in* $\mathfrak{A}$ *by* $x_0, \dots, x_p$,

$$\mathfrak{A} \models \varphi[x_0 \dots x_p],$$

if and only if φ is satisfied in $\mathfrak{A}$ by $x_0, \dots, x_p, \dots, x_q$ for some (or, equivalently, every) $x_{p+1}, \dots, x_q$.

Let φ be a sentence all of whose bound variables are among $v_0, \dots, v_q$. We say that $\mathfrak{A}$ *satisfies* φ, in symbols $\mathfrak{A} \models \varphi$, iff φ is satisfied in $\mathfrak{A}$ by some (or, equivalently, every) sequence $x_0, \dots, x_q$.

The proof of Proposition 1.3.16 is straightforward but tedious. We shall sketch it here as a first example of an inductive proof on the 'complexity' of formulas. We shall often omit similar easy inductive proofs in the future.

PROOF OF PROPOSITION 1.3.16

(i). If $t(v_0 \dots v_p)$ is a variable v_i, then

$$t[x_0 \dots x_q] = x_i = y_i = t[y_0 \dots y_r].$$

If $t(v_0 \dots v_p)$ is a constant symbol c, and x is the interpretation of c in $\mathfrak{A}$, then

$$t[x_0 \dots x_q] = x = t[y_0 \dots y_r].$$

Suppose $t(v_0 \dots v_p)$ is $F(t_1 \dots t_m)$, where F is an m-placed function symbol and the proposition holds for each of the terms $t_1, \dots, t_m$. This means that

$$t_i[x_0 \dots x_q] = t_i[y_0 \dots y_r], \qquad (i = 1, \dots, m).$$

Therefore, if G is the interpretation of F in $\mathfrak{A}$,

$$\begin{aligned} t[x_0 \dots x_q] &= G(t_1[x_0 \dots x_q] \dots t_m[x_0 \dots x_q]) \\ &= G(t_1[y_0 \dots y_r] \dots t_m[y_0 \dots y_r]) = t[y_0 \dots y_r]. \end{aligned}$$

This verifies (i) for all terms t.

(ii). If φ is an atomic formula $t_1 \equiv t_2$, then using (i) we see that

$$\begin{gathered} t_1[x_0 \dots x_q] = t_1[y_0 \dots y_r], \\ t_2[x_0 \dots x_q] = t_2[y_0 \dots y_r]. \end{gathered}$$

Therefore the following are equivalent:

$$\begin{gathered} \mathfrak{A} \models \varphi[x_0 \dots x_q], \\ t_1[x_0 \dots x_q] = t_2[x_0 \dots x_q], \\ t_1[y_0 \dots y_r] = t_2[y_0 \dots y_r], \\ \mathfrak{A} \models \varphi[y_0 \dots y_r]. \end{gathered}$$

Let φ be an atomic formula $P(t_1 \dots t_n)$, where P is an n-placed predicate symbol and $t_1, \dots, t_n$ are terms. Then, using (i), we see that the following are equivalent (where R is the interpretation of P in $\mathfrak{A}$):

$$\begin{gathered} \mathfrak{A} \models \varphi[x_0 \dots x_q], \\ R(t_1[x_0 \dots x_q] \dots t_n[x_0 \dots x_q]), \\ R(t_1[y_0 \dots y_r] \dots t_n[y_0 \dots y_r]), \\ \mathfrak{A} \models \varphi[y_0 \dots y_r]. \end{gathered}$$

Suppose now that ψ, θ are formulas, all of whose free and bound variables are among $v_0, \dots, v_p$, which satisfy part (ii) of the proposition.

If φ is $\psi \wedge \theta$, the following are equivalent:

$$\begin{gathered} \mathfrak{A} \models \varphi[x_0 \dots x_q], \\ \mathfrak{A} \models \psi[x_0 \dots x_q] \text{ and } \mathfrak{A} \models \theta[x_0 \dots x_q], \\ \mathfrak{A} \models \psi[y_0 \dots y_r] \text{ and } \mathfrak{A} \models \theta[y_0 \dots y_r], \\ \mathfrak{A} \models \varphi[y_0 \dots y_r]. \end{gathered}$$

If φ is $\neg\psi$, then the following are equivalent.

$$\begin{gathered} \mathfrak{A} \models \varphi[x_0 \dots x_q], \\ \text{not } \mathfrak{A} \models \psi[x_0 \dots x_q], \\ \text{not } \mathfrak{A} \models \psi[y_0 \dots y_r], \\ \mathfrak{A} \models \varphi[y_0 \dots y_r]. \end{gathered}$$

Finally, let φ be $(\forall v_i)\psi$, where $i \leqslant p$. Then the following are equivalent:

$$\begin{gathered} \mathfrak{A} \models \varphi[x_0 \dots x_q], \\ \text{for all } x \in A,\ \mathfrak{A} \models \psi[x_0 \dots x_{i-1} x x_{i+1} \dots x_q], \\ \text{for all } y \in A,\ \mathfrak{A} \models \psi[y_0 \dots y_{i-1} y y_{i+1} \dots y_r], \\ \mathfrak{A} \models \varphi[y_0 \dots y_r]. \end{gathered}$$

In this last part of the proof we used the fact that the free variables of ψ are just the free variables of φ and, perhaps, v_i. Our proof is now complete. ⊣

We shall state one more elementary proposition which deals with the behavior of the satisfaction relation under the substitution of variables by terms. We omit the proof, which is another tedious but straightforward induction.

PROPOSITION 1.3.18. *Let $\varphi(v_0 \dots v_p)$ be a formula and let $t_0(v_0 \dots v_p), \dots, t_p(v_0 \dots v_p)$ be terms. Suppose that no variable occurring in any of the terms $t_0, \dots, t_p$ occurs bound in φ. Let $x_0, \dots, x_p$ be a sequence of elements of A and let $\varphi(t_0 \dots t_p)$ be the formula obtained from φ by substituting t_i for v_i $(i = 0, \dots, p)$. Then*

$$\mathfrak{A} \models \varphi(t_0 \dots t_p)[x_0 \dots x_p] \textit{ if and only if } \mathfrak{A} \models \varphi[t_0[x_0 \dots x_p] \dots t_p[x_0 \dots x_p]].$$

This proposition is especially useful in the simple case that the terms $t_0, \dots, t_p$ are constant symbols $c_0, \dots, c_p$ whose interpretations in $\mathfrak{A}$ are $a_0, \dots, a_p$. In that case, $\varphi(c_0 \dots c_p)$ is a sentence, and the proposition shows that

$$\mathfrak{A} \models \varphi(c_0 \dots c_p) \text{ if and only if } \mathfrak{A} \models \varphi[a_0 \dots a_p].$$

Thus a sentence formed by replacing a tuple variables by constant symbols is true in a model if and only if the tuple of interpretations of the constant symbols satisfies the formula in the model.

We have now completed the project started several paragraphs back. Namely, we say that a sentence

$$\sigma \textit{ is true in } \mathfrak{A}$$

iff

$$\mathfrak{A} \models \sigma[x_0 \dots x_q] \text{ for some (or for every) sequence } x_0, \dots, x_q \text{ of } A.$$

We use the special notation $\mathfrak{A} \models \sigma$ to denote that σ is true in $\mathfrak{A}$. This last phrase is equivalent to each of the following phrases:

σ *holds in* $\mathfrak{A}$;
$\mathfrak{A}$ *satisfies* σ;
σ *is satisfied in* $\mathfrak{A}$;
$\mathfrak{A}$ *is a model of* σ.

When it is not the case that σ holds in $\mathfrak{A}$, we say that σ *is false in* $\mathfrak{A}$, or that σ *fails in* $\mathfrak{A}$, or $\mathfrak{A}$ *is a model of* $\neg\sigma$. Given a set Σ of sentences, we say that $\mathfrak{A}$ *is a model of* Σ iff $\mathfrak{A}$ is a model of each σ in Σ; it is convenient to use the notation $\mathfrak{A} \models \Sigma$ for this notion. A sentence σ that holds in every model for $\mathscr{L}$ is called *valid*. A sentence, or a set of sentences, is *satisfiable* iff it has at least one model. $\models \sigma$ denotes that σ is a valid sentence.

A sentence φ is a *consequence of another sentence* σ, in symbols $\sigma \models \varphi$, iff every model of σ is a model of φ. A sentence φ is a *consequence of a set of sentences* Σ, in symbols $\Sigma \models \varphi$, iff every model of Σ is a model of φ. It follows that

$$\Sigma \cup \{\sigma\} \models \varphi \text{ if and only if } \Sigma \models \sigma \to \varphi.$$

Two models $\mathfrak{A}$ and $\mathfrak{B}$ for $\mathscr{L}$ are *elementarily equivalent* iff every sentence that is true in $\mathfrak{A}$ is true in $\mathfrak{B}$, and vice versa. We express this relationship between models by $\equiv$. It is easy to see that $\equiv$ is indeed an equivalence relation. The symbol we have chosen to denote elementary equivalence is exactly the same as the identity symbol for the language $\mathscr{L}$. However, no confusion can ever arise because one is a relation between models for $\mathscr{L}$ and the other is a relation between terms of $\mathscr{L}$. If the context is clear, *equivalent* shall mean elementarily equivalent.

PROPOSITION 1.3.19. *If* $\mathfrak{A} \cong \mathfrak{B}$, *then* $\mathfrak{A} \equiv \mathfrak{B}$. *In case* $\mathfrak{A}$ *is finite, then the converse is also true.*

We conclude this section by stating a number of important results without proofs, but whose proofs will be given in the next chapter.

THEOREM 1.3.20 (Gödel's Completeness Theorem). *Given any sentence* σ, σ *is a theorem of* $\mathscr{L}$ *if and only if* σ *is valid.*

THEOREM 1.3.21 (Extended Completeness Theorem). *Let* Σ *be any set of sentences. Then* Σ *is consistent if and only if* Σ *has a model.*

THEOREM 1.3.22 (Compactness Theorem). *A set of sentences* Σ *has a model if and only if every finite subset of* Σ *has a model.*

As in Section 1.2, we conclude with a table of equivalent notions.

TABLE 1.3.1

Syntax	Semantics
φ is a theorem, $\vdash \varphi$	φ is valid, $\models \varphi$
Σ is consistent	Σ has a model
φ is deducible from Σ, $\Sigma \vdash \varphi$	φ is a consequence of Σ, $\Sigma \models \varphi$

EXERCISES

1.3.1. Prove that the isomorphism relation $\cong$ is an equivalence relation. Let α be any cardinal. Show that there are at most $2^{\alpha \cup \|\mathscr{L}\|}$ nonisomorphic models for $\mathscr{L}$ of power α.

