Proof Strategy for Lemmon's *Beginning Logic*

Familiarity with the rules of logic does not automatically confer fluency in their use. The simplest proofs are clear enough, but as we move through chapter 1 of Lemmon we rapidly encounter sequents that are not intuitive. That is not to say that the semantics are unpersuasive; we may even know these sequents by name as rules of inference from some other logic textbook. But Lemmon's original ten rules are not designed to maximize our proving power, and some of the derivations on page 41 in particular require some clever moves.

Once we get into chapter 2 and begin introducing sequents and theorems, our proving power will increase drastically and most proofs will become much simpler. (It may also become harder to keep track of all of those derived rules mentally, but after we have suffered through some of these early proofs that will seem a very small price to pay.) Here, in the meantime, is a brief introduction to proof strategy in Lemmon's system.

<u>Look at your conclusion</u>

First, ask yourself what you are being asked to prove. This involves looking at the conclusion to determine what sort of claim it is. Often this suggests a useful proof strategy immediately.

Conditionals

To prove a conditional, use conditional proof. Assume the antecedent of the conditional and attempt to derive the consequent. If the consequent is itself a conditional, make another assumption for conditional proof. At the end, use CP to put everything back together in the proper order.

Example: $((P \& Q) \rightarrow R) \vdash (P \rightarrow (Q \rightarrow R))$

1	(1) $((P \& Q) \rightarrow R)$	A
2	(2) P	A* (for CP, since $(P \rightarrow (Q \rightarrow R))$ is a conditional)

Our goal now will be to derive $(Q \rightarrow R)$.

3	(3) Q	A* (for CP, since $(Q \rightarrow R)$ is a conditional)

Our goal will be to derive R. Notice that with each successive assumption, the "target" is a simpler formula. Quite often we will discharge the assumptions in reverse order; the last assumption made will be the first to be discharged. This is not required by the rules, and occasionally we will run into a sequent where this rule of thumb fails. But it's a good thing to keep in mind.

2,3	(4) $(P \& Q)$	&I 2,3
1,2,3	(5) R	MP 1,4

Now we have derived the target for our assumption on line 3. Naturally, we get rid of that assumption to build a conditional that we recognize as another target.

1,2	(6) (Q → R)	CP 3,5 (getting rid of the assumption from line 3)

This, in turn, is the target we were aiming for with the assumption on line 2. One final CP discharges that assumption.

1	(7) (P → (Q → R))	CP 2,6 (getting rid of the assumption from line 2)

Conjunctions

Often the simplest way to prove a conjunction is to prove each conjunct separately, then use the rule &I to put them together at the end.

Example: – (– P v – Q) ⊢ (P & Q)

1	(1)	– (– P v – Q)	A

Noticing that our conclusion is a conjunction, we aim first to prove the left conjunct. Unfortunately, the premise we are given is not easily unpacked; none of Lemmon's ten basic rules lets us get inside of the negation of a disjunction. Since there is no obvious shortcut, we simply go about proving P by *reductio ad absurdum*.

2	(2)	– P	A* (for RAA, aiming to get P)
2	(3)	(– P v – Q)	vI 2
1,2	(4)	((– P v – Q) & – (– P v – Q))	&I 1,3
1	(5)	– – P	RAA 2,4
1	(6)	P	DN 5

Having come this far, we simply carry out the same procedure over again for the other half of the conjunction we are trying to prove.

7	(7)	– Q	A* (for RAA, aiming to get Q)
7	(8)	(– P v – Q)	vI 7
1,7	(9)	((– P v – Q) & – (– P v – Q))	&I 1,8
1	(10)	– – Q	RAA 7, 9
1	(11)	Q	DN 10

Now both halves of the derivation are complete and we simply put them together to finish the proof.

1	(12)	(P & Q)	&I 6,11

Disjunctions

If the conclusion is a disjunction, the first thing to do is to look at the sequent optimistically to see whether we can derive one of the disjuncts by itself from the premises. If so, we simply do that and then use vI to finish things off. When this works, it is almost always the easiest way to do the proof. The sequent (P & Q) ⊢ (P v R) is a simple example.

But often it does not work. In that case, until we have sequents available that turn disjunctions into related conditionals, we must treat the disjunction as a package. And unless there is a simple move or a rare trick, there is nothing to be done but to approach the sequent through *reductio ad absurdum*.

Proofs by *reductio* are generally tedious, even when we have to proceed by *reductio*, not all is lost. Sometimes a whole series of moves is repeated so that structurally they are less complex than their length would suggest. And in the case where there is just one premise, we can sometimes get a clearer view of the appropriate strategy if we "reverse" the proof, assuming the negation of the conclusion in order to prove something contradictory to the sole given premise. A *reductio* and (if necessary) a double negation will then wind up the proof.

Example: – (P & Q) ⊢ (– P v – Q)

1	(1)	– (P & Q)	A

The premise is very unfriendly: none of the ten basic rules will permit us to dig into the negation of a conjunction in a direct way. Of course if we had a wider array of rules at our disposal we could simply pick out a useful one, say, – (P & Q) ⊢ (P → – Q). But if we had that wide an array of rules, we would likely have de Morgan's laws among them. Instead, we're stuck here trying to prove one of de Morgan's laws with just the basic ten rules.

The trick here is to turn the problem inside out. Instead of trying to tackle our sequent directly, we'll prove the sequent – (– P v – Q) ⊢ (P & Q). Once we get that, we'll conjoin (P & Q) with our premise (1) here to get a contradiction and then apply RAA to get the negation of our assumption. Then DN will finish the job.

Why is this a good strategy? Because in our subproof, the conclusion is (P & Q), and we can derive *that* by the strategy for proving conjunctions, just proving each part separately.

2	(2)	– (– P v – Q)	A* (for RAA, aiming to prove (P & Q))

Here we start off on our subproof. The target now is (P & Q), so in accordance with our strategy for proving conjunctions we set a subgoal of proving P. There is no obvious way to do this, so we start another *reductio* inside of the global one.

3	(3)	– P	A* (for RAA)
3	(4)	(– P v – Q)	vI 3

2,3	(5)	((– P v – Q) & – (– P v – Q))	&I 2, 4
2	(6)	– – P	RAA 3, 5
2	(7)	P	DN 6

Now we are halfway done with the subproof. On the next five lines we apply the same rules in the same order to obtain Q resting on the same assumption.

8	(8)	– Q	A*
8	(9)	(– P v – Q)	vI 8
2,8	(10)	((– P v – Q) & – (– P v – Q))	&I 2, 9
2	(11)	– – Q	RAA 10
2	(12)	Q	DN 11

Still following the strategy for proving a conjunction, we complete the job by conjoining lines (7) and (12).

2	(13)	(P & Q)	&I 7, 12

Now we finish off the proof by creating a contradiction and discharging the assumption we made at line (2).

1,2	(14)	((P & Q) & – (P & Q))	&I 1, 13
1	(15)	– – (– P v – Q)	RAA 2, 14

Finally, DN cleans this up and yields what we originally set out to prove.

1	(16)	(– P v – Q)	DN 15