A Kakeya-Type Problem for Circles

Author(s): Thomas Wolff

Source: *American Journal of Mathematics*, Oct., 1997, Vol. 119, No. 5 (Oct., 1997), pp. 985-1026

Published by: The Johns Hopkins University Press

Stable URL: https://www.jstor.org/stable/25098564

A KAKEYA-TYPE PROBLEM FOR CIRCLES

By THOMAS WOLFF

Abstract. We prove full Hausdorff dimension in a variant of the Kakeya problem involving circles in the plane, and also sharp estimates for the relevant maximal function. These results can also be formulated in terms of the wave equation in two space variables. A novelty in our approach is the use of ideas from computational geometry.

The purpose of this paper is to prove a certain $L^3 \to L^3$ maximal inequality, Theorem 1 below.

Fix $\delta > 0$ and, for $a \in \mathbb{R}^2$ and $r \in [\frac{1}{2}, 2]$, let $C_\delta(a,r) = \{x \in \mathbb{R}^2 \colon r - \delta < |x-a| < r + \delta\}$. If $f \colon \mathbb{R}^2 \to \mathbb{R}$, and $\delta > 0$, then we define $M_\delta f \colon [\frac{1}{2}, 2] \to \mathbb{R}$ via

$$M_\delta f(r) = \sup_{a \in \mathbb{R}^2} \frac{1}{|C_\delta(a,r)|} \int_{C_\delta(a,r)} |f(x)| dx.$$

THEOREM 1. *If $\epsilon > 0$ then there is a constant A_ϵ such that*

$$||M_\delta f||_{L^3([\frac{1}{2},2])} \leq A_\epsilon \delta^{-\epsilon} ||f||_3$$

for all $\delta > 0$ and f.

An immediate consequence by an argument which we learned from [1] is the following

COROLLARY. *A Borel set in the plane which contains a circle of every radius must have Hausdorff dimension* 2.

We refer to [4] for background and motivation. In [4] a partial result in the direction of Theorem 1 is proved, which implies for example that a set as in the corollary must have dimension at least $\frac{11}{6}$. Theorem 1 as stated will follow by combining the approach in [4] with ideas from [3].

As is discussed in [4], Theorem 1 is related to a certain purely combinatorial problem: one is given N circles $C_1, \ldots, C_N$ and roughly speaking has to control

---

Manuscript received February 14, 1996.

Research supported in part by NSF grant DMS 93-07872.

American Journal of Mathematics 119 (1997), 985–1026.

the number of pairs (i,j) for which C_i and C_j are internally tangent, assuming, say, that no three are internally tangent at a point. The condition for two circles to be internally tangent is a quadratic equation in their centers and radii, which may be regarded as points of $\mathbb{R}^3$. Thus the tangency counting problem is a variant on the three dimensional unit distance problem considered in [3], and it is not hard to see that their techniques lead to a bound of the form $N^{\frac{3}{2}}a(N)$ where $a(N)$ grows extremely slowly.

On a heuristic level, this bound for the number of tangencies corresponds to Theorem 1, and the proof of Theorem 1 given below can be regarded as a "continuum" version of the argument in [3].

The paper is organized as follows: in Section 1 we prove some lemmas from elementary geometry which we need later on, and in Section 2 we present a convenient form of the construction from [3]. In Section 3 we make some preliminary reductions in the proof of Theorem 1, and in Sections 4 and 5 we carry out the main steps in the proof.

We want to note that the following variant on Theorem 1 can be proved by essentially the same argument. For $x_1 \in \mathbb{R}$, let

$$\mathcal{M}_\delta f(x_1) = \sup_{\substack{r\in[\frac{1}{2},2] \\ x_2\in\mathbb{R}}} \frac{1}{|C_\delta(x,r)|} \int_{C_\delta(x,r)} |f|$$

where $x = (x_1, x_2)$. Then

$$\forall \epsilon > 0\ \exists A_\epsilon\colon\ \|\mathcal{M}_\delta f\|_{L^3(\mathbb{R})} \leq C_\epsilon \delta^{-\epsilon} \|f\|_3. \tag{1}$$

Hence if $\alpha \leq 1$ and $F \subset \mathbb{R}^2$ is a set of Hausdorff dimension at least α, and E is a set which contains circles centered at all points of F, then E has Hausdorff dimension at least $1 + \alpha$.

The reason is that a circle is determined by three parameters, and the distinguished role played by the radial parameter in Sections 3–5 below can equally well be played by the x_1-coordinate of the center. We will not give an explicit proof of (1), but in Appendix 1, we will prove the Hausdorff dimension statement, since the argument is less immediate than the proof of the corollary to Theorem 1.

It is also possible to reformulate Theorem 1 in terms of the wave equation using spherical means. If one considers the solution to the wave equation in $2+1$ dimensions, $\Box u = 0$ with (say) initial conditions $u(\cdot, 0) = f$, $u_t(\cdot, 0) = 0$, then a standard argument which is sketched in [4] leads from Theorem 1 to the estimate (on $1 \leq t \leq 2$)

$$\|u\|_{L^3_t L^\infty_x} \lesssim \|f\|_{3,\frac{1}{2}+\epsilon},\quad \epsilon > 0 \tag{2}$$

where the norm on the right is the inhomogeneous Sobolev norm. We reproduce the argument in Appendix 2 for the convenience of the reader. Estimate (2) is an endpoint for various interpolation scales, e.g. if one interpolates between (2) and the energy estimate $\|u\|_{L_t^\infty L_x^2} \leq \|f\|_2$ one obtains the bound (on $1 \leq t \leq 2$) $\|u\|_{L_t^5 L_x^5} \lesssim \|f\|_{\frac{5}{2}, \frac{3}{10}+\epsilon}$ which was proved in the recent paper [8]. Estimate (2) is also related to old results of Pecher ([6]), as is described in [4].

Acknowledgment. I thank W. Schlag for comments on a preliminary version of this paper.

1. Geometric lemmas.

In this section we collect some facts about intersections of thin annuli which will be used below.

Notation will be as follows:

$C(x, \rho)$: the circle $\{y \in \mathbb{R}^2 \colon |y - x| = \rho\}$.

C^ϵ: the ϵ-neighborhood of the circle C, i.e., if $C = C(x, \rho)$, then $C^\epsilon = \{y \in \mathbb{R}^2 \colon \rho - \epsilon < |y - x| < \rho + \epsilon\}$.

$\Gamma(x, \rho)$: the light cone in $\mathbb{R}^3$ determined by x and ρ, i.e., $\Gamma(x, \rho) = \{(y, \tau) \in \mathbb{R}^3 \colon |y - x| = |\tau - \rho|\}$.

$d(C, \tilde{C})$: the distance between the points corresponding to the circles C and $\tilde{C}$ defined by $d(C(x, \rho), C(\tilde{x}, \tilde{\rho})) = |x - \tilde{x}| + |\rho - \tilde{\rho}|$.

$\Delta(C, \tilde{C})$: the distance from the point corresponding to the circle C to the light cone corresponding to $\tilde{C}$, or vice-versa, defined by

$$\Delta(C, \tilde{C}) = \big||x - \tilde{x}| - |\rho - \tilde{\rho}|\big|.$$

We will adopt the following conventions:

If, say, $\tilde{C}$ is a circle, then $\tilde{x}$ and $\tilde{\rho}$ will be understood to mean the center and radius of $\tilde{C}$. Similarly, x_j and ρ_j are the center and radius of C_j, etc.

The letter A will be used to denote large fixed constants.

We always assume that all circles $C(x, \rho)$ satisfy

$$\rho \in [1 - \alpha_0, 1 + \alpha_0], \ |x| \leq \alpha_0 \tag{3}$$

where α_0 is a small fixed constant; $\alpha_0 = \frac{1}{100}$ will work. The *τ-rectangle* of C^ϵ, $C = C(x, \rho)$, corresponding to an arc γ of C with length $2\pi\rho \cdot \tau$ is the set $\{y \in C^\epsilon;\ x + \rho \cdot \frac{y-x}{|y-x|} \in \gamma\}$.

We will say such a "rectangle" is *centered* at $a \in C^\epsilon$ if $x + \rho \cdot \frac{a-x}{|a-x|}$ is the center of γ.

LEMMA 1.1. *For any two circles C_1 and C_2 with $x_1 \neq x_2$, $C_1^\epsilon \cap C_2^\epsilon$ is contained in the union of at most A_1 τ-rectangles of C_1^ϵ, where*

$$\tau = \frac{\epsilon}{\sqrt{\max(d(C_1, C_2), \epsilon) \max(\Delta(C_1, C_2), \epsilon)}}. \tag{4}$$

Furthermore, the diameter of $C_1^\epsilon \cap C_2^\epsilon$ *is* $\lesssim \sqrt{\frac{\Delta(C_1,C_2)+\epsilon}{d(C_1,C_2)+\epsilon}}$*; in fact,* $C_1^\epsilon \cap C_2^\epsilon$ *is contained in an* $A_1\sqrt{\frac{\Delta(C_1,C_2)+\epsilon}{d(C_1,C_2)+\epsilon}}$ *rectangle of* C_1^ϵ *centered at the point* $x_1 + \rho_1 \,\mathrm{sgn}\,(\rho_1 - \rho_2)\frac{x_2-x_1}{|x_2-x_1|}$.

Proof. Notation aside, the lemma says that the intersection of the two annuli has diameter $\lesssim \sqrt{\frac{\Delta(C_1,C_2)+\epsilon}{d(C_1,C_2)+\epsilon}}$ and is contained in the union of finitely many (actually just two) sets with diameter $\lesssim \tau$. These facts are used in several other papers on similar subjects, e.g. [7], and are not hard to check, so we omit the proof. □

LEMMA 1.2. *Conversely, if* $x \in C_1^\epsilon \cap C_2^\epsilon$ *then* $C_1^\epsilon \cap C_2^{A_2\epsilon}$ *contains a* τ*-rectangle of* C_1^ϵ *centered at* x*, where* τ *is given by (4).*

Proof. This is again clear, since if we let $d = d(C_1, C_2)$ and $\Delta = \Delta(C_1, C_2)$, then the angle of intersection of C_1 and C_2 is $\approx \sqrt{\Delta d}$. However, we will give a detailed proof.

