How to Think Quantum-Logically

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# HOW TO THINK QUANTUM-LOGICALLY

The heart of the quantum-logical interpretation of quantum mechanics is the following 'proportion':

$$\frac{\text{GEOMETRY}}{\text{GENERAL RELATIVITY}}=\frac{\text{LOGIC}}{\text{QUANTUM MECHANICS}}.$$

The laws of geometry, it should be remembered, were regarded as necessary truths for thousands of years. If someone asserted that he had traveled in a straight line and without reversing the sense of his motion, and went on to claim that in the course of his trip he had visited the same point twice, he would have been accused of contradicting himself just as much as if he had violated some elementary law of logic. But the acceptance of general relativity theory has forced us to say that the state of affairs just described is conceivable, not inconceivable, and that things just as 'bad' as this actually happen.

The advocate of the quantum-logical interpretation of quantum mechanics claims that in exactly parallel fashion the laws of logic have been regarded as necessary truths for thousands of years, but with no more justification than was present in the case of geometry. More precisely, the laws of *Aristotelian* logic have been regarded as necessary truths, and the laws of *Euclidean* geometry have been regarded as necessary truths. And just as it is impossible to understand the true nature of space and time as long as it is assumed that 'space' obeys the laws of Euclidean geometry, so it is impossible to understand the true nature of microprocesses as long as it is assumed that physical propositions obey the laws of Aristotelian logic. Logic, we advocates of this interpretation claim, is just as empirical as geometry.

The statement that one cannot understand the true nature of space and time as long as one assumes the validity of Euclidean geometry does not at all mean that one cannot construct a physics which assumes Euclidean geometry, nor does it mean that such a physics must of necessity lead to

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false predictions. As Reichenbach long ago pointed out, one can stick to Euclidean geometry in one's physics provided one is willing to pay the price. And the price, as he showed in detail, is the acceptance of causal anomalies – mysterious forces, instantaneous actions at a distance, infinite reduplications, or some choice among or combination of these. In a Riemannian world, for example, if I visit the same place twice while traveling in a straight line, I can get around the apparent contradiction with Euclidean geometry by insisting that it was not really the *same* place that I visited, or that the line I traveled on was not really straight, etc.; but then a host of physical phenomena will receive very strange explanations.

In the same way, according to the advocates of quantum logic, one can stick to classical logic in one's physics provided one is willing to pay the price. And the price is very similar to the price that one pays if one sticks to classical geometry in one's physics: one ends up with mysterious forces, or with a systematic 'cut between the observer and the system' (which is just to say, 'what one observes depends on how one looks, but one cannot say *how* and *why* it depends on how one looks'), or something else equally silly.

## I. THE CORRECT WAY TO VIEW THE WORLD

The heart of the quantum-logical interrelation is that the logical relations among physical states of affairs – the relations of implication and incompatibility – are themselves an empirical matter. If we think of physical states of affairs as 'objects',[1] then we must think of these 'objects' as having a lattice structure (cf. Putnam, 1968). The ordering relation in this lattice is the relation of *implication*. And whereas the traditional view is that this relation of implication is given a priori, the quantum-logical view is that it is a factual, synthetic, empirical matter, to be determined by the construction of theories and the scientific testing of those theories in the context of trying to understand various physical processes, just as the geometry of the world is an empirical matter.

Let me now come down to some details. The world consists of particles (not of 'waves', nor of 'waveparticles'). I say this because I am quantizing a particle theory; if I were quantizing a field theory, I would say 'the world consists of fields'. Each of these particles has a position. And each of these particles has a momentum. In symbols, taking 'Oscar' to be the name of one of these particles:

> ($Er$) the position of Oscar is $r$ . ($Er'$) the momentum of Oscar is $r'$.

But it must not be concluded that each of these particles has a position *and* a momentum! For the following is *rejected*:

> ($Er$) ($Er'$) (the position of Oscar is $r$ . the momentum of Oscar is $r'$).

Since these two statements are equivalent in classical logic, while in quantum logic one of them is accepted and the other is rejected, we see that the equivalence of these two statements must also be rejected in quantum logic, and this is our first example of a logical law which holds in classical logic but not in quantum logic:

> $(Ex)\, Fx \,.\, (Ey)\, Gy$ is equivalent to $(Ex)\,(Ey)\, Fx \,.\, Gy$ (correct in classical logic, but not correct in general in quantum logic).