1.3.2. Let $\mathfrak{A} \underset{\sim}{\subset} \mathfrak{B}$ mean that $\mathfrak{A}$ is isomorphically embedded in $\mathfrak{B}$. Show that the relation $\underset{\sim}{\subset}$ is reflexive and transitive but not antisymmetric. Let N be the set of all natural numbers 0, 1, 2, ... Decide if the following are true or false:

$$\langle N, \leqslant, +, 0\rangle \underset{\sim}{\subset} \langle N, \leqslant, \cdot, 1\rangle,$$
$$\langle N, \leqslant, \cdot, 1\rangle \underset{\sim}{\subset} \langle N, \leqslant, +, 0\rangle,$$
$$\langle N-\{0\}, \leqslant, \cdot, 1\rangle \underset{\sim}{\subset} \langle N, \leqslant, +, 0\rangle,$$
$$\langle N-\{0\}, \cdot, 1\rangle \underset{\sim}{\subset} \langle N, +, 0\rangle,$$
$$\langle N-\{0\}, \cdot\rangle \underset{\sim}{\subset} \langle N, +\rangle.$$

Take $\leqslant$, $+$, and $\cdot$ as the usual ordering and operations on N.

1.3.3. Let $\varphi(v_0 \dots v_n)$ be a formula of $\mathscr{L}$, and $\mathfrak{A}$ be a model for $\mathscr{L}$. Prove that:

(i). The satisfaction relation $\mathfrak{A} \models \varphi[x_0 \dots x_n]$ has a precise definition in Zermelo-Fraenkel set theory.

(ii). If $\mathfrak{A}'$ is an expansion of $\mathfrak{A}$ and $x_0, \dots, x_n \in A$, then

$$\mathfrak{A} \models \varphi[x_0 \dots x_n] \quad \text{if and only if} \quad \mathfrak{A}' \models \varphi[x_0 \dots x_n].$$

1.3.4. Prove Proposition 1.3.19. Also construct a counterexample if $\mathfrak{A}$ is not finite.

1.3.5. A sentence φ is *universal* iff it is in prenex form and all of its quantifiers are universal, i.e., $\forall$. Prove that if φ is universal and $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{B} \models \varphi$, then $\mathfrak{A} \models \varphi$. A sentence is *existential* iff it is in prenex form and all of its quantifiers are existential, i.e., $\exists$. Prove that if φ is existential and $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{A} \models \varphi$, then $\mathfrak{B} \models \varphi$. Thus universal sentences are preserved under submodels and existential sentences are preserved under extensions.

1.3.6. There are at most $2^{\|\mathscr{L}\|}$ nonequivalent models for $\mathscr{L}$.

1.3.7. Let $\mathfrak{A}$ and $\mathfrak{B}$ be equivalent models for $\mathscr{L}$. Suppose that every element of A is a constant of A and the same is true for $\mathfrak{B}$. Then show that $\mathfrak{A} \cong \mathfrak{B}$. If the hypothesis is assumed only for $\mathfrak{A}$, show that $\mathfrak{A}$ is isomorphically embedded in $\mathfrak{B}$.

1.3.8*. Find a necessary and sufficient condition on $\mathscr{L}$ so that there will be *exactly* $2^{\|\mathscr{L}\|}$ nonequivalent models. Do the same for exactly 2^{α} nonisomorphic models of power α, for each infinite cardinal α.

1.3.9. Let $\mathfrak{A}$ be a model for $\mathscr{L}$ and let X be a nonempty subset of $\mathfrak{A}$. Let

$$B = \bigcap\{C : \mathfrak{C} \subset \mathfrak{A} \quad \text{and} \quad X \subset C\}.$$

Then there is a submodel $\mathfrak{B} \subset \mathfrak{A}$ with universe B. $\mathfrak{B}$ is called the *submodel generated* by X.

1.3.10. Let $\mathfrak{A}$, X be as above and let $\mathfrak{B}$ be the submodel generated by X. Then

$$B = \{t[x_1 \dots x_n] : t \quad \text{is a term of } \mathscr{L} \text{ and} \quad x_1, \dots, x_n \in X\}.$$

Moreover,

$$|X| \leqslant |B| \leqslant |X| \cup \|\mathscr{L}\|.$$

1.3.11. Suppose $X \subset A$ and X generates the whole model $\mathfrak{A}$. Let f be a one–one map on X into another model $\mathfrak{B}$. Then there is at most one isomorphic embedding g on $\mathfrak{A}$ into $\mathfrak{B}$ such that $f \subset g$.

1.3.12. Suppose $\mathscr{L}$ has no function or constant symbols. Then for every model $\mathfrak{A}$ for $\mathscr{L}$ and every nonempty subset $X \subset A$, $\mathfrak{A}$ has a submodel with universe X.

Suppose $\mathscr{L}$ has no function symbols. Then for every model $\mathfrak{A}$ for $\mathscr{L}$ and every nonempty $X \subset A$ which contains all the constants of $\mathfrak{A}$, $\mathfrak{A}$ has a submodel with universe X.

1.3.13. Verify all the claims in Table 1.3.1, assuming the extended completeness theorem.

1.3.14*. An element $a \in A$ of a model $\mathfrak{A}$ is said to be *definable* (in $\mathfrak{A}$) iff there is a formula $\varphi(x)$ of $\mathscr{L}$ such that a is the only element in A satisfying φ. For each $n \in \omega$, find a model $\mathfrak{A}_n$ for $\mathscr{L}$ and a language with only a finite number of symbols, which has exactly n undefinable elements. For $n = 0$ or $n > 1$, the examples are easy to find. For $n = 1$, it is much harder.

1.3.15*. Let $\mathscr{L}$ have only a finite number of relation and constant symbols, but no function symbols. Define the relations $\equiv_n$ on models for $\mathscr{L}$ as follows by induction:

$\mathfrak{A} \equiv_0 \mathfrak{B}$ iff the submodels of $\mathfrak{A}$ and $\mathfrak{B}$ generated by the constant elements are isomorphic, or else $\mathscr{L}$ has no constant symbols.

$\mathfrak{A} \equiv_{n+1} \mathfrak{B}$ iff for every $a \in A$ there exists $b \in B$ such that $(\mathfrak{A}, a) \equiv_n (\mathfrak{B}, b)$, and for every $b \in B$ there exists $a \in A$ such that $(\mathfrak{A}, a) \equiv_n (\mathfrak{B}, b)$.

Note that the definition of $\equiv_{n+1}$ depends on the fact that $\equiv_n$ has already been defined for all languages of the form $\mathscr{L} \cup \{c_1, \dots, c_m\}$. Prove that $\mathfrak{A} \equiv \mathfrak{B}$ if and only if for all n, $\mathfrak{A} \equiv_n \mathfrak{B}$.

[*Hint*: Show that for each n, there are finitely many $\equiv_n$ classes, and each $\equiv_n$ class is the class of all models of a sentence of $\mathscr{L}$.]

1.3.16. Let $\mathscr{L}$ be as in the previous exercise. Prove that if $\mathfrak{A} \equiv_n \mathfrak{B}$ then for every sentence φ of $\mathscr{L}$ in prenex form with at most n quantifiers,

$$\mathfrak{A} \models \varphi \quad \text{if and only if} \quad \mathfrak{B} \models \varphi.$$

1.3.17*. Let $\mathscr{L}$ be as in Exercise 1.3.15. Let K be a class of models for $\mathscr{L}$. Prove that the following are equivalent:

(i). There is a sentence φ of $\mathscr{L}$ such that K is the class of all models for $\mathscr{L}$ satisfying φ.

(ii). For some $n \in \omega$, K is closed under $\equiv_n$, i.e., if $\mathfrak{A} \in K$ and $\mathfrak{A} \equiv_n \mathfrak{B}$, then $\mathfrak{B} \in K$.

[*Hint*: For each n there are only finitely many nonequivalent prenex sentences with at most n quantifiers.]

1.3.18. Show that Exercise 1.3.15 is false if either $\mathcal{L}$ has function symbols, or $\mathcal{L}$ has infinitely many constant or relation symbols. However, it is still true that if for all n, $\mathfrak{A} \equiv_n \mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$.

1.3.19. Show that two models $\mathfrak{A}$, $\mathfrak{B}$ for $\mathcal{L}$ are equivalent if and only if for all finite $\mathcal{L}_0 \subset \mathcal{L}$ the reducts of $\mathfrak{A}$ and $\mathfrak{B}$ to $\mathcal{L}_0$ are equivalent. This exercise shows that a version of Exercise 1.3.15 can be given if $\mathcal{L}$ has only relation and constant symbols.

1.3.20*. As applications of Exercise 1.3.15, show that the following pairs of models $\mathfrak{A}$, $\mathfrak{B}$ are equivalent:

(i). $\mathfrak{A} = \langle A \rangle$, $\mathfrak{B} = \langle B \rangle$, where A and B are infinite.

(ii). $\mathfrak{A} = \langle A, \leqslant \rangle$, $\mathfrak{B} = \langle B, \leqslant \rangle$, where A and B are densely ordered by $\leqslant$ with no endpoints.

(iii). $\mathfrak{A} = \langle \omega, \leqslant \rangle$, $\mathfrak{B} = \langle \omega + \omega^* + \omega, \leqslant \rangle$, where $\omega + \omega^* + \omega$ is the order type of the natural numbers followed by a copy of the integers.

(iv). $\mathfrak{A} = \langle \omega^\omega, \leqslant \rangle$, $\mathfrak{B} = \langle \omega_1, \leqslant \rangle$, where ω^ω is the ordinal exponentiation of ω to the power ω.

(v). $\mathfrak{A} = \langle S_\omega(X), \subset \rangle$, $\mathfrak{B} = \langle S_\omega(Y), \subset \rangle$, where X and Y are infinite sets, $S_\omega(X)$, $S_\omega(Y)$ are the sets of finite subsets of X and Y, respectively, and $\subset$ is the inclusion relation.

1.3.21. Let T be a set of universal sentences. Assume $T \models \forall x \, \exists y \, P(x, y)$. Prove that there exist terms $t_1(x), \ldots, t_n(x)$ such that

$$T \models \forall x \bigvee_{m=1}^{n} P(x, t_m(x)).$$

1.3.22*. Let $\mathfrak{A}$ be a countable model for a countable language. Prove that if the simple expansion $(\mathfrak{A}, b)$ has more than one automorphism for each finite sequence b of elements of A, then $\mathfrak{A}$ has 2^ω automorphisms.

1.4. Theories and examples of theories

A (*first-order*) *theory* T of $\mathcal{L}$ is a collection of sentences of $\mathcal{L}$. T is said to be *closed* iff it is closed under the $\models$ relation. In view of Table 1.3.1, this is the same as requiring that T be closed under $\vdash$. Since theories are sets of sentences of $\mathcal{L}$, we may apply the expressions

a model of a theory,
consistent theory,
satisfiable theory,

as introduced in Section 1.3.

A theory T is called *complete* (in $\mathcal{L}$) iff its set of consequences is maximal consistent. If T is a theory of $\mathcal{L}$ and $\mathcal{L} \subset \mathcal{L}'$, $\mathcal{L} \neq \mathcal{L}'$, then T is not a closed theory of $\mathcal{L}'$. On the other hand, it is easy to see that if $\mathcal{L}' \subset \mathcal{L}$, then the *restriction of a closed theory T to $\mathcal{L}'$*, in symbols $T|\mathcal{L}'$, is always a closed theory of $\mathcal{L}'$. T is a *subtheory of T'* iff $T \subset T'$. If T is a subtheory of T', then T' is an *extension* of T.