We use complex notation, e.g. $\mathrm{re}z$ is the real part of z, and may assume that $x_1 = 0, \rho_1 = 1$, and x is on the positive real axis. Then $|x_2| + |1-\rho_2| \lesssim d$, $\big||x_2| - |1-\rho_2|\big| \lesssim \Delta$. Also $|x| \in (1-\epsilon, 1+\epsilon)$, $|x - x_2| \in (\rho_2 - \epsilon, \rho_2 + \epsilon)$. We claim that

$$(\mathrm{im}\, x_2)^2 \lesssim (d+\epsilon)(\Delta+\epsilon). \tag{5}$$

Namely, if $|1-\rho_2| \le 3\epsilon$ then $|x_2| \lesssim \Delta + \epsilon$ and (5) is obvious. If $|1-\rho_2| \ge 3\epsilon$ then consider the triangle $0xx_2$. Let ϕ be the angle at x. Then $(\mathrm{im}\, x_2)^2 \lesssim \sin^2\phi \lesssim 1-\cos\phi$. On the other hand, by the law of cosines $2|x||x-x_2|(1-\cos\phi) = |x_2|^2 - (|x| - |x-x_2|)^2$. Hence $1 - \cos\phi \lesssim |x_2|^2 - (|1-\rho_2| - 2\epsilon)^2 \lesssim (d+\epsilon)(\Delta+\epsilon)$ and (5) follows.

Now suppose $\theta \in [-\pi, \pi]$. Then

$$\begin{aligned} |e^{i\theta} - x_2|^2 - |1-x_2|^2 &= 2\,\mathrm{re}\,x_2(1-\cos\theta) - 2\,\mathrm{im}\,x_2 \sin\theta \\ &= \mathcal{O}(d\theta^2 + \theta\sqrt{(d+\epsilon)(\Delta+\epsilon)}) \end{aligned}$$

and therefore (using (3)) $\big||e^{i\theta} - x_2| - |1-x_2|\big| \lesssim d\theta^2 + \theta\sqrt{(d+\epsilon)(\Delta+\epsilon)}$. Hence if $|1-r| < \epsilon$ and $|\theta| < \tau$ then

$$\big||re^{i\theta} - x_2| - |1-x_2|\big| \lesssim \epsilon + d\tau^2 + \tau\sqrt{(d+\epsilon)(\Delta+\epsilon)} \lesssim \epsilon,$$

i.e., $|re^{i\theta} - x_2| = \rho_2 + \mathcal{O}(\epsilon)$, proving the lemma. □

LEMMA 1.3. *Assume that* $\epsilon \le t$ *and that* C_1, C_2 *and* $\tilde{C}$ *are circles with*

$$d(C_i, \tilde{C}) \le t$$
$$\Delta(C_i, \tilde{C}) \le \beta$$

for $i = 1, 2$ *and that* $\tilde{C}^\epsilon \cap C_1^\epsilon \cap C_2^\epsilon \neq \emptyset$. *Then it follows that* $\Delta(C_1, C_2) \leq A_3 \frac{t(\beta+\epsilon)}{d(C_1,C_2)}$.

Proof. Again this is easy to see by considering angles of intersection, but we give the proof. We use complex notation and normalize so that $\tilde{x} = 0, \tilde{\rho} = 1$, and $\tilde{C}^\epsilon \cap C_1^\epsilon \cap C_2^\epsilon$ contains a point of the positive real axis, necessarily in $(1-\epsilon, 1+\epsilon)$. From the previous proof (cf. (5)) we have

$$|\operatorname{im} x_i| \lesssim \sqrt{t(\beta+\epsilon)}, \ i = 1, 2$$

since $\epsilon \leq t$. Also for $i = 1, 2$, we have $|x_i - 1| = \rho_i + \mathcal{O}(\epsilon)$, $|x_i| = |1 - \rho_i| + \mathcal{O}(\beta)$, so

$$|x_i|^2 - 2 \operatorname{re} x_i + 1 = \rho_i^2 + \mathcal{O}(\epsilon)$$

$$|x_i|^2 = 1 - 2\rho_i + \rho_i^2 + \mathcal{O}(\beta t).$$

Subtracting these equations gives $\operatorname{re} x_i = 1 - \rho_i + \mathcal{O}(\beta t + \epsilon)$ and therefore, $\operatorname{re}(x_1 - x_2)^2 = (\rho_1 - \rho_2)^2 + \mathcal{O}(t(\beta t + \epsilon))$. Hence

$$\begin{aligned} |x_1 - x_2|^2 - (\rho_1 - \rho_2)^2 &= \operatorname{re}(x_1 - x_2)^2 - (\rho_1 - \rho_2)^2 + \operatorname{im}(x_1 - x_2)^2 \\ &= \mathcal{O}(t(\beta + \epsilon)), \end{aligned}$$

i.e., $\Delta(C_1, C_2) d(C_1, C_2) \lesssim t(\beta + \epsilon)$ as claimed. ☐

LEMMA 1.4. *Assume that* C_1, C_2 *and* $\tilde{C}$ *are circles such that, with* $t > 4\epsilon$ *and* $1 \geq \lambda \geq \sqrt{\frac{\epsilon}{t}}$,

(i) $\operatorname{sgn}(\tilde{\rho} - \rho_1) = \operatorname{sgn}(\tilde{\rho} - \rho_2)$.

(ii) $\Delta(C_i, \tilde{C}) \leq \epsilon$ *and* $d(C_i, \tilde{C}) \geq t$.

(iii) *There are points* $z_i \in C_i^\epsilon \cap \tilde{C}^\epsilon$ *with* $|z_1 - z_2| \leq \lambda$.

Then $C_1^\epsilon \cap C_2^\epsilon \cap \{\zeta \colon |\zeta - z_1| \leq A_4 \lambda\} \neq \emptyset$.

Proof. There are two symmetric cases; we will assume $\tilde{\rho} - \rho_1$ and $\tilde{\rho} - \rho_2$ are positive. It follows by (ii) that $\tilde{\rho} - \rho_1$ and $\tilde{\rho} - \rho_2$ are $> \epsilon$, since $t \geq 4\epsilon$ implies $|\rho - \rho_i| \geq \frac{1}{2} t$. (This type of argument will be used repeatedly.)

The assumption $\Delta(C_i, \tilde{C}) \leq \epsilon$ means that there are circles $\overline{C_i}$ concentric with C_i and contained in C_i^ϵ which are tangent to $\tilde{C}$, necessarily at points ζ_i with $|\zeta_i - z_i| \lesssim \sqrt{\frac{\epsilon}{t}}$, by Lemma 1.1. Note that $\overline{\rho}_i \leq \tilde{\rho}$. Let γ be the shorter of the two arcs of $\tilde{C}$ determined by ζ_1 and ζ_2. Since $\tilde{\rho} - \overline{\rho}_1$ and $\tilde{\rho} - \overline{\rho}_2$ are small (cf. (3)) it is then clear geometrically that $\overline{C_1}$ and $\overline{C_2}$ must intersect at a point p in the solid angle determined by γ (i.e. the set $\{z \colon \tilde{x} + \tilde{\rho}\frac{z-\tilde{x}}{|z-\tilde{x}|} \in \gamma\}$) and that $|p - \zeta_i| \leq \lambda$. We leave the details to the reader. Since $\overline{C_i} \subset C_i^\epsilon$ we are done. ☐

LEMMA 1.5. *Assume that* $C_1, C_2, \tilde{C}_1, \tilde{C}_2$ *are circles such that, with* $1 \geq \lambda \geq \sqrt{\frac{\epsilon}{t_1}}$

(i) $\operatorname{sgn}(\rho_i - \tilde{\rho}_j)$ *is independent of j for each i.*

(ii) $d(C_1, \tilde{C}_j) \in [\frac{t_1}{4}, 4t_1]$ *and* $d(C_2, \tilde{C}_j) \geq t_2$.

(iii) $\Delta(C_i, \tilde{C}_j) \leq \epsilon$.

Assume that $\operatorname{dist}(C_1^\epsilon \cap \tilde{C}_1^\epsilon,\ C_1^\epsilon \cap \tilde{C}_2^\epsilon) \leq \lambda$. *Then also* $\operatorname{dist}(C_2^\epsilon \cap \tilde{C}_1^\epsilon,\ C_2^\epsilon \cap \tilde{C}_2^\epsilon) \leq A_5(\lambda \frac{t_1}{t_2} + \sqrt{\frac{\epsilon}{t_2}})$.

Proof. We note first of all that hypothesis (iii) implies $C_i^\epsilon \cap \tilde{C}_j^\epsilon \neq \emptyset$.