This is an instance of a distributive law, and it is such laws – distributive laws – which are for the most part given up in quantum logic. For example, the law of propositional calculus,

> $(p \vee q) \,.\, r$ is equivalent to $(p \,.\, r) \vee (q \,.\, r)$,

which is intimately related to the above law of classical predicate calculus, is one which fails in quantum logic.

We can now list some features of the quantum-logical view of the world.

(1) *Measurement only determines what is already the case: it does not bring into existence the observable measured, or cause it to 'take on a sharp value' which it did not already possess.*

The statement 'Oscar has a position' is just the statement '($Er$) the position of Oscar is $r$', and this statement is accepted at all times, whether we *know* the position of Oscar or not. 'Measurement' is just a physical interaction, obeying the same laws as any other physical interaction; indeed the term 'measurement' plays no role in the axiomatization of physics, on this view.

Inasmuch as a measurement, say, a position measurement, is just an interaction which enables me to find out something which was there to be found out prior to my making the measurement, this interpretation might be called a 'hidden variable' interpretation. But it differs from the usual hidden variable theories in two ways:

(a) Complementarity is fully retained. For any particular $r$ and any particular $r'$ the statement 'Oscar has the position $r$ and Oscar has the momentum $r'$' is a logical contradiction. It is, of course, just the sacrifice of the distributive law that we mentioned a few moments ago that enables us to simultaneously retain the objective conception of measurement as finding out something which exists independently and the objective conception of complementarity as a prohibition on the simultaneous existence of certain states of affairs, and not just as a prohibition on simultaneous knowledge.

(b) The laws of quantum mechanics admit of no supplementation by laws of motion of the traditional Hamiltonian or Lagrangian kind. The statement 'the position of Oscar is $r$' is a *logically strongest consistent statement* in quantum logic. To conjoin any other 'information' to this statement would have to lead to either a redundancy or a logical contradiction. (Thus the 'completeness' of the quantum-mechanical description, as von Neumann terms it, is retained in this interpretation.) In particular, laws which require as inputs statements of the form 'Oscar has position $r$. Oscar has momentum $r'$' cannot possibly be consistent with quantum mechanics. Moreover, given a logically strongest consistent statement about the state of Oscar at any one time $t_0$, the laws of quantum mechanics enable one to determine some logically strongest consistent statement about the state of Oscar at an arbitrary later or earlier time $t_1$; so the laws of motion of quantum mechanics cannot be supplemented in any way at all without redundancy or contradiction. (In view of (b), this interpretation is *not* a hidden variable theory, contrary to what was said a moment ago; but it is better to say that quantum logic resolves the whole dispute between hidden variable theorists and von Neumann theorists by keeping what is correct in each view, than to try to classify the quantum-logical interpretation as 'hidden variable' or not.)

(2) *Probability enters into quantum theory just as it enters into classical physics: through considering large ensembles.*

This will be discussed in a future paper, and reasons of space do not permit me to give even a brief discussion here. However, I will mention one respect in which quantum physics appears to be superior to classical physics: on this interpretation, the laws of motion which apply to all systems have as a logical consequence the statistical laws which describe the distribution of a property in a large ensemble of noninteracting systems which all satisfy some logically strongest consistent statement. Thus the question as to where such statistical distributions 'come from' receives an unexpected and very satisfying answer from quantum logic.

(3) *The Hilbert spaces used in quantum mechanics are simply mathematical representations of various 'logical spaces'; namely, the lattices of physical propositions involved in treating the particular systems at issue in the various problems handled.*

At first blush, the use of Hilbert spaces (and more general linear spaces) in quantum mechanics seems like black magic. Once one observes that the subspaces of a Hilbert space form a lattice under the relation 'subspace of', and that that lattice is isomorphic to the lattice formed by the physical propositions about the system whose Hilbert space that Hilbert space is (i.e., isomorphic to the lattice formed by certain physical propositions under the relation of implication), then the appearance of black magic disappears. In classical physics also each well-defined system corresponded to a 'space', the so-called phase space. Each instantaneous state description corresponded, in the classical case, to a *point* in phase space. Thus the points in phase space were the mathematical counterparts of *logically strongest consistent statements*. An arbitrary consistent statement could be mathematically represented by a *set* of points, namely, the set of state descriptions in which that statement was true. The lattice of subsets of phase space was then isomorphic, under a well-known isomorphism, to the lattice of physical propositions about the system in question.