A *set of axioms of a theory* T is a set of sentences with the same consequences as T. Clearly, T is a set of axioms of T, and the empty set is a set of axioms of T if and only if T is a set of valid sentences of $\mathcal{L}$. Every set of sentences Σ is a set of axioms for the closed theory $T = \{\varphi : \Sigma \models \varphi\}$. A theory T is *finitely axiomatizable* iff it has a finite set of axioms.

The most convenient and standard way of giving a theory T is by listing a finite or infinite set of axioms for it. Another way to give a theory is as follows: Let $\mathfrak{A}$ be a model for $\mathcal{L}$; then the *theory of* $\mathfrak{A}$ is the set of all sentences which hold in $\mathfrak{A}$. The theory of any model $\mathfrak{A}$ is obviously a complete theory.

Historically, the importance of theories stems from the following two facts. Once the axioms of a theory are given, then by using the relation $\vdash$ we can find out, in a syntactical manner, all the consequences of T. On the other hand, by using the satisfaction relation, we can also study all the models of T.

By the extended completeness theorem, these two approaches give basically the same results about consequences of T. However, owing to the fact that models of T also have non-first-order properties, such as isomorphism, submodels, extensions, plus many others, the second approach leads to the field now known as model theory.

We shall give in the rest of this section some examples of theories and their models to show the intimate connections that model theory has with other branches of mathematics. In each example we describe a closed theory by a set of axioms. Some classical results will be stated without proof.

1.4.1. Let $\mathcal{L}$ consist of the single 2-placed relation symbol $\leqslant$. Using the usual notation for $\leqslant$, we write $x \leqslant y$ for $\leqslant(xy)$. The theory of *partial order* has three axioms:

(1) $(\forall xyz)(x \leqslant y \wedge y \leqslant z \rightarrow x \leqslant z)$,
(2) $(\forall xy)(x \leqslant y \wedge y \leqslant x \rightarrow x \equiv y)$,
(3) $(\forall x)(x \leqslant x)$.

They are, respectively, the transitive, antisymmetric, and reflexive properties of partial orders. Any model $\langle A, \leqslant \rangle$ of this theory consists of a nonempty set A and a partial order relation $\leqslant$ on A. If we add the comparability axiom

(4) $(\forall xy)(x \leqslant y \vee y \leqslant x)$,

we obtain the theory of *simple order* (also called *linear order*). A model $\langle A, \leqslant\rangle$ for this theory is a simply-ordered set. Adding two more axioms (writing $x \not\equiv y$ for $\neg\,(x \equiv y)$):

(5) $(\forall xy)(x \leqslant y \wedge x \not\equiv y \to (\exists z)(x \leqslant z \wedge z \not\equiv x \wedge z \leqslant y \wedge z \not\equiv y))$,
(6) $(\exists xy)(x \not\equiv y)$,

we then have the theory of *dense* (*simple*) *order*. The rationals with the usual $\leqslant$ is an example of a model of this theory. The theory of dense order has no finite models. If we wish to consider only dense orders *without endpoints*, we add the axioms

(7) $(\forall x)(\exists y)(x \leqslant y \wedge x \not\equiv y)$,
(8) $(\forall x)(\exists y)(y \leqslant x \wedge x \not\equiv y)$.

Proposition 1.4.2. *Any two countable models of the theory of dense order without endpoints are isomorphic.*

Example 1.4.3. Let $\mathscr{L} = \{+, \cdot, ^{-}, \mathit{0}, \mathit{1}\}$, where $+, \cdot$ are 2-placed function symbols, $^{-}$ is a 1-placed function symbol, and $\mathit{0}$ and $\mathit{1}$ are constant symbols. The theory of *Boolean algebras* has the following axioms (where we shall assume that the following formulas all have their free variables universally quantified in front).

Associativity of $+$ and $\cdot$:
$x+(y+z) \equiv (x+y)+z, \qquad x\cdot(y\cdot z) \equiv (x\cdot y)\cdot z.$
Commutativity of $+$ and $\cdot$:
$x+y \equiv y+x, \qquad x\cdot y \equiv y\cdot x.$
Idempotent laws:
$x+x \equiv x, \qquad x\cdot x \equiv x.$
Distributive laws:
$x+(y\cdot z) \equiv (x+y)\cdot(x+z), \qquad x\cdot(y+z) \equiv x\cdot y+x\cdot z.$
Absorption laws:
$x+(x\cdot y) \equiv x, \qquad x\cdot(x+y) \equiv x.$
De Morgan laws:
$\overline{x+y} \equiv \bar{x}\cdot\bar{y}, \qquad \overline{x\cdot y} \equiv \bar{x}+\bar{y}.$
Laws of zero and one:
$x+\mathit{0} \equiv x, \qquad x\cdot\mathit{0} \equiv \mathit{0},$
$x+\mathit{1} \equiv \mathit{1}, \qquad x\cdot\mathit{1} \equiv x,$
$\mathit{0} \not\equiv \mathit{1},$
$x+\bar{x} \equiv \mathit{1}, \qquad x\cdot\bar{x} \equiv \mathit{0}.$

Law of double negation:
$\bar{\bar{x}} \equiv x.$

A model $\mathfrak{A} = \langle A, +, \cdot, ^{-}, 0, 1\rangle$ of this theory is called a Boolean algebra. (Strictly speaking, we should write $+_{\mathfrak{A}}, \cdot_{\mathfrak{A}}, -_{\mathfrak{A}}, 0_{\mathfrak{A}}, 1_{\mathfrak{A}}$ in the above model. But following our convention we shall drop the subscripts.) A partial order $\leqslant$ can be defined on A by: $x \leqslant y$ if and only if $x+y = y$. It can be shown that $\leqslant$ has a largest element, namely 1, a smallest element, namely 0, and, given any two elements $x, y \in A$, the l.u.b. (least upper bound) of x and y is $x+y$, and the g.l.b. (greatest lower bound) of x and y is $x\cdot y$.

A *field of sets* S is a collection of subsets of a nonempty set X such that both the empty set $\emptyset$ and the set X are in S and S is closed under $\cup$, $\cap$ and $^{-}$ with respect to X. It is easy to see that if S is a field of sets, then

$$\langle S, \cup, \cap, ^{-}, \emptyset, X\rangle$$

is a Boolean algebra. Conversely, we have:

Proposition 1.4.4 (Representation Theorem for Boolean algebras). *Every Boolean algebra is isomorphic to a field of sets.*

An *atom* of a Boolean algebra is an element $x \neq 0$ such that there is no element y which lies properly between 0 and x, i.e., not $0 \leqslant y \leqslant x$, $0 \neq y$, $y \neq x$. A Boolean algebra is *atomic* iff every nonzero element x includes an atom. A Boolean algebra is *atomless* iff it has no atoms. There are Boolean algebras which are neither atomic nor atomless. Adding the axiom (writing $x \leqslant y$ for $x+y \equiv y$)

$(\forall x)\ (\mathit{0} \not\equiv x \to (\exists y)\ (y \leqslant x \wedge \mathit{0} \not\equiv y \wedge (\forall z)\ (z \leqslant y \to z \equiv \mathit{0} \vee z \equiv y)))$

gives us the theory of *atomic Boolean algebras*; while adding the axiom

$\neg\,(\exists y)\ (\mathit{0} \not\equiv y \wedge (\forall z)\ (z \leqslant y \to z \equiv \mathit{0} \vee z \equiv y))$

gives us the theory of *atomless Boolean algebras*.

Proposition 1.4.5. *Any two countable atomless Boolean algebras are isomorphic.*

Some other relevant facts about Boolean algebras can be found in the exercises.

Example 1.4.6. Let $\mathscr{L} = \{+, \mathit{0}\}$, where $+$ is a 2-placed function symbol and $\mathit{0}$ is a constant symbol. The theory of *groups* has the following axioms:

(1) $x+(y+z) \equiv (x+y)+z$ (associativity),
(2) $x+\mathit{0} \equiv x$, $\mathit{0}+x \equiv x$ (identity),
(3) $(\exists y)\ (x+y \equiv \mathit{0} \wedge y+x \equiv \mathit{0})$ (existence of inverse).

A model $\langle G, +, 0\rangle$ of this theory is a *group*. We obtain the theory of *Abelian groups* when we add the axiom

(4) $x+y \equiv y+x$ (commutativity).

The *order* of an element x of a group is the least n such that $x+x+\ \dots\ +x$ (n times) $= 0$. If no such n exists, the order of x is infinity. For a fixed $n \geqslant 1$, we can write down the abbreviation nx for the expression

$$x+(x+(\dots(x+x)\dots)),\ n \text{ times}.$$

Suppose p is a prime. The theory of *Abelian groups with all elements of order p* has the extra axiom

(5_p) $px \equiv 0$.

Proposition 1.4.7. *Any two models of the theory of Abelian groups with all elements of order p of the same power are isomorphic.*

To obtain the theory of *Abelian groups with all elements of order* ∞ (*torsion-free*) we need an infinite list of axioms: for each $n \geqslant 1$, we add the axiom

(6_n) $x \not\equiv 0 \to nx \not\equiv 0$.

This theory is our first example of a nonfinitely axiomatizable theory. If we add a further infinite list of axioms, one for each $n \geqslant 1$,

(7_n) $(\exists y)(ny \equiv x)$,

we have the theory of *divisible torsion-free Abelian groups*.

Proposition 1.4.8. *Any two uncountable divisible torsion-free Abelian groups of the same power are isomorphic. There are countably many such groups which are countable and not isomorphic.*

Example 1.4.9. Let $\mathscr{L} = \{+, \cdot, 0, 1\}$, where $+$ and $\cdot$ are 2-placed function symbols and 0, 1 are constant symbols. The theory of *commutative rings* (*with unit*) has the axioms (1)–(4) listed above plus the axioms (8)–(11) given below:

(8) $1 \cdot x \equiv x \wedge x \cdot 1 \equiv x$ (1 is a unit),
(9) $x \cdot (y \cdot z) \equiv (x \cdot y) \cdot z$ (associativity of $\cdot$),
(10) $x \cdot y \equiv y \cdot x$ (commutativity of $\cdot$),
(11) $x \cdot (y+z) \equiv (x \cdot y)+(x \cdot z)$ (distributivity of $\cdot$ over $+$).

Adding one more axiom

(12) $x \cdot y \equiv 0 \to x \equiv 0 \vee y \equiv 0$ (no zero divisors),

gives us the theory of *integral domains*. Adding the two axioms

(13) $0 \not\equiv 1$,
(14) $x \not\equiv 0 \to (\exists y)(y \cdot x \equiv 1)$ (existence of multiplicative inverse),

gives the important theory of *fields*. For a fixed prime p, if we add the axiom

(15_p) $p1 \equiv 0$,

we have the theory of *fields of characteristic p*. On the other hand, if we add for all primes p the negation of (15_p), namely, all the axioms

(16) $p1 \not\equiv 0$, with p a prime,

we have the theory of *fields of characteristic zero*, each field has a unique characteristic, either prime or zero. We now introduce the abbreviation x^n for the expression

$$x \cdot (x \cdot (x \dots x)\dots),\ n \text{ times}.$$

The infinite list of axioms, one for each $n \geqslant 1$,

(17_n) $(\exists y)(x_n \cdot y^n + x_{n-1} \cdot y^{n-1} + \dots + x_1 \cdot y + x_0 \equiv 0) \vee x_n \equiv 0$,

when added to the theory of fields, gives us the theory of *algebraically closed fields*.