By hypotheses (iii) and (i), for each i there is $\epsilon_i \in \{\pm 1\}$ such that $\left||x_i - \tilde{x}_j| + \epsilon_i(\rho_i - \tilde{\rho}_j)\right| \leq \epsilon$ for each j. Hence

$$|x_i - \tilde{x}_1| - |x_i - \tilde{x}_2| = \epsilon_i(\tilde{\rho}_1 - \tilde{\rho}_2) + \mathcal{O}(\epsilon)$$

so that

$$\big|\, |x_1 - \tilde{x}_1| - |x_1 - \tilde{x}_2| \,\big| = \big|\, |x_2 - \tilde{x}_1| - |x_2 - \tilde{x}_2| \,\big| + \mathcal{O}(\epsilon)$$

and therefore, since $|\tilde{\rho}_1 - \tilde{\rho}_2| \leq 8t_1$,

$$(|x_1 - \tilde{x}_1| - |x_1 - \tilde{x}_2|)^2 = (|x_2 - \tilde{x}_1| - |x_2 - \tilde{x}_2|)^2 + \mathcal{O}(\epsilon t_1). \tag{6}$$

Now consider the triangles $\tilde{x}_1 x_i \tilde{x}_2$. Let $\theta_i \in [0, \pi]$ be the angle at x_i. By assumption and the last statement in Lemma 1.1, $\theta_1 \lesssim \lambda$. On the other hand, by the law of cosines

$$1 - \cos\theta_i = \frac{|\tilde{x}_1 - \tilde{x}_2|^2 - (|\tilde{x}_1 - x_i| - |\tilde{x}_2 - x_i|)^2}{2|\tilde{x}_1 - x_i|\,|\tilde{x}_2 - x_i|}$$

so by (6)

$$\begin{aligned}(1 - \cos\theta_2)|\tilde{x}_1 - x_2|\,|\tilde{x}_2 - x_2| &= (1 - \cos\theta_1)|\tilde{x}_1 - x_1|\,|\tilde{x}_2 - x_1| + \mathcal{O}(\epsilon t_1) \\ &\lesssim \lambda^2 t_1^2.\end{aligned}$$

By the form of the statement we can assume $\frac{\epsilon}{t_2}$ is small. Then $|\tilde{x}_j - x_2| \geq \frac{1}{2}t_2$ by (ii), hence $\theta_2 \lesssim \lambda\frac{t_1}{t_2}$. The conclusion now follows from Lemma 1.1. □

The next two lemmas are versions of the 3-circle principle of Marstrand ([5]). We first give a discussion which is relevant to both of them.

Fix three circles C_1, C_2, C_3. Then in the first place there are at most two circles $C = C(z, s)$ such that $\Delta(C, C_i) = 0$, $i = 1, 2, 3$, and such that no two of the points of tangency $x_i + \rho_i \operatorname{sgn}(\rho_i - s)\frac{z - x_i}{|z - x_i|}$ coincide. We call these tangent circles.

Now consider the map G: $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$G(x,\rho) = \begin{pmatrix} |x - x_1| - |\rho - \rho_1| \\ \\ |x - x_2| - |\rho - \rho_2| \\ \\ |x - x_3| - |\rho - \rho_3| \end{pmatrix}.$$

What is actually shown in the proofs of the 3-circle lemma ([5]), Lemma 5.2 is the following. Assume $\lambda \geq A_6\sqrt{\frac{\epsilon}{t}}$.

Consider the set $\Omega_{\epsilon t\lambda}$ of all circles $C = C(x,\rho)$ such that $\Delta(C, C_i) < \epsilon$, $i = 1, 2, 3$, and also $d(C, C_i) \geq t$ and $\operatorname{dist}(C^\epsilon \cap C_i^\epsilon,\ C^\epsilon \cap C_j^\epsilon) \geq \lambda$. Then for each tangent circle $C(z,s)$ contained in $\Omega_{\epsilon\frac{t}{2}\frac{\lambda}{2}}$ there is an ellipsoid $E(z,s) \subset \mathbb{R}^3$ centered at (z,s) with volume $\lesssim \lambda^{-3}\epsilon^3$ and diameter $\lesssim \lambda^{-2}\epsilon$, such that $\Omega_{\epsilon t\lambda}$ is contained in the union of these (two at most) ellipsoids. Furthermore, on each ellipsoid $E(z,s)$, G is boundedly conjugate to its linear part, i.e.,

$$|G(x,\rho)| \approx |DG(z,s)\,(x - z, \rho - s)| \tag{7}$$

with the constant being universal.

This may be seen by reading between the lines in [5] or [4]. We give a brief sketch following [4]. Namely, one checks that on $\Omega_{\epsilon\frac{t}{2}\frac{\lambda}{2}}$ there are estimates $|\det DG(x,\rho)| \gtrsim \lambda^3$, $\|DG(x,\rho)^{-1}\| \lesssim \lambda^{-2}$, and that the second derivatives of G are $\lesssim t^{-1}$. Since t is large compared with $\frac{\epsilon}{\lambda^2}$, the statement then follows using the quantitative form of the inverse function theorem.

An immediate consequence (also used in [4]) is

LEMMA 1.6. *If* $\lambda \geq A_6\sqrt{\frac{\epsilon}{t}}$ *the set*

$$\{\rho \in \left[\frac{1}{2}, 2\right] : \exists x \in \mathbb{R}^2 \text{ with } (x,\rho) \in \Omega_{\epsilon t\lambda}\}$$

is contained in the union of two intervals of length $A_7\lambda^{-2}\epsilon$.

Another consequence is

LEMMA 1.7. *If* $\lambda \geq A_6\sqrt{\frac{\epsilon}{t}}$ *then for each i, the set*

$$\bigcup_{(x,\rho)\in\Omega_{\epsilon t\lambda}} C^\epsilon(x,\rho) \cap C_i^\epsilon$$

is contained in the union of A_8 $\sqrt{\frac{\epsilon}{t}}$*-rectangles of* C_i^ϵ.

Proof. It suffices to consider the part of $\Omega_{\epsilon t\lambda}$ which is contained in a single ellipsoid $E(z,s)$. Let $e_i = \operatorname{sgn}(\rho_i - s)\frac{z-x_i}{|z-x_i|}$, so that $x_i + \rho_i e_i$ is the tangency point between $C(z,s)$ and C_i. Then the ith row of the matrix $DG(z,s)$ has the form $\pm(e_i, 1)$ so if $(x,\rho) \in \Omega_{\epsilon t\lambda}$, then

$$|(e_i, 1)\cdot(x-z, \rho - s)| \leq \epsilon$$

by (7). Consequently, for $i \neq j$,

$$|(e_i - e_j)\cdot(x-z)| \leq 2\epsilon. \tag{8}$$

We will prove the lemma with $i=1$. Let $e_1^* \in \mathbb{R}^2$ be a unit vector perpendicular to e_1. The assumptions together with Lemma 1.1 imply $|e_i - e_j| \geq \lambda$ for all i and j. A little linear algebra shows that therefore $e_1^* = \alpha(e_1 - e_2) + \beta(e_1 - e_3)$ with $|\alpha| + |\beta| \lesssim \lambda^{-1}$. We conclude by (8) that $|e_1^* \cdot (x-z)| \lesssim \frac{\epsilon}{\lambda}$, hence also $|e_1^* \cdot (x - x_1)| \lesssim \frac{\epsilon}{\lambda}$ since $z - x_1$ is parallel to e_1. So

$$\left| e_1^* \cdot \frac{x-x_1}{|x-x_1|}\right| \lesssim \frac{\epsilon}{\lambda t}$$

or in other words for an appropriate choice of ±

$$\left| e_1 \pm \frac{x-x_1}{|x-x_1|}\right| \lesssim \frac{\epsilon}{\lambda t} \lesssim \sqrt{\frac{\epsilon}{t}}.$$

The lemma follows. □

2. The cell decomposition. Assume we are given a set of n "circles," i.e., of points $(x,\rho) \in \mathbb{R}^2 \times \mathbb{R}^+$. We will assume that x and ρ satisfy (3). The purpose of this section is to prove

PROPOSITION 2.1. *Fix $\epsilon > 0$, $r < n$. Suppose that we randomly choose r circles $(x,\rho) \in \mathcal{C}$, independently with equal probability. We call these dividing circles. Then, with probability at least $\frac{1}{2}$, $\mathcal{C}$ may be partitioned as follows:*

$$\mathcal{C} = \mathcal{C}^* \cup \left(\bigcup_{i=1}^{R} \mathcal{C}_i\right) \quad \textit{(disjoint union)}$$

where $\mathcal{C}^* = \{(y,t) \in \mathcal{C}\colon ||x-y| - |\rho - t|| < \epsilon$ for some dividing circle $(x,\rho)\}$ and

$$R \leq r^3 a(r) \text{ with } a(r) = \mathcal{O}((\log r)^{\alpha}) \ \ \forall \alpha > 0 \tag{9}$$

$$n_i \leq A\frac{n\log n}{r} \text{ for each } i \tag{10}$$

where $n_i = |\{(y,\tau) \in \mathcal{C}\colon \; ||x-y| - |\rho - \tau|| < \epsilon \text{ for some } (x,\rho) \in \mathcal{C}_i\}|$. *We will refer to* $\mathcal{C}_i$ *as a* cell *and will say that these* n_i *circles* enter $\mathcal{C}_i$.

The main part of the proof of Proposition 2.1 may be summarized as follows: go through the argument on pp. 144–150 of [3] replacing spheres by light cones. We will sketch this argument below, leaving out some of the details since we see no reason to repeat [3] verbatim. It leads to the following Lemma 2.2. (Added July 1996: Lemma 2.2 actually is explicitly known—more generally the "vertical decomposition" in [3] has been extended to the case of algebraic surfaces of fixed degree in $\mathbb{R}^3$. See the recent book [9].) Here Q_0 is the unit square in $\mathbb{R}^2$. At the end of the section we show how Lemma 2.2 implies Proposition 2.1.

Remark. The conclusion (10) can be supplemented by other similar conclusions which are stronger in some ways, for example, that $\sum_{i=1}^{R} |\mathcal{C}_i| n_i \leq A\frac{n^2}{r}$. This will be clear to the reader who is familiar with [3]. However, (10) is the easiest to prove and suffices for our purposes, basically because in Theorem 1 we do not care about logarithmic factors.

LEMMA 2.2. *Assume the points of* $\mathcal{C}$ *are in general position (i.e., belong to a suitable dense open subset of* $(\mathbb{R}^2 \times \mathbb{R}^+)^n$*), and that we randomly choose r dividing circles. Then with probability at least* $\frac{1}{2}$*, there is a decomposition of* $Q_0 \times \mathbb{R}$ *into (open) cells* $\Omega_1 \cdots \Omega_R$ *together with their boundaries, such that*

(i) *The boundary of each cell is contained in light cones* $\Gamma(x,t)$*, where* (x,t) *is a dividing circle, and "vertical" submanifolds, i.e., submanifolds of the form* $\gamma \times \mathbb{R}$ *where* γ *is a curve in the x plane.*

(ii) $R \leq r^3 a(r)$.