In quantum mechanics, we who advocate the quantum-logical interpretation maintain, the corresponding lattice of physical propositions is not 'Boolean'; in particular, distributive laws fail. So it *cannot* be isomorphic to any Boolean lattice, and in particular not to the lattice of sub*sets* of any space. But it can be isomorphic to the lattice of sub*spaces* of a suitable linear space because *that* lattice has the same structure as the lattice of physical propositions. And the whole function of the linear spaces used in quentum mechanics is to provide a convenient mathematical representation of the lattice of physical propositions, and to enable one to give a convenient mathematical representation of time development laws (i.e., laws of the form: 'if $p$ is true at time $x$, then $q$ will be true at time $x+t$').

## II. STATE VECTORS AND QUANTUM LOGIC

We can now say what the famous '$\psi$-function' (also called the 'state vector' or 'state function') of quantum mechanics really is. Each state vector corresponds to a one-dimensional subspace of the Hilbert space (namely, the one-dimensional subspace in which that vector lies). And a one-dimensional subspace of a linear space has no subspaces except itself and the null space. So the physical proposition corresponding to a state vector must be one which is implied by no proposition except itself and the logically false proposition: that is to say, it must be a logically strongest consistent proposition. And that is indeed the case: each $\psi$-function corresponds to a statement which is implied only by itself and by the logically false statement, in quantum logic. So we say, the state vector is nothing but the mathematical representation of a logically strongest consistent statement.

Many writers write as if a system had one and only one state vector at a given time, however. Can this be true, if quantum logic is assumed? The answer is 'no'. For let $\psi_1, \psi_2, \ldots, \psi_n$ be all the state vectors which are eigenvectors of position; that is to say (identifying each state vector with the corresponding physical proposition), let $\psi_1, \psi_2, \ldots, \psi_n$ be all the statements of the form 'Oscar has the position $r$'. (Of course, these are non-denumerably infinite in number; but we shall simplify by pretending that there are only finitely many possible positions.) Similarly, let $\phi_1, \phi_2, \ldots, \phi_n$ be all the statements of the form 'Oscar has the momentum $r$'. Then, as we have already stated, the conjunction

$$(\psi_1 \vee \psi_2 \vee \ldots \psi_n) \cdot (\phi_1 \vee \phi_2 \vee \ldots \phi_n)$$

(i.e., 'Oscar has a position . Oscar has a momentum') is true, and this may be expressed also by writing

$$(E\psi)\ (\psi \text{ is an eigenvector of position . Oscar has } \psi).$$
$$(E\phi)\ (\phi \text{ is an eigenvector of momentum . Oscar has } \phi).$$

Thus, on the quantum logic interpretation, contrary to what is often maintained, a system has *many* state vectors; it has a state vector of each nondegenerate observable, in fact.

Now then, let $\psi_1$ be a state vector which is an eigenvector of position and let $\phi_1$ be a state vector which is an eigenvector of momentum. Then the statement

$$\text{Oscar has } \psi_1 \text{ . Oscar has } \phi_1$$

is a logical contradiction! For this is just another way of saying that Oscar has a definite position $r$ and a definite momentum $r'$ in violation of complementarity.

Thus we get: a system *has* more than one state vector, on the quantum-logic interpretation, but one can never *assign* more than one state vector! Or, to drop talk of 'state vectors' altogether, since that talk tends to be highly misleading on the view of the world sketched in the preceding part, we may say: A system has a position *and* it has a momentum. But if you *know* the position (say, $r$), you cannot *know* the momentum. For if you did, say, know that the momentum was $r'$, you would know 'Oscar has the position $r$ . Oscar has the momentum $r'$', which is a logical contradiction.

The logic by itself does not say exactly how any particular position measurement will make momentum uncertain. But we know that in each case the physical laws will in some way have to say that the position measurement makes the momentum uncertain, because physical laws have to be compatible with logic – that is to say, they have to be compatible with the *true* logic, which is quantum logic.

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## NOTE

[1] In my view this is a permissible but not a necessary way of talking. Cf. Putnam (1967).

## REFERENCES

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