Proposition 1.4.10. *Any two uncountable algebraically closed fields of the same characteristic and power are isomorphic.*

Each axiom (17_n) says that every polynomial of degree n has a root. The theory of *real closed fields* has as axioms all the axioms for fields plus the axiom

(18) $(\forall x)(\exists y)(y^2 \equiv x \vee y^2+x \equiv 0)$,

and two infinite lists of axioms. One is the infinite list (17_n) for all odd n, and the other is the infinite list that says that 0 is not a sum of nontrivial squares:

(18_n) $x_0^2+x_1^2+\ \dots\ +x_n^2 \equiv 0 \to x_0 \equiv 0 \wedge x_1 \equiv 0 \wedge\ \dots\ \wedge x_n \equiv 0$.

The theory of *ordered fields* is formulated in the language $\mathscr{L} = \{\leqslant, +, \cdot, 0, 1\}$. It has all the field axioms, the linear order axioms, and the additional axioms

$$x \leqslant y \to x+z \leqslant y+z,$$
$$x \leqslant y \wedge 0 \leqslant z \to x \cdot z \leqslant y \cdot z.$$

The ordered fields of rational numbers and of real numbers are examples.

Of the examples of theories we have discussed so far, the following are complete: dense order without endpoints, atomless Boolean algebras, infinite Abelian groups with all elements of order p, torsion-free divisible Abelian groups, algebraically closed fields of a given characteristic, and real closed fields. The various propositions show that each of these complete theories, except the last one, enjoys the unusual property that in some (sometimes all) infinite powers all models of the given theory of that power are isomorphic.

EXAMPLE 1.4.11. Let $\mathscr{L} = \{+, \cdot, S, 0\}$, where $+$, $\cdot$ are 2-placed function symbols, S is a 1-placed function symbol (called the successor function), and 0 is a constant symbol. *Number theory* (or *Peano arithmetic*) has the following list of axioms:

(1) $0 \not\equiv Sx$ (0 has no predecessor),
(2) $Sx \equiv Sy \rightarrow x \equiv y$ (S is one–one),
(3) $x+0 \equiv x$,
(4) $x+Sy \equiv S(x+y)$,
(5) $x \cdot 0 \equiv 0$,
(6) $x \cdot Sy \equiv (x \cdot y)+x$,

and, finally, for each formula $\varphi(v_0 \dots v_n)$ of $\mathscr{L}$, where v_0 does not occur bound in φ, the axiom

$$(7_\varphi)\quad \varphi(0v_1 \dots v_n) \wedge (\forall v_0)(\varphi(v_0 v_1 \dots v_n) \rightarrow \varphi(Sv_0 v_1 \dots v_n)) \rightarrow (\forall v_0)\varphi(v_0 \dots v_n).$$

Axioms (3) and (4) are the usual recursive definition of $+$ in terms of 0 and S, and axioms (5) and (6) are the recursive definition of $\cdot$ in terms of 0, S and $+$. The whole list of axioms (7_φ), one for each φ, is called the *axiom schema of induction.*

The *standard model* of number theory is $\langle \omega, +, \cdot, S, 0 \rangle$, where S is the successor function and $+, \cdot, 0$ have their usual meaning. All other (non-isomorphic) models are called *nonstandard. Complete number theory* (or *complete arithmetic*) is the set of all sentences φ of $\mathscr{L}$ that hold in the standard model.

There are several deep results about number theory:

Gödel's (1931) incompleteness theorem states that number theory is not complete; therefore, complete number theory is a proper extension of number theory.

No finite extension (that is, by adding a finite number of new axioms) of number theory is complete; therefore complete number theory is not finitely axiomatizable over number theory, whence it is certainly not finitely axiomatizable.

Number theory itself is not finitely axiomatizable. This was proved by Ryll-Nardzewski (1952) by the use of nonstandard models. The existence of nonstandard models of complete number theory was first shown by Skolem (1934).

We mention a number of interesting subtheories of number theory. For instance, if the induction schema (7_φ) is replaced by the single axiom

(8) $(\forall x)(x \not\equiv 0 \rightarrow (\exists y)(x \equiv Sy))$,

we obtain a finitely axiomatizable subtheory of number theory (the theory Q of Tarski, Mostowski and Robinson, 1953) which is incomplete, and no finite extension of it is complete.

In the language $\mathscr{L}' = \{S, 0\}$ obtained by leaving out the symbols $+$ and $\cdot$, the subtheory of number theory given by axioms (1), (2) and the schema (7_φ), restricted of course to formulas of $\mathscr{L}'$, is complete. However, it is still not finitely axiomatizable, as can be shown by using the compactness theorem.

In the language $\mathscr{L}'' = \{+, S, 0\}$, the axioms (1)–(4) and the schema (7_φ), again restricted to formulas of $\mathscr{L}''$, give the *additive number theory* (or *Pressburger arithmetic*). This theory is not finitely axiomatizable, but it is complete (Presburger, 1929); the completeness of the theory $\mathscr{L}'$ in the previous paragraph follows from the proof given by Presburger.

EXAMPLE 1.4.12. We shall now discuss some examples of set theories.

There are two quite different reasons to include a discussion of set theories in a book on model theory. The first reason is that, if we wish to be completely precise, we should formulate our whole treatment of model theory within an appropriate system of axiomatic set theory. Actually, we are taking the more practical approach of formulating things in an informal set theory, but it is still important that, *in principle*, we could do it all in an axiomatic set theory. We have left for the Appendix an outline of the informal set theory we are using. The other reason for discussing set theories is that they are among the most interesting and important examples of theories. The second reason is the one which concerns us at this time. The theory of models is particularly well suited to the study of models of set theory. In the Appendix we have listed the axioms for four of the most familiar set theories: Zermelo, Zermelo–Fraenkel, Bernays, and Bernays–Morse. The first two of them are formulated in the language $\mathscr{L} = \{\epsilon\}$, while the other two are formulated in the language $\mathscr{L}' = \{\epsilon, V\}$, where ϵ is a binary relation symbol and V is a unary relation symbol. Zermelo set theory is a subtheory of Zermelo–Fraenkel, and Bernays set theory is a subtheory of Bernays–Morse.

The deepest results in set theory use constructions of models. However, these constructions are often of a special nature, for models of set theory only, and are therefore outside the scope of this book. For instance, the notion of constructible sets was used by Gödel (1939) to show that if Bernays set theory is consistent, then it remains consistent if we add to it the axiom of choice and the generalized continuum hypothesis; in other words, if Bernays set theory has a model, then it has a model in which the axiom of choice and the generalized continuum hypothesis are true. The same proofs and results are also well known to hold for Zermelo–Fraenkel set theory. Cohen's forcing construction has been used by Cohen and others to obtain a remarkable series of additional consistency results (see Cohen, 1963). For example, if Bernays (or Zermelo–Fraenkel) set theory has a model, then it has a model in which the axiom of choice is false, and another model in which the axiom of choice is true but the generalized continuum hypothesis is false.

In the rest of our discussion let us use the abbreviation ZF for 'Zermelo–Fraenkel set theory'. Whether or not we can prove that ZF is consistent depends on just how much we are assuming in our intuitive set theory. If our intuitive set theory is just a replica of ZF, then we cannot prove the consistency of ZF, even if we allow the use of the axiom of choice. Similarly, for any of the other set theories T we have introduced in the Appendix, we cannot prove the consistency of T if our intuitive set theory is a replica of T. These assertions follow from the Gödel incompleteness theorem. On the other hand, in Bernays–Morse set theory we can prove the consistency of Bernays set theory and of ZF. In ZF we can prove the consistency of Zermelo set theory. If we assume the existence of an inaccessible cardinal, then we can prove that Bernays–Morse set theory as well as ZF are consistent. Bernays set theory and ZF are very close to each other, and we can prove that one is consistent if and only if the other is. We shall leave the last three results above for exercises.

Neither Zermelo set theory, nor ZF, nor Bernays–Morse set theory is finitely axiomatizable (assuming that they are consistent). But, surprisingly, Bernays set theory is finitely axiomatizable (Bernays, 1937). With its finite axiomatization it is sometimes called Bernays–Gödel set theory. Each of the four set theories in our discussion, like number theory, has the following property: if the theory is consistent, then it is not complete, and no finite extension of it is complete. This is another consequence of the Gödel incompleteness theorem.

There is no completely satisfactory notion of a 'standard' model of set theory. The closest thing to it is the notion of a *natural model*. Natural models, roughly, are models of the form $\langle M, \epsilon\rangle$, where M is a set of sets formed by starting with the empty set and repeating the operations of union and power set, while ϵ is the ϵ-relation restricted to M. More precisely, we define for each ordinal α the set $R(\alpha)$ by

$$R(0) = 0,$$
$$R(\alpha+1) = S(R(\alpha)),$$

and

$$R(\alpha) = \bigcup_{\beta<\alpha} R(\beta) \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Then a *natural model* of ZF (or of Zermelo set theory) is a model of the form $\langle R(\alpha), \epsilon\rangle$. A natural model of Bernays set theory is a model of the form $\langle R(\alpha+1), \epsilon, R(\alpha)\rangle$.

None of our set theories has any countable natural models. For this reason, a somewhat weaker notion of 'standard' model is also important. A model $\langle M, \epsilon\rangle$ is said to be a *transitive model* iff ϵ is the ϵ-relation restricted to M and every element of an element of M is an element of M. For models of the language $\mathscr{L}' = \{\epsilon, V\}$ we make a similar definition. The countable transitive models are the most important models for Cohen's forcing construction.

Since number theory has just one standard model and is not complete, it has consistent extensions which have no standard models. If ZF has any transitive model at all, then it has many nonequivalent transitive models. Nevertheless, if ZF is consistent, then it has consistent extensions which have no transitive models at all. Moreover, in ZF plus the axiom of choice, we cannot prove the following: if ZF has a model, then ZF has a transitive model.

EXERCISES

1.4.1. Is there a theory of well order in the first-order language $\{\leqslant\}$?

1.4.2. Find two dense orders without endpoints of the same power which are not isomorphic.

1.4.3. Every finite Boolean algebra is atomic. If it has n atoms, then it has exactly 2^n elements. Any two finite Boolean algebras with the same number of elements are isomorphic.

1.4.4. Every finite subset of a Boolean algebra generates a finite sub-Boolean algebra.

1.4.5. Find two atomless nonisomorphic Boolean algebras of the same power.

1.4.6. Prove the following weak form of the representation theorem: Every atomic Boolean algebra is isomorphic to a field of sets.