(iii) $\bar{n}_i \leq A\frac{n \log n}{r}$, *where* $\bar{n}_i = |\{(x,t) \in \mathcal{C}\colon \; \Gamma(x,t) \cap \Omega_i \neq \emptyset\}|$.

Proof. ([3]) It is convenient to be able to consider the two branches of the light cone $\Gamma(x,t)$ separately, so we introduce the following notation: $\mathcal{C} = \{x_j, \rho_j\}_{j=1}^{n}$, $\Gamma_j = \Gamma(x_j, \rho_j)$, and for $\omega \in \{\pm 1\}$,

$$\begin{aligned} \Gamma_j^{\omega} &= \Gamma_j \cap \{(x,t)\colon \; \omega(t-\rho_j) \geq 0\} \\ &= \{(x,t)\colon \; |x-x_j| = \omega(t-\rho_j)\}. \end{aligned}$$

Thus each Γ_j^{ω} is the graph of a function f_j^{ω},

$$\Gamma_j^{\omega} = \{(x,t)\colon \; t = f_j^{\omega}(x)\},$$

where

$$f_j^{\omega}(x) = \rho_j + \omega|x-x_j|.$$

For $i \neq j$ we also let $\gamma_{ij}^{\omega\tilde{\omega}}$ = projection of $\Gamma_i^\omega \cap \Gamma_j^{\tilde{\omega}}$ on the x plane, i.e.

$$\gamma_{ij}^{\omega\tilde{\omega}} = \{x \in \mathbb{R}^2 \colon \ \omega|x - x_i| + \rho_i = \tilde{\omega}|x - x_j| + \rho_j\}. \tag{11}$$

Note we do not define $\gamma_{ij}^{\omega\tilde{\omega}}$ when $i = j$.

For points in general position, the curves $\gamma_{ij}^{\omega\tilde{\omega}}$ will be conic sections or the empty set. Furthermore, the following properties will hold.

(i) $(x_k, \rho_k) \notin \Gamma_i \quad (i \neq k)$.

(ii) $x_k \notin \gamma_{ij}^{\omega\tilde{\omega}} \quad (i \neq j)$.

(iii) $\gamma_{ij}^{\omega\tilde{\omega}}$ and $\gamma_{kl}^{\zeta\tilde{\zeta}}$ are not tangent at a point, except of course when $\{(i,\omega),\ (j,\tilde{\omega})\} = \{(k,\zeta),\ (l,\tilde{\zeta})\}$. In particular, $\gamma_{ij}^{\omega\tilde{\omega}}$ and $\gamma_{kl}^{\zeta\tilde{\zeta}}$ are not identical, except when $\{(i,\omega),\ (j,\tilde{\omega})\} = \{(k,\zeta),\ (l,\tilde{\zeta})\}$ or when both are the empty set.

For (iii), we note that $\gamma_{ij}^{\omega\tilde{\omega}}$ and $\gamma_{ij}^{\zeta\tilde{\zeta}}$ do not intersect if (i) holds and $\rho_i \neq \rho_j$. This may be seen from the defining equations (11). The remaining cases where $\{i,j\} \neq \{k,l\}$ are easily handled by perturbing one of the relevant points.

Now suppose a collection of r dividing circles is given. For the time being, any collection will work and we will order so that they are $C_1 \dots C_r$; the random choice will only become relevant at the end of the proof. In the following arguments it is understood that all indices i, j, k, l etc. are between 1 and r, i.e. correspond to dividing circles.

Fix a curve $\gamma_{ij}^{\omega\tilde{\omega}}$ and define "upper and lower envelopes" $f_\pm \colon \gamma_{ij}^{\omega\tilde{\omega}} \to \mathbb{R} \cup \{\pm\infty\}$,

$$f_+(x) = \min\left(\{f_k^\zeta(x) \colon \ (k,\zeta) \notin \{(i,\omega),(j,\tilde{\omega})\},\ f_k^\zeta(x) \geq f_i^\omega(x)\}\right)$$

and

$$f_+(x) = +\infty \text{ if no such } f_k^\zeta \text{ exists.}$$

$$f_-(x) = \max\left(\{f_k^\zeta(x) \colon \ (k,\zeta) \notin \{(i,\omega),(j,\tilde{\omega})\},\ f_k^\zeta(x) \leq f_i^\omega(x)\}\right)$$

and

$$f_-(x) = -\infty \ \text{ if no such } f_k^\zeta \text{ exists.}$$

Define a + subdivision point to be a point of $\gamma_{ij}^{\omega\tilde{\omega}}$ where either $f_+ = f_i^\omega$, or else $f_+ > f_i^\omega$ but there are two different pairs (k,ζ) and $(l,\tilde{\zeta})$ with $f_k^\zeta = f_+ = f_l^{\tilde{\zeta}}$. Define a $-$ subdivision point analogously.

For each pair $\{(k,\zeta),(l,\tilde{\zeta})\} \neq \{(i,\omega),(j,\tilde{\omega})\}$ there are at most four points on $\gamma_{ij}^{\omega\tilde{\omega}}$ where $f_k^\zeta = f_l^{\tilde{\zeta}}$, since these points are intersection points between two conic sections. Furthermore, because of (iii), the sign of $f_k^\zeta - f_l^{\tilde{\zeta}}$ must change when we move along $\gamma_{ij}^{\omega\tilde{\omega}}$ past any one of these points. Therefore, by the discussion involving Davenport-Schinzel sequences on pp. 148–9 and 137 of [3] we have the following:

(12) There are at most $ra(r)$ + and $-$ subdivision points on each $\gamma_{ij}^{\omega\tilde{\omega}}$.

We divide $\gamma_{ij}^{\omega\tilde{\omega}} \cap Q_0$ into subarcs σ using the + subdivision points and the points where $\gamma_{ij}^{\omega\tilde{\omega}}$ intersects ∂Q_0. Corresponding to each arc σ, we form the submanifold ("type 1 vertical wall")

$$\{(x,t) \in \mathbb{R}^3 \colon \ x \in \sigma, f_i^{\omega}(x) < t < f_+(x)\}.$$

We also perform the corresponding $-$ operation: we subdivide $\gamma_{ij}^{\omega\tilde{\omega}} \cap Q_0$ using the $-$ subdivision points and the points where $\gamma_{ij}^{\omega\tilde{\omega}}$ intersects ∂Q_0, and then form the type 1 vertical walls

$$\{(x,t) \in \mathbb{R}^3 \colon \ x \in \sigma, f_i^{\omega}(x) > t > f_-(x)\}.$$

In view of (12) and the obvious fact that there are $\mathcal{O}(r^2)$ curves $\gamma_{ij}^{\omega\tilde{\omega}}$, we have erected only $r^3a(r)$ vertical walls. Together with the cones Γ_i they divide $Q_0 \times \mathbb{R}$ into cells each of which has a unique top and bottom (these are cones Γ_i^{ω} or the plane at ∞) and the rest of whose boundary consists of vertical walls. Using the definition, one checks that two vertical walls can only intersect at points (x,t) such that x is an endpoint of both corresponding arcs σ. It follows by topology that each vertical wall is part of the boundary of just two cells, and then that there are at most $r^3a(r)$ cells whose boundary contains some vertical wall. Also, for each pair of cones there is a bounded number of cells with these cones as top and bottom and with no vertical wall, since these cells are components of the complement of the two cones in $Q_0 \times \mathbb{R}$. We conclude that there are at most $r^3a(r)$ cells altogether.

Now consider the x-projection $\pi(C)$ of one of these cells C. Its boundary $\partial\pi(C)$ consists of arcs of curves $\gamma_{ij}^{\omega\tilde{\omega}}$, segments of ∂Q_0 and possibly also a point component if the top and bottom are $\Gamma_i^{\pm}$. We refer to these arcs, segments and point as the pieces of $\partial\pi(C)$.

Following [3] we consider points p which are either an intersection point between two pieces of $\partial\pi(C)$, or else an x_1-extreme point of a piece (i.e., a point which is a local extremum for the function $x \to x_1$). We note that intersection points between two pieces must be + or $-$ subdivision points or appropriate points of ∂Q_0. Through each point p we draw the maximal line segment l parallel to the x_2 axis such that $l \setminus \{p\} \subset \pi(C)$. These line segments decompose $\pi(C)$ into subregions, each of which has piecewise smooth boundary consisting of at most two line segments parallel to the x_2 axis and at most two other pieces, which can be either arcs of curves $\gamma_{ij}^{\omega\tilde{\omega}}$ or horizontal segments of ∂Q_0.

It requires a bit of work to see this last statement. Let Σ be one of the subregions and $m = \max(x_1\colon\ x \in \bar{\Sigma})$. Consider the set $\partial\Sigma \cap \{x\colon\ m-\delta < x_1 < m\}$, for small δ. It consists of finitely many nonintersecting curves. We can choose two of these curves, σ_1 and σ_2 so that the vertical segments with abcissa x_1 connecting σ_1 and σ_2 belong to Σ when $m - \delta < x_1 < m$. Now take such a vertical segment and slide it to the left keeping the endpoints on σ_1 and σ_2 and the interior in Σ until no longer possible. When this happens, either one of the σ_i's has an x_1-minimum point, one of the σ_i's intersects another piece of ∂C or else the interior of the segment must touch a piece of ∂C, necessarily at an x_1-maximum point. In any of these cases, the limiting position of the segment is contained in a line l (or vertical segment of Q_0) or is a singleton $\{p\}$. It follows that the region swept out by the segment must be all of Σ, and clearly its boundary is contained in arcs of σ_1 and σ_2 and (possibly degenerate) segments of l and of $\{x\colon\ x_1 = m\}$.