1.4.7. Prove that the theory of *infinite models* whose axioms are the infinite list of sentences σ_n, where each sentence σ_n says that there are at least n distinct elements, is not finitely axiomatizable.

1.4.8. Prove that there is no theory T such that $\mathfrak{A}$ is a model of T if and only if $\mathfrak{A}$ is finite.

1.4.9. Let $\mathfrak{A} = \langle A, +, \cdot, {}^-, 0, 1\rangle$ be a Boolean algebra. A (proper) *filter* on $\mathfrak{A}$ is a subset $D \subset A$ such that $D \neq \emptyset$, $D \neq A$, and whenever $x, y \in D$ and $x \leqslant z$, then $x \cdot y \in D$ and $z \in D$. A subset E of A is said to have the *finite intersection property* iff for all $x_1, \ldots, x_n \in E$ and all n, $x_1 \cdot x_2 \cdot \ldots \cdot x_n \neq 0$. Prove that every subset E with the finite intersection property generates a filter D in the following sense:

$$x \in D \quad \text{iff} \quad x \geqslant y_1 \cdot \ldots \cdot y_n \quad \text{for some} \quad y_1, \ldots, y_n \in E.$$

A filter D on $\mathfrak{A}$ is said to be *principal* iff for some element $a \neq 0$ in A,

$$x \in D \quad \text{iff} \quad a \leqslant x.$$

D is an *ultrafilter* on $\mathfrak{A}$ iff no proper extension of D is a filter on $\mathfrak{A}$. Prove that the only principal ultrafilters on $\mathfrak{A}$ are generated by the atoms of $\mathfrak{A}$, i.e., D is a principal ultrafilter iff for some atom $a \in A$,

$$x \in D \quad \text{iff} \quad a \leqslant x.$$

Thus, if $\mathfrak{A}$ is atomless, then all ultrafilters on $\mathfrak{A}$ are nonprincipal. Prove the following:

(i). Every nonzero element of A belongs to an ultrafilter, and more generally, every filter on $\mathfrak{A}$ can be extended to an ultrafilter.

(ii). If D is an ultrafilter on $\mathfrak{A}$, then

$$x + y \in D \quad \text{iff either} \quad x \in D \quad \text{or} \quad y \in D, \qquad x \in D \quad \text{iff} \quad \bar{x} \notin D.$$

(iii). Let X be the set of all ultrafilters on $\mathfrak{A}$. For each $a \in A$, define

$$h_a = \{D \in X : a \in D\}.$$

Show that h is an isomorphism of $\mathfrak{A}$ onto the field of sets

$$\langle \{h_a : a \in A\}, \cup, \cap, {}^-, \emptyset, X\rangle.$$

This gives a proof of Proposition 1.4.4.

1.4.10. Let $\mathscr{L}$ be a first-order language and consider the equivalence relation $\vdash \varphi \leftrightarrow \psi$ on the formulas of $\mathscr{L}$. Let

$$(\varphi) = \{\psi : \vdash \varphi \leftrightarrow \psi\},$$
$$B_{\mathscr{L}} = \{(\varphi) : \varphi \text{ a formula of } \mathscr{L}\},$$
$$0_{\mathscr{L}} = (\varphi \wedge \neg \varphi),\ 1_{\mathscr{L}} = (\varphi \vee \neg \varphi).$$

Define

$$(\varphi) + (\psi) = (\varphi \vee \psi), \qquad (\varphi) \cdot (\psi) = (\varphi \wedge \psi), \qquad \overline{(\varphi)} = (\neg \varphi).$$

Then $\mathfrak{B}_{\mathscr{L}} = \langle B_{\mathscr{L}}, +, \cdot, {}^-, 0_{\mathscr{L}}, 1_{\mathscr{L}}\rangle$ is a Boolean algebra and it is known as the *Lindenbaum algebra* of $\mathscr{L}$. We shall drop the subscript $\mathscr{L}$ if it is understood. $\mathfrak{B}$ has several important subalgebras. For each $n \in \omega$, we can define

$$B_n = \{(\varphi) : \varphi \text{ has at most the variables } v_0, v_1, \ldots, v_{n-1} \text{ free}\}.$$

Then each B_n determines a sub-Boolean algebra $\mathfrak{B}_n$ of $\mathfrak{B}$. In particular, $\mathfrak{B}_0$ is the *Lindenbaum algebra of all sentences* of $\mathscr{L}$. Prove that the following are equivalent:

(i). Lindenbaum's theorem for $\mathscr{L}$, Proposition 1.3.11.

(ii). Every filter on the Lindenbaum algebra $\mathfrak{B}_0$ can be extended to an ultrafilter.

1.4.11. Let T be any theory of $\mathscr{L}$ and define

$$D_T = \{(\varphi) : T \vdash \varphi\}.$$

Prove that:

(i). T is consistent iff D_T is a filter on $\mathfrak{B}_0$.

(ii). T is consistent and finitely axiomatizable iff D_T is a principal filter on $\mathfrak{B}_0$.

(iii). T is complete iff D_T is an ultrafilter in $\mathfrak{B}_0$.

(iv). T is complete and finitely axiomatizable iff D_T is a principal ultrafilter on $\mathfrak{B}_0$.

The converse of the above four equivalences also holds in the following sense. Let D be a subset of B_0 and define

$$T_D = \{\varphi : \varphi \text{ is a sentence and } (\varphi) \in D\}.$$

Then (i)–(iv) hold with T replaced by T_D and D_T by D. If T is complete, then, of course, the quotient algebra $\mathfrak{B}_0/D_T$ is the two-element algebra. In general, there is a one-to-one correspondence between complete closed extensions of T and ultrafilters in $\mathfrak{B}_0/D_T$. Show, without using the completeness theorem, that the above results still hold when the notions of closed theory, finitely axiomatizable and complete are replaced by their syntactical analogues.

1.4.12. Let $\mathscr{S}$ be the language of Section 1.2. By considering the equivalence relation $\vdash \varphi \leftrightarrow \psi$ on sentences of $\mathscr{S}$, we can define the exact analogue of $\mathfrak{B}_{\mathscr{L}}$, namely

$$\mathfrak{B}_{\mathscr{S}} = \langle B_{\mathscr{S}}, +, \cdot, {}^{-}, 0_{\mathscr{L}}, 1_{\mathscr{S}} \rangle.$$

The following shows that there is a close relation between completeness theorems and representations of Boolean algebras. Prove that the following are equivalent:

(i). The completeness theorem for $\mathscr{S}$, Theorem 1.2.7.

(ii). The Lindenbaum algebra $\mathfrak{B}_{\mathscr{S}}$ is isomorphic to a field of sets.

There is an analogue of this result for $\mathscr{L}$ and $\mathfrak{B}_{\mathscr{L}}$ which we shall discuss in the exercises for Section 2.1.

1.4.13. Show that in the language $\{S, 0\}$ the theories given by the two sets of axioms (see Example 1.4.11)

(A) axioms (1), (2), and schema (7_{φ}),

(B) axioms (1), (2), (8), and the schema

(9_n) $Sv_0 \not\equiv v_1 \vee Sv_1 \not\equiv v_2 \vee \ldots \vee Sv_n \not\equiv v_0$

are equivalent.

1.4.14. Show that if $\omega < \alpha$ and α is a limit ordinal, then $\langle R(\alpha), \epsilon \rangle$ is a model of Zermelo set theory. Hence in ZF we can prove that Zermelo set theory is consistent.

1.4.15. Let θ be an (uncountable) inaccessible cardinal. Show that $\langle R(\theta), \epsilon \rangle$ is a model of ZF, and that $\langle R(\theta+1), \epsilon, R(\theta) \rangle$ is a model of Bernays–Morse set theory.

1.4.16. Which axioms of ZF are true in the model $\langle R(\omega), \epsilon \rangle$? And in the model $\langle R(\omega+\omega), \epsilon \rangle$?

1.4.17*

(i). Let $\langle A, E, U \rangle$ be an arbitrary model of Bernays set theory. Prove that $\langle U, E \cap (U \times U) \rangle$ is a model of ZF. Hence if Bernays set theory is consistent, so is ZF.

(ii). Let $\mathfrak{A} = \langle A, E \rangle$ be an arbitrary model of ZF. We may assume that no subset of A belongs to A. Let us say that a subset $X \subset A$ is *definable* in $\mathfrak{A}$ iff there is a formula $\varphi(v_n \ldots v_n)$ and elements $x_1, \ldots, x_n \in A$ such that

$$X = \{x \in A : \mathfrak{A} \models \varphi[xx_1 \ldots x_n]\}.$$

Let B be the set of all definable subsets X of A such that there is no $y \in A$ with $X = \{x \in A : xEy\}$, and let E' be the set of all pairs $x \in A$, $X \in B$ such that $x \in X$. Prove that $\langle A \cup B, E \cup E', A \rangle$ is a model of Bernays set theory. Hence, if ZF is consistent, so is Bernays set theory.

1.4.18*

(i). Let $\langle A, \epsilon \rangle$ be any model of the axiom of extensionality, where ϵ is the ϵ-relation restricted to A. Prove that $\langle A, \epsilon \rangle$ is isomorphic to a transitive model, called a *transitive realization* of $\langle A, \epsilon \rangle$.

(ii). Show that any two transitive models of the axiom of extensionality which are isomorphic are equal.

1.4.19. Prove that there exists a complete theory T which has arbitrarily large natural models. T may be taken to be an extension of Zermelo set theory.

1.4.20. Let $\mathfrak{A}$ and $\mathfrak{B}$ be infinite simple orderings. Prove that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy exactly the same universal sentences.

1.5. Elimination of quantifiers

Each model $\mathfrak{A}$ of a theory T gives rise to a complete theory, namely the set of all sentences holding in $\mathfrak{A}$, which is an extension of T. For this reason, it is important to know something about the complete extensions of a theory. In a few fortunate cases, it is possible to give a simple description of all the complete extensions of a theory by using the method of elimination of quantifiers.

This method applies only to very special theories. Moreover, each time the method is applied to a new theory we must start from scratch in the proofs, because there are few opportunities to use general theorems about models. On the other hand, the method is extremely valuable when we want to beat a particular theory into the ground. When it can be carried out, the method of elimination of quantifiers gives a tremendous amount of information about a theory. For instance, it tells us about the behavior of all formulas, as well as all sentences, relative to the theory. Usually it also gives a uniform way of deciding whether or not a sentence belongs to the theory; in other words, it gives a proof that the theory is decidable.

The question of the decidability of a theory lies outside the scope of this book, since it is not usually considered model theory. However, it is a very important question, and in fact the most striking applications of the elimination of quantifiers are to show that certain theories are decidable. The method is also valuable as a source of examples of thoroughly understood theories, which are useful for testing conjectures and for illustrating results. The method may be thought of as a direct attack on a theory. Later on, especially in §3.5, we shall learn of several more indirect attacks on theories, which work more often but give less information in particular cases.