For each of the line segments l just drawn, we erect a vertical wall over it extending through C, i.e., form the vertical wall

$$\{(x,t) \in C\colon\ x \in l\}.$$

The resulting cell decomposition will be that of Lemma 2.2. Note that we have erected at most $r^3a(r)$ new vertical walls, since each arc in the + or $-$ subdivision of $\gamma_{ij}^{\omega\tilde{\omega}}$ has at most two boundary points and (being generous) four x_1-extreme points, and corresponding to each point, we have drawn at most line segments into at most two cells C. It follows that the cell decomposition still has at most $r^3a(r)$ cells.

Each cell has a top and bottom, and now also has at most four vertical walls. Consequently each cell would also be a cell in the decomposition resulting from a fixed finite number (six) of dividing circles. Namely, at most two circles are needed for the top and bottom, then at most two more to determine curves $\gamma_{ij}^{\omega\tilde{\omega}}$ producing the two possible type 1 vertical walls, and at most two more to determine intersection points p whose line segments produce the remaining vertical walls.

Now suppose that the r dividing circles are chosen at random; we will show that (iii) of Lemma 2.2 is valid with high probability (this argument is from [2]). Fix a number ν and consider $P(\nu)$, the probability that $\bar{n}_i \geq \nu$ for some i. Clearly there are $\lesssim n^6$ sets $\Omega \subset \mathbb{R}^3$ which are a cell in the decomposition resulting from six or fewer dividing circles, and by the preceding remarks only these sets can be a cell in the decomposition resulting from r dividing circles. For each Ω, let $S(\Omega) = \{i\colon\ \Gamma(x_i, \rho_i) \cap \Omega \neq \emptyset\}$ and $\bar{n}(\Omega) = |S(\Omega)|$. If Ω is to be a cell in the decomposition resulting from a given set of r dividing circles then clearly none of these r circles can belong to $S(\Omega)$, so the probability that Ω is a cell is at most

$(1 - \frac{\overline{n}(\Omega)}{n})^r$. Consequently

$$P(\nu) \lesssim \sum_{\Omega:\ \overline{n}(\Omega) \geq \nu} \left(1 - \frac{\overline{n}(\Omega)}{n}\right)^r$$
$$\lesssim n^6 \left(1 - \frac{\nu}{n}\right)^r$$

which is small if $\nu = A\frac{n \log n}{r}$ with A large.

Proof of Proposition 2.1. We may clearly assume the points of $\mathcal{C}$ are in general position and may therefore apply Lemma 2.2. The key observation is now the following.

Claim. Suppose Ω_i is one of the cells from Lemma 2.2, and $\Gamma(y, \tau)$ is a light cone and (x, t) is a point of Ω_i such that

(i) $\Delta((x, t), (y, \tau)) < \epsilon$.

(ii) $\Delta((x, t), (x_j, \rho_j)) \geq \epsilon$ for all dividing circles (x_j, ρ_j).

Then $\Gamma(y, \tau) \cap \Omega_i \neq \emptyset$.

Proof. This follows because $\Gamma(y, \tau)$ is transverse to the vertical boundaries of Ω_i. Namely, $\Gamma(y, \tau)$ must contain a point (x, t') with $|t' - t| < \epsilon$; in fact $t' = t \pm \Delta((x, t), (y, \tau))$. If we move vertically from (x, t) to (x, t'), then of course we cannot cross any vertical submanifold. On the other hand, assumption (ii) implies we do not cross any dividing $\Gamma(x_j, \rho_j)$. Hence by (i) of Lemma 2.2, (x, t') must still belong to the cell Ω_i proving the claim.

Let $\mathcal{C}_i$ be the points $(x, t) \in \mathcal{C}$ which belong to the closure of Ω_i and satisfy (ii) of the claim (if a point of $\mathcal{C}$ satisfies (ii) and belongs to the closure of more than one Ω_i, then we arbitrarily assign it to one of the possible $\mathcal{C}_i$'s). The claim then implies that n_i (in Proposition 2.1) is $\leq \overline{n}_i$ (in Lemma 2.2). Hence (10) follows from (iii) of Lemma 2.2 and the proof is complete. □

3. Preliminaries. Given a family of circles $\mathcal{C}$ with centers and radii satisfying (3), we define a multiplicity $\mu_\epsilon^{\mathcal{C}}$ as follows

$$\mu_\epsilon^{\mathcal{C}}(x) = \sum_{C \in \mathcal{C}} \chi_{C^\epsilon}(x) \tag{13}$$

where χ_E is the characteristic function of E.

For fixed $C = C(x,\rho) \in \mathcal{C}$, t and β we define

$$\begin{aligned}
\mathcal{C}_t^C &= \{\tilde{C} \in \mathcal{C}:\ t \le |\rho - \tilde{\rho}| \le 2t\} \\
\mathcal{C}_{t\beta}^C &= \{\tilde{C} \in \mathcal{C}_t^C:\ \Delta(C,\tilde{C}) \le \beta\} \\
\mathcal{C}_{t\beta *}^C &= \mathcal{C}_t^C \setminus \mathcal{C}_{t\beta}^C
\end{aligned}$$

and will also consider the multiplicities $\mu_\epsilon^{\mathcal{C}_t^C}$, etc., defined by (13).

MAIN LEMMA. *For any $\eta_0 > 0$ there is $A < \infty$ such that if $\mathcal{C}$ is any collection of circles with δ-separated radii, $\delta < A^{-1}$, then there is a subcollection $\mathcal{A} \subseteq \mathcal{C}$ with $|\mathcal{A}| \ge A^{-1}|\mathcal{C}|$ such that*

$$|\{x \epsilon C^\delta:\ \mu_\delta^{\mathcal{A}_t^C}(x) \ge \delta^{-\eta_0}\lambda^{-2}\}| \le \delta\lambda \tag{14}$$

for all t and λ.

If we assume the main lemma then it is easy to prove Theorem 1. Namely, it suffices to prove the corresponding restricted weak type estimate

$$|\{\rho:\ M_\delta\chi_E(\rho) \ge \lambda\}| \le A_\epsilon \delta^{-\epsilon}\frac{|E|}{\lambda^3} \tag{15}$$

and we may restrict to circles with centers and radii satisfying (3).

Fix E and λ. Then there is a collection $\mathcal{C}$ of circles with δ-separated radii such that $|E \cap C^\delta| \ge \lambda |C^\delta|$ for all $C \in \mathcal{C}$, and $|\mathcal{C}| \gtrsim \delta^{-1}|\{\rho:\ M_\delta\chi_E(\rho) \ge \lambda\}|$.

Choose a subfamily $\mathcal{A}$ by the lemma ($\eta_0 = \frac{\epsilon}{4}$) so that $|\mathcal{A}| \ge A_\epsilon^{-1}|\mathcal{C}|$ and

$$|\{x \in C^\delta:\ \mu_\delta^{\mathcal{A}_t^C}(x) \ge A_\epsilon \delta^{-3\epsilon/4}\lambda^{-2}\}| \le \delta^{\frac{\epsilon}{4}}\delta\lambda$$

for all t if $C \in \mathcal{A}$. Then also

$$|\{x \in C^\delta:\ \mu_\delta^{\mathcal{A}}(x) \ge \delta^{-\epsilon}\lambda^{-2}\}| \le \frac{1}{2}\lambda|C^\delta|$$

if δ is small, by a simple pigeonhole argument. So

$$\begin{aligned}
|E| &\ge |E \cap \{x:\ \mu_\delta^{\mathcal{A}}(x) \le \delta^{-\epsilon}\lambda^{-2}\}| \\
&\ge (\delta^{-\epsilon}\lambda^{-2})^{-1} \sum_{C \in \mathcal{A}} |C^\delta \cap E \cap \{\mu_\delta^{\mathcal{A}} \le \delta^{-\epsilon}\lambda^{-2}\}|
\end{aligned}$$

$$\begin{aligned} &\geq (\delta^{-\epsilon}\lambda^{-2})^{-1}\frac{\lambda}{2}\sum_{C\in\mathcal{A}}|C^{\delta}| \\ &\approx \delta^{\epsilon}\lambda^{3}\delta|\mathcal{A}| \end{aligned}$$

which is (15). $\square$

The rest of the paper is the proof of the Main Lemma.

LEMMA 3.1. (Measure estimates imply entropy estimates.) *Assume that* $\lambda \leq 1$, $\delta \leq \epsilon \leq \min(\beta, t)$ *and that*

$$|\{x \in C^{\epsilon}\colon \mu_{A_0\epsilon}^{\mathcal{C}^C_{t\beta}} \geq m\}| \leq \lambda\epsilon$$

for a certain number $m \geq \frac{\epsilon}{\delta}\lambda^{-1}$. *Then the cardinality of a set of disjoint* $\frac{\epsilon}{\sqrt{t\beta}}$ *rectangles of* C^{ϵ} *which each intersect at least* $2m$ $\tilde{C}^{\epsilon}$*'s,* $\tilde{C} \in \mathcal{C}^C_{t\beta}$ *cannot exceed* $A\lambda\frac{\sqrt{t\beta}}{\epsilon}$.

Proof. First assume $\beta \leq \frac{t}{4}$. Let R be an $\frac{\epsilon}{\sqrt{t\beta}}$ rectangle of C^{ϵ} which intersects C_i^{ϵ} for m values of i. Since $\beta \leq \frac{t}{4}$ we know $d(C_i, C) \leq t$. Since $\epsilon \leq \min(\beta, t)$, Lemma 1.2 implies that $C_i^{A_0\epsilon}$ contains R for each i, i.e., $\mu_{A_0\epsilon}^{\mathcal{C}^C_{t\beta}} \geq m$ on R. Hence the sum of measures of a disjoint set of R's cannot exceed $\lambda\epsilon$ and we're done.