Before describing the method, we need some more notation. In Section 1.3, we introduced the notion of a sentence φ being a consequence of a set Σ of sentences, in symbols $\Sigma \models \varphi$. What meaning shall we give to $\Sigma \models \varphi$ if φ is a formula? We shall say that a formula $\varphi(v_0 \dots v_n)$ is a *consequence of* Σ, symbolically $\Sigma \models \varphi$, iff for every model $\mathfrak{A}$ of Σ and every sequence $a_0, \dots, a_n \in A$, $a_0, \dots, a_n$ satisfies φ. It follows that the *formula* $\varphi(v_0 \dots v_n)$ is a consequence of Σ if and only if the *sentence* $(\forall v_0 \dots v_n)\varphi(v_0 \dots v_n)$ is a consequence of Σ. We say that two formulas φ, ψ are *Σ-equivalent* iff $\Sigma \models \varphi \leftrightarrow \psi$.

In general, the method of elimination of quantifiers is as follows: First, depending on the theory T, we pick out an appropriate set of formulas, called *basic formulas*. By a *Boolean combination* of basic formulas we mean a formula obtained from basic formulas by repeated application of the connectives $\neg$, $\wedge$. The main result to be proved is that *every formula is T-equivalent to a Boolean combination of basic formulas*. The key step in the proof is the step where we 'eliminate quantifiers'. In fact, we may state at once a simple but general lemma which shows why the name 'elimination of quantifiers' is given to the method (the name is due to Tarski, 1935).

LEMMA 1.5.1. *Let T be a theory and let Σ be a set of formulas, called basic formulas. In order to show that every formula is T-equivalent to a Boolean combination of basic formulas, it is sufficient to show the following*:

(i). *Every atomic formula is T-equivalent to a Boolean combination of basic formulas.*

(ii). *If θ is a Boolean combination of basic formulas, then $(\exists v_m)\theta$ is T-equivalent to a Boolean combination of basic formulas.*

PROOF. Let Ψ be the set of all formulas which are T-equivalent to a Boolean combination of basic formulas. We show by induction that every formula φ belongs to Ψ. If φ is an atomic formula, then $\varphi \in \Psi$ by (i). If φ is $\neg\psi$ and $\psi \in \Psi$, it is obvious that $\varphi \in \Psi$. Similarly, if φ is $\psi_1 \wedge \psi_2$ and $\psi_1, \psi_2 \in \Psi$, then $\varphi \in \Psi$. If φ is $(\exists v_m)\psi$ and $\psi \in \Psi$, then ψ is T-equivalent to a Boolean combination θ of basic formulas. Moreover, φ is T-equivalent to $(\exists v_m)\theta$. By (ii), $(\exists v_m)\theta \in \Psi$, so $\varphi \in \Psi$. ⊣

We shall illustrate the method with two simple examples. Our first example is the theory of dense simple order without endpoints (Example 1.4.1). Let us temporarily (in this section only) call this theory Δ. As we mentioned in Section 1.4, the theory Δ is complete. The method of elimination of quantifiers is one of several ways which we shall come across for proving that theories are complete. The completeness of Δ will follow from our results below. The elimination of quantifiers was applied to the theory Δ very early, by Langford (1927).

As basic formulas we shall take the atomic formulas

$$v_m \equiv v_n, \qquad v_m \leqslant v_n.$$

The Boolean combinations of atomic formulas are precisely the formulas which have no quantifiers. In any language, formulas which have no quantifiers are called *open formulas*. We wish to prove that every formula φ is Δ-equivalent to an open formula ψ. As we carry out our arguments, we shall also keep track of which variables occur in the open formula which is Δ-equivalent to a given formula. This will be useful for applications. Before we can eliminate any quantifiers, we must take a close look at the open formulas. For convenience, we use the abbreviation

$$v_m < v_n \quad \text{for} \quad v_m \leqslant v_n \wedge \neg v_m \equiv v_n.$$

Let us consider $n+1$ variables $v_0, \dots, v_n$, $n > 0$. By an *arrangement* of the variables $v_0, \dots, v_n$ we mean a finite conjunction of the form

$$\theta_0 \wedge \theta_1 \wedge \dots \wedge \theta_{n-1},$$

where $u_0, \dots, u_n$ is a renumbering of $v_0, \dots, v_n$ and each formula θ_i is either $u_i < u_{i+1}$, or else $u_i \equiv u_{i+1}$. The lemma below allows us to put every open formula into a 'normal form' built up from arrangements of the variables.

LEMMA 1.5.2. *Every open formula $\varphi(v_0 \dots v_n)$ is Δ-equivalent either to one of the formulas $v_0 < v_0$, $v_0 \equiv v_0$, or else to the disjunction of finitely many arrangements of the variables $v_0, \dots, v_n$.*

PROOF. First, we consider the case $n = 0$. In this case, the open formula $\varphi(v_0)$ is built up from the atomic formulas $v_0 \leqslant v_0$, $v_0 \equiv v_0$. Since $\Delta \models v_0 \leqslant v_0$ and $\Delta \models v_0 \equiv v_0$, we must have either $\Delta \models \varphi$ and $\Delta \models \varphi \leftrightarrow v_0 \equiv v_0$, or else $\Delta \models \neg\varphi$ and $\Delta \models \varphi \leftrightarrow v_0 < v_0$.

Let us now make three observations about arrangements (we assume that $n > 0$):

(1). There are only finitely many different arrangements of the variables $v_0, \dots, v_n$.

(2). For each simply-ordered structure $\mathfrak{A}$, each sequence $a_0, \dots, a_n$ satisfies some arrangement of $v_0, \dots, v_n$.

(3). Let $\varphi(v_0 \dots v_n)$ be an open formula and let ψ be an arrangement of $v_0, \dots, v_n$. Then one or both of the formulas $\psi \to \varphi$, $\psi \to \neg\varphi$, is a consequence of the theory of simple order.

(1) should be obvious, while (2) follows easily from the fact that in a simply-ordered structure, exactly one of the relations $a < b$, $a = b$, $b < a$ holds between two elements a, b. (3) is proved by induction on the length of the open formula φ, and is left to the reader.

Now let $\varphi(v_0 \dots v_n)$ be an open formula. If $\Delta \models \neg\varphi$, then φ is Δ-equivalent to the formula $v_0 < v_0$. Assume the other possibility, that it is not the case that $\Delta \models \neg\varphi$. Consider any model $\mathfrak{A}$ of Δ and sequence $a_0, \dots, a_n$ which satisfies φ in $\mathfrak{A}$. By (2), $a_0, \dots, a_n$ also satisfies some arrangement ψ of $v_0, \dots, v_n$ in $\mathfrak{A}$. Thus we cannot have $\Delta \models \psi \to \neg\varphi$, and, by (3), we must have $\Delta \models \psi \to \varphi$. Form the disjunction θ of all arrangements ψ of $v_0, \dots, v_n$ for which $\Delta \models \psi \to \varphi$. θ is the disjunction of at least one, but only finitely many formulas, in view of (1). It follows from our remarks above that $\Delta \models \varphi \to \theta$, and from the definition of θ we see that $\Delta \models \theta \to \varphi$. So φ and θ are Δ-equivalent, and our proof is complete. ⊣

We observe that actually the above lemma is true for the theory of simple order as well as for the theory Δ. The reader may check this by going carefully through the proof, noticing that the only axioms of Δ which we actually made use of are the axioms of simple order. In the next theorem, however, we need all of the axioms of Δ.

Theorem 1.5.3. *Every formula φ is Δ-equivalent to an open formula ψ. Moreover, if all the free variables of φ are among $v_0, \dots, v_n$, $n \geqslant 0$, then ψ can be chosen so that all its variables are among $v_0, \dots, v_n$.*

Proof. We first prove that every formula φ is Δ-equivalent to an open formula ψ. By Lemma 1.5.1, it suffices to prove that for every open formula $\psi(v_0 \dots v_n)$, the formula $(\exists v_m)\psi$ is Δ-equivalent to an open formula. If $m > n$, then v_m does not occur at all in ψ, so $(\exists v_m)\psi$ is Δ-equivalent to ψ. We may thus assume that $m \leqslant n$. By renaming the variables we can even make $m = n$.

Using Lemma 1.5.2, we may suppose that ψ is either $v_0 < v_0$, $v_0 \equiv v_0$, or a disjunction of finitely many arrangements of $v_0, \dots, v_n$. If ψ is either $v_0 < v_0$ or $v_0 \equiv v_0$, then obviously $(\exists v_n)\psi$ is Δ-equivalent to ψ. In the remaining case, let

$$\psi = \theta_0 \vee \dots \vee \theta_p,$$

where each θ_i is an arrangement of $v_0, \dots, v_n$. Then

$$\Delta \models (\exists v_n)\psi \leftrightarrow (\exists v_n)\theta_0 \vee \dots \vee (\exists v_n)\theta_p.$$

We may eliminate the quantifier $(\exists v_n)$ in the following way: If $n = 1$, the only possibilities for the formulas $(\exists v_1)\theta_i$ are

$$(\exists v_1)v_0 < v_1, \qquad (\exists v_1)v_0 \equiv v_1, \qquad (\exists v_1)v_1 < v_0.$$

Each of these is a consequence of Δ, and it follows that $(\exists v_1)\psi$ is a consequence of Δ and is Δ-equivalent to $v_0 \equiv v_0$.

Let $n > 1$. Then, from each arrangement θ_i of $v_0, \dots, v_n$, we may form in a natural way an arrangement θ_i^* of $v_0, \dots, v_{n-1}$ obtained by leaving out v_n. It is easy to see that

$$\Delta \models (\exists v_n)\theta_i \leftrightarrow \theta_i^*, \qquad i = 0, \dots, p,$$

and hence

$$\Delta \models (\exists v_n)\psi \leftrightarrow \theta_0^* \vee \dots \vee \theta_p^*.$$

We have shown in each case that $(\exists v_m)\psi$ is Δ-equivalent to an open formula.

We now prove the second clause of the theorem. Our proof given above actually shows that if $\psi(v_0 \dots v_n)$ is an open formula, $n > 0$, then $(\exists v_n)\psi$ is Δ-equivalent to an open formula of the form $\theta(v_0 \dots v_{n-1})$. Let $\varphi(v_0 \dots v_n)$ be an arbitrary formula, $n \geqslant 0$. Then φ is Δ-equivalent to some open formula $\psi(v_0 \dots v_n \dots v_{n+m})$. But φ is also Δ-equivalent to $(\exists v_{n+1}) \dots (\exists v_{n+m})\varphi$, and hence to $(\exists v_{n+1}) \dots (\exists v_{n+m})\psi$. The latter formula is Δ-equivalent to an open formula of the form $\theta(v_0 \dots v_n)$, and thus φ is Δ-equivalent to θ. Our proof is complete. ⊣

The proof of the theorem also gives a decision procedure for the theory Δ. Very briefly, the decision procedure is as follows. We are given an arbitrary sentence φ and we wish to determine whether φ belongs to the theory Δ. Our first step is to put φ into prenex normal form, say (after renumbering variables),

$$(\mathrm{Q}_0 v_0)(\mathrm{Q}_1 v_1) \dots (\mathrm{Q}_n v_n)\psi,$$

where $\mathrm{Q}_0, \dots, \mathrm{Q}_n$ are quantifier symbols $\exists$ or $\forall$, and ψ is open. We may assume further that Q_n is $\exists$, for otherwise we may work with $\neg\varphi$. Next, we put ψ into one of the forms $v_0 < v_0$, $v_0 \equiv v_0$, or a disjunction of finitely many arrangements of $v_0, \dots, v_n$. Then we eliminate the quantifier $(\exists v_n)$, that is, we replace $(\exists v_n)\psi$ by a Δ-equivalent open formula $\theta(v_0 \dots v_{n-1})$ by the process explained in the proof. After that, we repeat the process until all the variables except v_0 are eliminated. When we finish, we can tell at once whether the resulting sentence $(Q_0 v_0)\theta(v_0)$ belongs to Δ. Of course, the decision procedure can be streamlined very much if it is really going to be used.

We now obtain another consequence of the theorem.

Corollary 1.5.4. *The theory of dense simple order without endpoints is complete.*

Proof. Let φ be an arbitrary sentence. By Theorem 1.5.3, φ is Δ-equivalent to an open formula $\psi(v_0)$. But for any open formula $\psi(v_0)$, we have either $\Delta \models \psi$ or $\Delta \models \neg \psi$. Hence either $\Delta \models \varphi$ or $\Delta \models \neg \varphi$, and Δ is complete. $\dashv$

Note that Corollary 1.5.4 is only concerned with sentences, but to prove the corollary via Theorem 1.5.3 we had to use an induction concerned with arbitrary formulas. This happens time and again in model theory, because the notion of a sentence is defined using the recursive definition of a formula. Theorem 1.5.3 also tells us something about the theories formed by adding new constant symbols to the language and taking Δ as a set of axioms. We leave this application of the theorem as an exercise.

Theorem 1.5.3 can be improved a little by taking for our basic formulas only the formulas $v_m \leqslant v_n$.

Corollary 1.5.5. *Every formula $\rho(v_0 \dots v_n)$ is Δ-equivalent to a Boolean combination of formulas of the form $v_m \leqslant v_p$, where $m = 0, \dots, n$ and $p = 0, \dots, n$.*

Proof. In view of Theorem 1.5.3, it is enough to observe that

$$\Delta \models v_m \equiv v_p \leftrightarrow v_m \leqslant v_p \wedge v_p \leqslant v_m. \dashv$$

We now take up our second example of the elimination of quantifiers. We shall obtain a full description of all complete closed theories in the pure identity language (see Section 1.3), which has no predicate, function or constant symbols at all. In other words, we shall describe all complete closed extensions of the theory with the empty set of axioms in the pure identity language.

As in the case of dense simple order, we begin with a lemma about arrangements. What should we mean by an arrangement this time? An arrangement of $v_0, \dots, v_n$ will be a formula which tells which variables are equal to each other and which are unequal. To be precise, we let e be an equivalence relation over the set $\{0, 1, \dots, n\}$ of indices of the variables $v_0, \dots, v_n$. We define the *arrangement of $v_0, \dots, v_n$ given by e* to be the conjunction of all the formulas

$$v_i \equiv v_j, \quad iej; \quad \text{and} \quad \neg v_i \equiv v_j, \quad \text{not} \quad iej.$$

Lemma 1.5.6. *Every open formula $\varphi(v_0 \dots v_n)$ is either inconsistent or is equivalent to a disjunction of finitely many arrangements of $v_0, \dots, v_n$.*

The proof is very similar to that of Lemma 1.5.2, so we leave it as an exercise.

We now must decide on our set of basic formulas. It should be clear that the atomic formulas are not enough. For instance, the sentence $(\forall v_0 v_1)(v_0 \equiv v_1)$ cannot be expressed by an open formula.

For our basic formulas we take all atomic formulas

$$v_m \equiv v_n,$$

together with the sentences σ_n which state that 'there are more than n distinct elements'. Formally, σ_n, $n > 0$, may be written

$$(\forall v_1 \dots v_n)(\exists v_0)(\neg v_0 \equiv v_1 \wedge \dots \wedge \neg v_0 \equiv v_n).$$

For good measure, we shall define σ_0 to be a valid sentence, say $(\exists v_0)(v_0 \equiv v_0)$.

Theorem 1.5.7. *Every formula φ in the pure identity language is equivalent to a Boolean combination ψ of basic formulas. Moreover, if all the free variables of φ are among $v_0, \dots, v_n$, then ψ may be chosen so that all its free variables are among $v_0, \dots, v_n$. In particular, if φ is a sentence, then so is ψ.*

Proof. We first show that every formula is equivalent to a Boolean combination of basic formulas. Let $\psi(v_0 \dots v_n)$ be an arbitrary Boolean combination of basic formulas. By Lemma 1.5.1, it suffices to prove that $(\exists v_m)\psi$ is equivalent to a Boolean combination of basic formulas. First, we note that ψ is equivalent to a formula of the form

$$(\psi_0 \wedge \theta_0) \vee \dots \vee (\psi_p \wedge \theta_p)$$

where each ψ_i is an open formula and each θ_i is a Boolean combination of the sentences $\sigma_0, \sigma_1, \sigma_2, \dots$. Still better, using Lemma 1.5.6, we may make each ψ_i be either the inconsistent sentence $\neg\sigma_0$ or else a disjunction of finitely many arrangements of $v_0, \ldots, v_n$.

As in the previous theorem, we may assume without loss of generality that $m = n$. In the case $n = 0$, the only arrangement of v_0 is the valid formula $v_0 \equiv v_0$, so each ψ_i is either valid, in which case it may be replaced by σ_0, or else it is the inconsistent formula $\neg\sigma_0$. Thus ψ is equivalent to a Boolean combination of the sentences $\sigma_0, \sigma_1, \ldots$, and so is $(\exists v_n)\psi$.

Assume that $n > 0$. For each arrangement ψ_i, $i \leqslant p$, form ψ_i^* by deleting all the equations and inequalities in which v_n occurs. Then ψ_i^* is an arrangement of the remaining variables $v_0, \ldots, v_{n-1}$. (If ψ_i happens to be $\neg\sigma_0$, we simply let ψ_i^* also be $\neg\sigma_0$.) Note that $(\exists v_n)\psi_i$ is *not*, in general, equivalent to ψ_i^*. (Why?) However, if e_i is the equivalence relation from which the arrangement ψ_i comes, and r_i is the number of equivalence classes in e_i, then we easily see that $(\exists v_n)\psi_i$ is equivalent to $\sigma_{r_i - 1} \wedge \psi_i^*$. Also $(\exists v_n)\psi$ is equivalent to the formula

$$(\theta_0 \wedge (\exists v_n)\psi_0) \vee \ldots \vee (\theta_p \wedge (\exists v_n)\psi_p).$$

It follows that $(\exists v_n)\psi$ is equivalent to

$$(\theta_0 \wedge \sigma_{r_0 - 1} \wedge \psi_0^*) \vee \ldots \vee (\theta_p \wedge \sigma_{r_p - 1} \wedge \psi_p^*).$$

This is indeed a Boolean combination of basic formulas, and the first part of the theorem is proved.

Now, using exactly the same trick as we used at the end of the proof of Theorem 1.5.3, we can obtain the full statement of the theorem – that each formula $\varphi(v_0 \ldots v_n)$ is equivalent to a Boolean combination $\psi(v_0 \ldots v_n)$ of basic formulas, and, if φ is a sentence, then so is ψ. ⊣

We are now ready to describe clearly all the closed theories in the pure identity language. It is easy to see that for every finite set N of positive natural numbers, there is a sentence $\sigma(N)$ whose models are precisely those $\mathfrak{A}$ such that $|A| \in N$. The reader should check that for each N, $\sigma(N)$ is a pure identity sentence, and, in fact, is a Boolean combination of $\sigma_0, \sigma_1, \ldots$. We now can conclude that, up to equivalence, the sentences $\sigma(N)$ and their negations are the only pure identity sentences.

COROLLARY 1.5.8. *For every pure identity sentence φ, there is a finite set N of positive natural numbers such that φ is equivalent either to $\sigma(N)$ or to $\neg\sigma(N)$.*

We now take up the theories. It is also easy to see that for each finite or infinite set N of positive natural numbers, there is a closed theory $\Delta(N)$ whose models are precisely those $\mathfrak{A}$ such that either $|A| \in N$ or $\mathfrak{A}$ is infinite. Again, the reader should check that each theory $\Delta(N)$ has a set of pure identity sentences for axioms. To make our notation more complete, we may as well write $\Sigma(N)$ for the closed theory which has the single axiom $\sigma(N)$, where N is finite. The next corollary shows that the $\Delta(N)$ and $\Sigma(N)$ are the only closed theories in the pure identity language.

COROLLARY 1.5.9.

(i). *The finitely axiomatizable closed theories in the pure identity language are precisely the theories $\Sigma(N)$, where N is finite, and $\Delta(N)$, where $\omega - N$ is finite.*

(ii). *The nonfinitely axiomatizable closed theories in the pure identity language are precisely the theories $\Delta(N)$, where $\omega - N$ is infinite.*

PROOF. (i). The theories $\Sigma(M)$, M finite, and $\Delta(N)$, $\omega - N$ finite, are finitely axiomatizable. Indeed, $\Sigma(M)$ has the single axiom $\sigma(M)$, and $\Delta(N)$ the single axiom $\neg\sigma(\omega - N)$. By Corollary 1.5.8, any finitely axiomatizable theory has a single axiom of the form $\sigma(N)$, or else $\neg\sigma(N)$, N finite. This proves (i).

(ii). Now let T be an arbitrary closed theory in the pure identity language. Let N be the set of all positive natural numbers such that T has a model of power in N. Then for each finite model $\mathfrak{A}$, $\mathfrak{A}$ is a model of T if and only if $|A| \in N$. If one of the sentences $\sigma(M)$, M finite, belongs to T, then all models of T are finite and $N \subset M$, and thus $T = \Sigma(N)$.

Assume now that T is not of the form $\Sigma(N)$. It follows that every sentence $\varphi \in T$ is equivalent to a sentence of the form $\neg\sigma(M)$. Let N' be the union of all sets M such that $T \models \neg\sigma(M)$. Then clearly $\mathfrak{A}$ is a model of T if and only if $\mathfrak{A}$ is infinite or $|A| \in \omega - N'$. Therefore $T = \Delta(\omega - N')$. Note also that $\omega - N' = N$, so T is $\Delta(N)$. Finally, if T is not finitely axiomatizable, then by (i) the set $\omega - N$ is infinite. This proves (ii). ⊣

Note that by our last corollary, there is no theory at all whose models consist precisely of all finite models. Likewise, if N is an arbitrary infinite set of positive natural numbers, there is no theory whose models consist of all $\mathfrak{A}$ such that $|A| \in N$. In other words, in the pure identity language, *any theory which has arbitrarily large finite models has an infinite model.* We shall see later that this is true in every other first-order language as well.