If $\beta \geq \frac{t}{4}$ then we write

$$\mathcal{C}^C_{t\beta} = \mathcal{C}^C_{t\frac{t}{4}} \cup (\mathcal{C}^C_{t\frac{t}{4}*} \cap \mathcal{C}^C_{t\beta}).$$

It suffices to show there are $\lesssim \lambda\frac{\sqrt{t\beta}}{\epsilon}$ disjoint $\frac{\epsilon}{\sqrt{\beta t}}$ rectangles intersecting m $\tilde{C}^{\epsilon}$'s from $\mathcal{C}^C_{t\frac{t}{4}}$ and similarly with $\mathcal{C}^C_{t\frac{t}{4}*} \cap \mathcal{C}^C_{t\beta}$.

For $\mathcal{C}^C_{t\frac{t}{4}}$ we use that any $\frac{\epsilon}{\sqrt{\beta t}}$-rectangle R is contained in an $\frac{\epsilon}{\sqrt{\frac{t}{4}t}}$-rectangle $\tilde{R}$ and that $\lesssim \sqrt{\frac{\beta}{t}}$ R's are in any given $\tilde{R}$, and then apply the $\beta = \frac{t}{4}$ case.

For $\mathcal{C}^C_{t\frac{t}{4}*} \cap \mathcal{C}^C_{t\beta}$, we use that $|\mathcal{C}^C_t| \lesssim \frac{t}{\delta}$ and that any $\tilde{C}^{\epsilon}$ intersects at most a bounded number of $\frac{\epsilon}{t}$ rectangles of C^{ϵ} (this follows from Lemma 1.1, since $d(C, \tilde{C}) \geq t$ and $\Delta(C, \tilde{C}) \geq \frac{t}{4}$). It follows that there can be at most $\frac{t}{\delta m}$ disjoint $\frac{\epsilon}{t}$-rectangles which each intersect m $\tilde{C}^{\epsilon}$'s. Hence there can be at most $\sqrt{\frac{\beta}{t}}\frac{t}{\delta m} = \frac{\sqrt{\beta t}}{\delta m}$ disjoint $\frac{\epsilon}{\sqrt{\beta t}}$-rectangles which each intersect m $\tilde{C}^{\epsilon}$'s. With our assumptions $\frac{\sqrt{\beta t}}{\delta m} \leq \frac{\sqrt{\beta t}}{\epsilon}\lambda$ as claimed. $\square$

We will be interested in estimates of the following type:

$$\left|\left\{x \in C^{\epsilon}\colon\ \mu_{A_0\epsilon}^{C_{t\beta}^{C}}(x) \geq \delta^{-\eta_1}\frac{\epsilon}{\delta}\lambda^{-2}\max\left(1, \lambda\sqrt{\frac{t}{\epsilon}}\right)^{\alpha}\right\}\right| \leq \lambda\epsilon. \tag{16}$$

Here A_0 is the constant from Lemma 3.1 and we always assume η_1 is a positive constant and $\alpha \in [0, \frac{1}{3}]$. We also always assume that $\epsilon \geq \delta$.

LEMMA 3.2. *Assume that $\epsilon \leq \min(\beta, t)$ and that (16) holds for a certain circle $C \in \mathcal{C}$, ϵ, β, t, and all $\lambda \in [\lambda_0, 1]$. Then there is $k \lesssim \lambda_0\frac{\sqrt{\beta t}}{\epsilon}$ and a collection $\{R_j\}_{j=1}^{k}$ of $\frac{\epsilon}{\sqrt{\beta t}}$-rectangles of C^{ϵ} such that at most*

$$A\delta^{-\eta_1}\frac{\epsilon}{\delta}\lambda_0^{-2}\max\left(\lambda_0\sqrt{\frac{t}{\epsilon}}, 1\right)^{\alpha}\lambda_0\frac{\sqrt{t\beta}}{\epsilon}$$

$\tilde{C}^{\epsilon}$'s, $\tilde{C} \in \mathcal{C}_{t\beta}^{C}$, intersect C^{ϵ} outside the union of the R_j's.

Proof. We will use the notation

$$m(\lambda) = \delta^{-\eta_1}\frac{\epsilon}{\delta}\lambda^{-2}\max\left(1, \lambda\sqrt{\frac{t}{\epsilon}}\right)^{\alpha}$$

and also define $\lambda(m)$ by inverting this equation, i.e.,

$$\lambda(m) = \left(\begin{array}{ll} \delta^{-\frac{\eta_1}{2}}(\frac{\epsilon}{\delta})^{1/2}m^{-1/2} & \text{if } m \geq m_* \\ \delta^{-\frac{\eta_1}{2-\alpha}}(\frac{\epsilon}{\delta})^{\frac{1}{2-\alpha}}(\frac{t}{\epsilon})^{\frac{\alpha}{2-\alpha}}m^{-\frac{1}{2-\alpha}} & \text{if } m \leq m_* \end{array}\right. \tag{17}$$

for a certain m_* whose value is irrelevant.

Evidently $m(\lambda) \geq \frac{\epsilon}{\delta}\lambda^{-2}$ so by Lemma 3.1 we know there are at most $A\lambda\frac{\sqrt{t\beta}}{\epsilon}$ disjoint $\frac{\epsilon}{\sqrt{t\beta}}$ rectangles of C^{ϵ} which intersect more than $2m(\lambda)$ $\tilde{C}^{\epsilon}$'s, $\tilde{C} \in \mathcal{C}_{t\beta}^{C}$.

We subdivide C^{ϵ} in disjoint $\approx \frac{\epsilon}{\sqrt{t\beta}}$ rectangles R_j and for each R_j we let $n_j = |\{\tilde{C} \in \mathcal{C}_{t\beta}^{C}\colon\ \tilde{C}^{\epsilon} \cap R_j \neq \emptyset\}|$. By the preceding discussion,

$$|\{j\colon\ n_j \geq n\}| \lesssim \lambda(n)\frac{\sqrt{t\beta}}{\epsilon} \tag{18}$$

if $n \leq m(\lambda_0)$. Let $\{R_j\}_{j=1}^{k}$ be the rectangles with $n_j \geq m(\lambda_0)$. Then $k \lesssim \lambda_0\frac{\sqrt{t\beta}}{\epsilon}$

by (18). On the other hand, the number of $\tilde{C}^{\epsilon}$'s intersecting C^{ϵ} outside $\bigcup_{j=1}^{k} R_j$ is at most

$$\sum_{2^l \le m(\lambda_0)} 2^l |\{j\colon\ n_j \ge 2^l\}| \lesssim \sum_{2^l \le m(\lambda_0)} 2^l \lambda(2^l) \frac{\sqrt{t\beta}}{\epsilon}. \tag{19}$$

Since the powers of m in (17) are > -1 we have $\sum_{2^l \le m(\lambda_0)} 2^l \lambda(2^l) \lesssim m(\lambda_0)\lambda_0$. So (19) is $\lesssim m(\lambda_0)\lambda_0 \frac{\sqrt{t\beta}}{\epsilon}$ which is Lemma 3.2. ☐

Next we want to record some trivial cases of (16).

LEMMA 3.3. *Estimate* (16) *is valid (with* $\alpha = 0$, *hence for all* α*) if* $\lambda\sqrt{\frac{t}{\epsilon}} \le A^{-1}\delta^{-\frac{\eta_1}{2}}$ *for a suitable constant* A.

(b) *If* $\lambda\sqrt{\frac{t}{\beta}} \le \delta^{-\frac{\eta_1}{2}}$ *then* (16) *is valid with* $\alpha = 0$ *if* $\mathcal{C}_{t\beta}^{C}$ *is replaced by* $\mathcal{C}_{t\beta*}^{C}$.

Proof. (a) The cardinality of $\mathcal{C}_{t\beta}^{C}$ is $\le A\frac{t}{\delta}$, so (everywhere) $\mu_{A_0\epsilon}^{\mathcal{C}_{t\beta}^{C}} \le A\frac{t}{\delta} \le \frac{\epsilon}{\delta}\delta^{-\eta_1}\lambda^{-2}$, i.e. the set in (16) is empty.

(b) If $\tilde{C} \in \mathcal{C}_{t\beta*}^{C}$ then by Lemma 1.1, $|C^{\epsilon} \cap \tilde{C}^{A_0\epsilon}| \lesssim \frac{\epsilon}{\sqrt{t\beta}} \cdot \epsilon$. Hence, by Tchebyshev's inequality, for any m,

$$\begin{aligned} |\{x \in C^{\epsilon}\colon\ \mu_{A_0\epsilon}^{\mathcal{C}_{t\beta*}^{C}}(x) \ge m\}| &\le m^{-1} \sum_{\tilde{C} \in \mathcal{C}_{t\beta*}^{C}} |C^{\epsilon} \cap \tilde{C}^{A_0\epsilon}| \\ &\le m^{-1} |\mathcal{C}_t^{C}| \frac{\epsilon}{\sqrt{t\beta}} \cdot \epsilon. \end{aligned} \tag{20}$$

Now $|\mathcal{C}_t^{C}| \lesssim \frac{t}{\delta}$, so taking $m = \delta^{-\eta_1}\frac{\epsilon}{\delta}\lambda^{-2}$, we get

$$\begin{aligned} |\{x \in C^{\epsilon}\colon\ \mu_{A_0\epsilon}^{\mathcal{C}_{t\beta*}^{C}}(x) \ge \delta^{-\eta_1}\frac{\epsilon}{\delta}\lambda^{-2}\}| &\lesssim \delta^{\eta_1}\lambda^2\sqrt{\frac{t}{\beta}} \cdot \epsilon \\ &\lesssim \delta^{\frac{\eta_1}{2}}\lambda\epsilon. \end{aligned}$$

☐

We will now reduce the number of parameters we have to deal with:

LEMMA 3.4. (tangent case suffices) *Assume that* (16) *holds for a certain* $\alpha, \eta_1, \mathcal{C}, C$, *and* t, *for all* λ, β, ϵ *satisfying the following condition*

$$\epsilon = \beta.$$

Then in fact, if $\eta > 0$ and δ is small enough,

$$\left|\left\{x \in C^{\epsilon}\colon\ \mu_{A_0\epsilon}^{\mathcal{C}_t^C}(x) \geq \delta^{-\eta_1-\eta}\frac{\epsilon}{\delta}\lambda^{-2}\max\left(1, \lambda\sqrt{\frac{t}{\epsilon}}\right)^{\alpha}\right\}\right| \leq \lambda\epsilon$$

for any ϵ and λ.

Proof. If $\lambda\sqrt{\frac{t}{2A_0\epsilon}} \leq \delta^{-\frac{\eta_1}{2}}$ then this follows from Lemma 3.3(a) (take $\beta = 1$ there). So we may assume $\lambda\sqrt{\frac{t}{2A_0\epsilon}} \geq \delta^{-\frac{\eta_1}{2}}$. Set $m = 2A_0\delta^{-\eta_1}\frac{\epsilon}{\delta}\lambda^{-2}(\lambda\sqrt{\frac{t}{\epsilon}})^{\alpha}$, and decompose $\mathcal{C}_t^C$ as follows

$$\mathcal{C}_t^C = \mathcal{C}^C_{t\,\lambda^2 t*} \cup (\cup_\beta \mathcal{A}_\beta) \cup \mathcal{C}^C_{t\,2A_0\epsilon}. \tag{21}$$

Here $\mathcal{A}_\beta = \mathcal{C}^C_{t\beta} \cap \mathcal{C}^C_{t\frac{\beta}{2}*}$, and β runs over dyadic numbers between $2A_0\epsilon$ and $2\lambda^2 t$. By

Lemma 3.3(b) and the assumption (applied with β and ϵ in (16) equal to $2A_0\epsilon$)

$$|\{x \in C^{\epsilon}\colon\ \mu_{A_0\epsilon}^{\mathcal{C}^C_{t\,\lambda^2 t*}} \geq m\}| \lesssim \lambda\epsilon \tag{22}$$

$$|\{x \in C^{\epsilon}\colon\ \mu_{A_0\epsilon}^{\mathcal{C}^C_{t\,2A_0\epsilon}} \geq m\}| \lesssim \lambda\epsilon \tag{23}$$

and we will now show that

$$|\{x \in C^{\epsilon}\colon\ \mu_{A_0\epsilon}^{\mathcal{A}_\beta} \geq m\}| \lesssim \lambda\epsilon \tag{24}$$

for all β. Namely, apply Lemma 3.2 with the indicated value of β, taking (in Lemma 3.2), $\lambda_0 = \lambda$, $\epsilon = \beta$, and using the assumption. Since $\beta \lesssim \lambda^2 t$ we have $\lambda\sqrt{\frac{t}{\beta}} \gtrsim 1$. So we obtain a collection of $\sqrt{\frac{\beta}{t}}$-rectangles $\{\overline{R}_j\}_{j=1}^k$ of C^β, with $k \lesssim \lambda\sqrt{\frac{t}{\beta}}$, such that at most $A\delta^{-\eta_1}\frac{\beta}{\delta}\lambda^{-2}(\lambda\sqrt{\frac{t}{\beta}})^{\alpha+1}$ $\tilde{C}^\beta$'s with $\tilde{C} \in \mathcal{C}^C_{t\beta}$ intersect C^β outside $\cup_j\overline{R}_j$. Let $R_j = \overline{R}_j \cap C^\epsilon$ and $E = \cup_j R_j$. Then $|R_j| \lesssim \epsilon \cdot \sqrt{\frac{\beta}{t}}$, so $|E| \lesssim \lambda\epsilon$. On the other hand, by Lemma 1.1, since $\mathcal{A}_\beta \subseteq \mathcal{C}^C_{t\frac{\beta}{2}*}$,

$$\begin{aligned}\sum_{\tilde{C}\in\mathcal{A}_\beta} |\tilde{C}^{A_0\epsilon} \cap (C^\epsilon \backslash E)| &\lesssim \epsilon \cdot \frac{\epsilon}{\sqrt{t\beta}}|\{\tilde{C} \in \mathcal{A}_\beta\colon\ \tilde{C}^{A_0\epsilon} \cap (C^\epsilon\backslash E) \neq \emptyset\}| \\ &\lesssim \epsilon \cdot \frac{\epsilon}{\sqrt{t\beta}} \cdot \delta^{-\eta_1}\frac{\beta}{\delta}\lambda^{-2}\left(\lambda\sqrt{\frac{t}{\beta}}\right)^{\alpha+1}\end{aligned}$$

$$= \delta^{-\eta_1}\frac{\epsilon}{\delta}\lambda^{-2}\left(\lambda\sqrt{\frac{t}{\beta}}\right)^{\alpha}\cdot\lambda\epsilon$$

$$\lesssim m\lambda\epsilon$$

since $\beta \geq \epsilon$. It now follows by Tchebyshev's inequality that

$$|\{x \in C^{\epsilon}\backslash E\colon\ \mu_{A_{0\epsilon}}^{\mathcal{A}_\beta}(x) \geq m\}| \lesssim \lambda\epsilon$$

and therefore

$$|\{x \in C^{\epsilon}\colon\ \mu_{A_{0\epsilon}}^{\mathcal{A}_\beta}(x) \geq m\}| \lesssim \lambda\epsilon + |E| \lesssim \lambda\epsilon$$

which is (24). Combining (21), (22), (23), and (24), we get

$$\left|\left\{x \in C^{\epsilon}\colon\ \mu_{A_{0\epsilon}}^{\mathcal{C}_t^C}(x) \geq A\log\frac{1}{\delta}\,\delta^{-\eta_1}\frac{\epsilon}{\delta}\lambda^{-2}\max\left(1, \lambda\sqrt{\frac{t}{\epsilon}}\right)^{\alpha}\right\}\right| \lesssim \log\frac{1}{\delta}\,\lambda\epsilon$$

for a suitable constant A, since there are only logarithmically many terms in (21). Applying this with λ replaced by (say) $\lambda\delta^{\frac{\eta}{3}}$ gives the result. □

The next lemma says in particular that (16) is true when $\alpha = \frac{1}{3}$. This result is essentially taken from Section 4 of [4] and we refer to [4] for the details of the proof. It should be regarded as a "Canham threshhold" for our problem in the sense of [3].

LEMMA 3.5. *If* $\eta_1 > 0$, δ *is small enough,* $\mathcal{C}$ *is a family of circles with* δ*-separated radii then there is a subfamily* $\mathcal{A} \subseteq \mathcal{C}$ *with* $|\mathcal{A}| \geq \frac{1}{2}|\mathcal{C}|$ *such that*

$$\left|\left\{x \in C^{\epsilon}\colon\ \mu_{A_{0\epsilon}}^{\mathcal{A}_t^C}(x) \geq \delta^{-\eta_1}\frac{\epsilon}{\delta}\lambda^{-2}\max\left(1, \lambda\sqrt{\frac{t}{\epsilon}}\right)^{1/3}\min\left(1, \frac{\delta|\mathcal{C}|}{t}\right)^{2/3}\right\}\right| \leq \lambda\epsilon$$

for all values of the parameters t, ϵ, λ, *and all* $C \in \mathcal{A}$. *In particular, (16) holds for* $\mathcal{A}$ *with* $\alpha = \frac{1}{3}$.

Sketch of proof. Define

$$m = m(t, \lambda, \epsilon) = \delta^{-\eta_1}\frac{\epsilon}{\delta}\lambda^{-2}\max\left(1, \lambda\sqrt{\frac{t}{\epsilon}}\right)^{1/3}\min\left(1, \frac{\delta|\mathcal{C}|}{t}\right)^{2/3}.$$

If $\lambda \leq \delta^{-\frac{\eta_1}{10}}\sqrt{\frac{\beta}{t}}$ then, using (20) and then the trivial bounds

$$m(t, \lambda, \epsilon) \geq \delta^{-\eta_1}\frac{\epsilon}{\delta}\lambda^{-2}\min\left(1, \frac{\delta|\mathcal{C}|}{t}\right)$$

and $|\mathcal{C}_t^C| \lesssim \min(|\mathcal{C}|, \frac{t}{\delta})$ we have

$$\begin{aligned}
|\{x \in C^\epsilon\colon \mu_{A_0\epsilon}^{\mathcal{C}^C_{t\beta*}}(x) \geq m\}| &\leq m^{-1}|\mathcal{C}_t^C| \frac{\epsilon}{\sqrt{\beta t}} \cdot \epsilon \\
&\lesssim \delta^{\eta_1} \frac{\delta}{\epsilon} \lambda^2 \frac{\min(|\mathcal{C}|, \frac{t}{\delta})}{\min(1, \frac{\delta|\mathcal{C}|}{t})} \frac{\epsilon}{\sqrt{\beta t}} \cdot \epsilon \\
&\approx \delta^{\eta_1} \lambda^2 \epsilon \sqrt{\frac{t}{\beta}} \\
&\leq \delta^{\frac{4}{5}\eta_1} \lambda \epsilon
\end{aligned}$$

for any $C \in \mathcal{C}$. Accordingly, if the lemma fails, then there must be $\frac{|\mathcal{C}|}{2}$ circles $C \in \mathcal{C}$ such that, with $\beta_\lambda = \delta^{\frac{\eta_1}{5}} \lambda^2 t$,

$$|\{x \in C^\epsilon\colon \mu_{A_0\epsilon}^{\mathcal{C}^C_{t\beta_\lambda}}(x) \geq m\}| \geq \frac{\lambda}{2}\epsilon$$

for some t, λ, ϵ depending on C. By pigeonholing we can then find a subfamily $\overline{\mathcal{C}} \subset \mathcal{C}$ with $|\overline{\mathcal{C}}| \geq (\log \frac{1}{\delta})^{-10}|\mathcal{C}|$ and fixed parameter values $t, \epsilon, \lambda, \beta$ with $\epsilon \leq \beta \leq \max(\epsilon, \beta_\lambda)$ so that the following holds.