There are several very important theories which have been analyzed using the elimination of quantifiers. For example, additive number theory (Presburger, 1929), the theory of Abelian groups (Szmielew, 1955), the theory of Boolean algebras (Tarski, 1949), the theory of all well-ordered models (Mostowski and Tarski, 1949), and the theories of real closed fields and of algebraically closed fields (Tarski, 1948). As might be guessed from our two simple examples, the elimination of quantifiers becomes quite difficult in some of the more substantial cases mentioned above. In each of those cases, the method gives a decision procedure for the theory, as well as a useful classification of all formulas and all complete extensions of the theory.

Most of the interesting theories which arise in mathematics are undecidable (e.g., number theory, set theory, groups, fields, partial order), and the method of elimination of quantifiers does not work for these theories.

EXERCISES

1.5.1. Let $\mathscr{L}(n)$ be the language $\{\leqslant, c_0, \dots, c_{n-1}\}$ obtained from the language $\{\leqslant\}$ by adding n constant symbols.

(i). Show that the set Δ of sentences is not complete in the language $\mathscr{L}(n)$, for $n > 1$. Show that all the complete extensions are finitely axiomatizable.

(ii)*. Describe all the complete extensions of Δ in the language $\mathscr{L}(\omega)$.

1.5.2*. Let Γ be the theory of dense simple order. Prove that Γ has exactly four complete closed extensions, which come from one of the four additional axioms:

there are no endpoints;
there is a left endpoint but no right endpoint;
there is a right endpoint but no left endpoint;
there are a right and a left endpoint.

Hint: As a set of basic formulas, take the set of all atomic formulas together with the formulas which state:

v_m is a left endpoint;
v_m is a right endpoint;
there is a left endpoint;
there is a right endpoint.

Modify the proof of Theorem 1.5.3 to show that every formula is Γ-equivalent to a Boolean combination of basic formulas.]

1.5.3*. Show by elimination of quantifiers that the theory of atomless Boolean algebras is complete.

1.5.4. Which are the complete theories in the pure identity language? State a simple criterion for two models $\mathfrak{A}$, $\mathfrak{B}$ of that language to be equivalent.

1.5.5. Describe all the complete theories in the language which has n constant symbols but no relation or function symbols. Do the same for the language with ω constant symbols.

1.5.6. Outline a decision procedure for deciding whether a given pure identity sentence is valid.

1.5.7*. Analyze the following theories using the method of elimination of quantifiers:

(i). The theory with no axioms in the language with one 1-placed relation symbol and no other symbols (Behmann, 1922).
(ii). The theory of Abelian groups with all elements being of order 3.
(iii). The theory of divisible Abelian groups.
(iv). Monadic first-order logic.

1.5.8*. This and the succeeding exercises are very long and tedious, if complete proofs are written out. The theory of one *successor function* has the axioms (1), (2) and $(7)_\varphi$ from Example 1.4.11 in the language $\mathscr{L} = \{S, 0\}$. Prove that this theory is complete by elimination of quantifiers.

1.5.9*. Prove that additive number theory (from Example 1.4.11) in the language $\mathscr{L} = \{+, S, 0\}$ is complete by elimination of quantifiers.

1.5.10*. The theory of *one equivalence relation* in the language $\mathscr{L} = \{E\}$ has the following axioms:

xEx;
$xEy \wedge yEz \rightarrow xEz$;
$xEy \rightarrow yEx$.

Give a decision procedure for this theory by the method of elimination of quantifiers.

1.5.11**. Give a decision procedure for the theory of Abelian groups (Example 1.4.6) by elimination of quantifiers. Use this to describe all complete extensions of this theory.

CHAPTER 2

MODELS CONSTRUCTED FROM CONSTANTS

2.1. Completeness and compactness

In this section, we prove the basic completeness theorem first proved by Gödel (1930). The proof we give is due to Henkin (1949) and it applies to situations somewhat more general than Gödel's original proof. This extension was already noted by Malcev (1936).

The result we prove is that every consistent set of sentences T in a language $\mathscr{L}$ has a model or, in other words, is satisfiable. The proof proceeds in two stages. We shall first show that T can be extended to another consistent set of sentences $\overline{T}$ in an expanded language $\overline{\mathscr{L}}$, having certain desirable features. Then we show that every T having these desirable features has a model. It will make no difference which of the two steps we prove first.

Definition. Let T be a set of sentences of $\mathscr{L}$ and let C be a set of constant symbols of $\mathscr{L}$. (C might be a proper subset of the set of all constant symbols of $\mathscr{L}$.) We say that C is a *set of witnesses* for T in $\mathscr{L}$ iff for every formula φ of $\mathscr{L}$ with at most one free variable, say x, there is a constant $c \in C$ such that

$$T \vdash (\exists x)\varphi \rightarrow \varphi(c).$$

We say that T *has witnesses in* $\mathscr{L}$ iff T has some set C of witnesses in $\mathscr{L}$.

The meaning and usage of $\varphi(c)$ should be quite clear here and in all succeeding places in this chapter: $\varphi(c)$ is obtained from φ by replacing simultaneously all free occurrences of x in φ by the constant c. We shall be careful to use $\varphi(c)$ only when it has been made clear from the context which variable x is to be replaced by c. Otherwise the notation $\varphi(c)$ would be ambiguous. For example, if φ is a formula with the free variables x, y, we have to indicate whether $\varphi(c)$ is obtained from φ by replacing x by c or by replacing y by c. An alternative notation which is completely unambiguous is to write $\varphi(c/x)$ for the formula obtained by replacing all free occurrences of x in φ by c. However, we prefer to use $\varphi(c)$ and rely on the context for clarity rather than use the more cluttered notation $\varphi(c/x)$.

LEMMA 2.1.1. *Let T be a consistent set of sentences of $\mathscr{L}$. Let C be a set of new constant symbols of power $|C| = \|\mathscr{L}\|$, and let $\overline{\mathscr{L}} = \mathscr{L} \cup C$ be the simple expansion of $\mathscr{L}$ formed by adding C. Then T can be extended to a consistent set of sentences $\overline{T}$ in $\overline{\mathscr{L}}$ which has C as a set of witnesses in $\overline{\mathscr{L}}$.*

PROOF. Let $\alpha = \|\mathscr{L}\|$. For each $\beta < \alpha$, let c_β be a constant symbol which does not occur in $\mathscr{L}$ and such that $c_\beta \neq c_\gamma$ if $\beta < \gamma < \alpha$. Let $C = \{c_\beta : \beta < \alpha\}$, $\overline{\mathscr{L}} = \mathscr{L} \cup C$. Clearly $\|\overline{\mathscr{L}}\| = \alpha$, so we may arrange all formulas of $\overline{\mathscr{L}}$ with at most one free variable in a sequence φ_ξ, $\xi < \alpha$. We now define an increasing sequence of sets of sentences of $\overline{\mathscr{L}}$:

$$T = T_0 \subset T_1 \subset \ldots \subset T_\xi \subset \ldots, \qquad \xi < \alpha,$$

and a sequence d_ξ, $\xi < \alpha$, of constants from C such that:

(i). each T_ξ is consistent in $\overline{\mathscr{L}}$;

(ii). if $\xi = \zeta + 1$, then $T_\xi = T_\zeta \cup \{(\exists x_\zeta)\varphi_\zeta \to \varphi_\zeta(d_\zeta)\}$; x_ζ is the free variable in φ_ζ if it has one, otherwise $x_\zeta = v_0$;

(iii). if ξ is a limit ordinal different from 0, then $T_\xi = \bigcup_{\zeta<\xi} T_\zeta$.

Suppose that T_ζ has been defined. Note that the number of sentences in T_ζ which are not sentences of $\mathscr{L}$ is smaller than α, i.e., the cardinal of the set of such sentences is less than α. Furthermore, each such sentence contains at most a finite number of constants from C. Therefore, let d_ζ be the first element of C which has not yet occurred in T_ζ. For instance, $d_0 = c_0$. We show that

$$T_{\zeta+1} = T_\zeta \cup \{(\exists x_\zeta)\varphi_\zeta \to \varphi_\zeta(d_\zeta)\}$$

is consistent. If this were not the case, then

$$T_\zeta \vdash \neg((\exists x_\zeta)\varphi_\zeta \to \varphi_\zeta(d_\zeta)).$$

By propositional logic,

$$T_\zeta \vdash (\exists x_\zeta)\varphi_\zeta \wedge \neg\, \varphi_\zeta(d_\zeta).$$

As d_ζ does not occur in T_ζ, we have by predicate logic,

$$T_\zeta \vdash (\forall x_\zeta)((\exists x_\zeta)\varphi_\zeta \wedge \neg\, \varphi_\zeta(x_\zeta)),$$

$$T_\zeta \vdash (\exists x_\zeta)\varphi_\zeta \wedge \neg\, (\exists x_\zeta)\varphi_\zeta,$$

which contradicts the consistency of T_ζ. If ξ is a nonzero limit ordinal, and each member of the increasing chain T_ζ, $\zeta < \xi$, is consistent, then obviously $T_\xi = \bigcup_{\zeta<\xi} T_\zeta$ is consistent. This completes the induction.

Now we let $\overline{T} = \bigcup_{\xi<\alpha} T_\xi$. It is evident that $\overline{T}$ is consistent in $\overline{\mathscr{L}}$ and $\overline{T}$ is an extension of T. Suppose that φ is a formula of $\overline{\mathscr{L}}$ with at most the variable x free. Then we may suppose that $\varphi = \varphi_\xi$ and $x = x_\xi$ for some $\xi < \alpha$. Whence the sentence

$$(\exists x_\xi)\varphi_\xi \to \varphi_\xi(d_\xi)$$

belongs to $T_{\xi+1}$ and so to $\overline{T}$. ⊣

The idea of the next lemma is just as simple, but its proof is more involved and tedious.

LEMMA 2.1.2. *Let T be a consistent set of sentences and C be a set of witnesses for T in $\mathscr{L}$. Then T has a model $\mathfrak{A}$ such that every element of $\mathfrak{A}$ is an interpretation of a constant $c \in C$.*

PROOF. First, note that if a set of sentences T has a set C of witnesses in $\mathscr{L}$, then C is also a set of witnesses for every extension of T. Second, if an extension of T has a model $\mathfrak{A}$, then $\mathfrak{A}$ is also a model of T. So we may as well assume that T is maximal consistent in $\mathscr{L}$.

For two constants $c, d \in C$, define

$$c \sim d \quad \text{iff} \quad c \equiv d \in T.$$

Because T is maximal consistent, we see that for $c, d, e \in C$,

$c \sim c$;

if $c \sim d$ and $d \sim e$, then $c \sim e$;

if $c \sim d$ then $d \sim c$.

So $\sim$ is an equivalence relation on C. For each $c \in C$, let

$$\tilde{c} = \{d \in C : d \sim c\}$$

be the equivalence class of c. We propose to construct a model $\mathfrak{A}$ whose set of elements A is the set of all these equivalence classes $\tilde{c}$, for $c \in C$; so we define

(1) $A = \{\tilde{c} : c \in C\}$.

We now define the relations, constants, and functions of $\mathfrak{A}$.