For each $C \in \overline{\mathcal{C}}$, the set

$$\Big\{x \in C^\epsilon\colon \Big|\Big\{\tilde{C} \in \mathcal{C}\colon x \in \tilde{C}^{A_0\epsilon}, \frac{\beta}{2} \leq \max(\epsilon, \Delta(C, \tilde{C})) \leq \beta, t \leq |\rho - \tilde{\rho}| \leq 2t\Big\}\Big|$$

$$\geq \left(\log \frac{1}{\delta}\right)^{-10} m\Big\}$$

has measure $\geq (\log \frac{1}{\delta})^{-10} \lambda\epsilon$.

It then follows by Lemma 1.1 and a counting argument (cf. [4]) that there are at least

$$\left(\log \frac{1}{\delta}\right)^{-10} |\mathcal{C}| \cdot \left(m \left(\log \frac{1}{\delta}\right)^{-10} \lambda \frac{\sqrt{t\beta}}{\epsilon}\right)^3$$

quadruples of the form $(C, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3)$ with $C, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \in \mathcal{C}$ and $\tilde{C}_i^{A_0\epsilon} \cap C^\epsilon \neq \emptyset$, $\frac{\beta}{2} \leq \max(\epsilon, \Delta(C, \tilde{C}_i)) \leq \beta$ for each i, $t \leq |\rho - \tilde{\rho}_i| \leq 2t$ for each i and $\mathrm{dist}(\tilde{C}_i^{A_0\epsilon} \cap C^\epsilon, \tilde{C}_j^{A_0\epsilon} \cap C^\epsilon) \geq (\log \frac{1}{\delta})^{-11}\lambda$ when $i \neq j$. (Roughly speaking, this is because for each $C \in \overline{\mathcal{C}}$, Lemma 1.1 implies that $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ may be chosen in at least $(m(\log \frac{1}{\delta})^{-10}\lambda\frac{\sqrt{t\beta}}{\epsilon})^3$ ways—details are in [4].)

On the other hand (cf. [4]) Lemma 1.6 implies there are at most

$$|\mathcal{C}| \min\left(|\mathcal{C}|, \frac{t}{\delta}\right)^2 \left[\left(\log \frac{1}{\delta}\right)^{-11} \lambda\right]^{-2} \frac{\beta}{\delta}$$

such quadruples. (This is because the three circles $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ must satisfy $|\tilde{\rho}_i - \tilde{\rho}_j| \leq 4t$ so may be chosen in at most $|\mathcal{C}| \min(|\mathcal{C}|, \frac{t}{\delta})^2$ ways. Once they are chosen, if we set $\overline{\lambda} = (\log \frac{1}{\delta})^{-11}\lambda$ then C must belong to the corresponding set $\Omega_{\beta t \overline{\lambda}}$. Note that $\overline{\lambda} \gtrsim \delta^{-\frac{\eta_1}{10}}(\log \frac{1}{\delta})^{-11}\sqrt{\frac{\beta}{t}}$. Hence Lemma 1.6 implies there are $\lesssim \overline{\lambda}^{-2}\frac{\beta}{\delta}$ ways to choose C.)

Comparing the upper and lower bounds leads to

$$\begin{aligned} m^3 &\leq \lambda^{-5} \min\left(|\mathcal{C}|, \frac{t}{\delta}\right)^2 \left(\frac{\epsilon}{\beta}\right)^{1/2} \frac{\epsilon}{\delta} \left(\frac{\epsilon}{t}\right)^{3/2} \left(\log \frac{1}{\delta}\right)^{62} \\ &= \lambda^{-6}\lambda\sqrt{\frac{t}{\epsilon}} \min\left(\frac{\delta|\mathcal{C}|}{t}, 1\right)^2 \left(\frac{\epsilon}{\beta}\right)^{1/2} \left(\frac{\epsilon}{\delta}\right)^3 \left(\log \frac{1}{\delta}\right)^{62}. \end{aligned}$$

The factor $(\frac{\epsilon}{\beta})^{1/2}$ is ≤ 1 and may be dropped. Hence

$$\begin{aligned} m &\leq \lambda^{-2} \left(\lambda\sqrt{\frac{t}{\epsilon}}\right)^{1/3} \min\left(\frac{\delta|\mathcal{C}|}{t}, 1\right)^{2/3} \frac{\epsilon}{\delta} \left(\log \frac{1}{\delta}\right)^{21} \\ &= \delta^{\eta_1} \left(\log \frac{1}{\delta}\right)^{21} \cdot m. \end{aligned}$$

which is a contradiction for small δ. □

4. Proof of Main Lemma.

We will prove the Main Lemma by induction on the parameter t. The following lemma will give the inductive step. Let $\eta > 0$ be very small and fixed, depending on η_0 only.

LEMMA 4.1. *Assume $\mathcal{C}$ is a family of circles with δ-separated radii. Fix $\lambda_0 \in [\delta, 1], t_0 \in [\delta, 1], \epsilon_0 \in [\delta, 1]$. Make the following inductive hypotheses*

$$\left|\left\{x \in C^{\epsilon}\colon\ \mu_{A_0\epsilon}^{\mathcal{C}_{t\,\epsilon}^{C}}(x) \geq \delta^{-\eta_0}\frac{\epsilon}{\delta}\lambda^{-2}\right\}\right| \leq \epsilon\lambda \tag{25}$$

for all $C \in \mathcal{C},\ t \leq t_0, \lambda \leq 1,\ \epsilon \in [\delta, 1]$, and also

$$\left|\left\{x \in C^{\epsilon}\colon\ \mu_{A_0\epsilon}^{\mathcal{C}_{t\,\epsilon}^{C}}(x) \geq \delta^{-\eta_0+\eta}\frac{\epsilon}{\delta}\lambda^{-2} \max\left(1, \lambda\sqrt{\frac{t}{\epsilon}}\right)^{\alpha}\right\}\right| \leq \epsilon\lambda \tag{26}$$

for all $C \in \mathcal{C}$, $t \in [t_0, \delta^{-\eta} t_0]$, $\lambda \leq 1$, $\epsilon \in [\delta, 1]$, *for a certain* $\alpha \in (0, \frac{1}{3}]$. *Assume also that*

$$|\rho - \tilde{\rho}| \leq \delta^{-\eta} t_0 \tag{27}$$

for all $C, \tilde{C} \in \mathcal{C}$. *Then there is a subfamily* $\mathcal{A} \subset \mathcal{C}$ *with* $|\mathcal{A}| \geq \frac{1}{4}|\mathcal{C}|$ *such that*

$$\left| \left\{ x \in C^{\epsilon} \colon \mu_{A_0 \epsilon}^{\mathcal{A}^C_{t\,\epsilon}}(x) \geq \delta^{-\eta_0 + \eta} \frac{\epsilon}{\delta} \lambda^{-2} \max\left(1, \lambda \sqrt{\frac{t}{\epsilon}}\right)^{\beta} \right\} \right| \leq \epsilon \lambda \tag{28}$$

for all $C \in \mathcal{A}$, $t \in [t_0, \delta^{-\eta} t_0]$, $\lambda \in [\lambda_0, \delta^{-\eta}\lambda_0]$, $\epsilon \in [\epsilon_0, \delta^{-\eta}\epsilon_0]$. *Here* $\beta = \frac{2}{5}\frac{\alpha}{2-\alpha}$.

Proof of Main Lemma assuming Lemma 4.1. Suppose first that t_0 is given and that (25) holds when $t \leq t_0$ and that $|\rho - \tilde{\rho}| \leq \delta^{-\eta} t_0$ $\forall C, \tilde{C} \in \mathcal{C}$. We claim there is a subfamily $\mathcal{A}$ with $|\mathcal{A}| \geq A_\eta^{-1}|\mathcal{C}|$ such that (25) holds also when $t \leq \delta^{-\eta} t_0$. Namely, by Lemma 3.5 there is a subfamily (still called $\mathcal{C}$) such that (26) holds for all values of the parameters with $\alpha = \frac{1}{3}$. Suppose now that we are given a further subfamily so that (26) holds for all $\lambda \leq 1$ and all $\epsilon \in [\delta, 1]$, for a certain α. (28) is trivial for any β if $\lambda < \delta$, say, and the interval $[\delta, 1]$ can be covered by $\lesssim \frac{1}{\eta}$ intervals of the form $[\lambda_0, \delta^{-\eta}\lambda_0]$. By applying Lemma 4.1 for each (λ_0, ϵ_0) in such a covering we obtain a subfamily with cardinality $\geq 4^{-A\eta^{-2}}|\mathcal{C}|$ so that (28) holds for all λ and ϵ. Next, the map $\alpha \to \beta(\alpha) \overset{def}{=} \frac{2}{5}\frac{\alpha}{2-\alpha}$ satisfies $\beta(\alpha) < \alpha$ if $\alpha \in (0, \frac{1}{3}]$ and has zero as an attracting fixed point. Accordingly, by finitely many iterations of (26)$\Rightarrow$(28) starting from the case $\alpha = \frac{1}{3}$ we obtain a subfamily with cardinality $\geq A_\eta^{-1}|\mathcal{C}|$ so that (28) holds for all λ and ϵ, with $\beta < \eta$. This means that (25) holds for $t \leq \delta^{-\eta} t_0$.

We will now remove the hypothesis that $|\rho - \tilde{\rho}| \leq \delta^{-\eta} t_0$ $\forall C, \tilde{C} \in \mathcal{C}$, i.e., we will show the following

Claim. Assume that $\mathcal{C}$ is a family of circles with δ-separated radii and (25) holds when $t \leq t_0$. Then there is a subfamily $\mathcal{A}$ with $|\mathcal{A}| \geq A_\eta^{-1}|\mathcal{C}|$ so that (25) holds when $t \leq \delta^{-\frac{\eta}{2}} t_0$.

To prove the claim, let A be the constant A_η in the preceding paragraph. We can cover $[\frac{1}{2}, 2]$ with intervals I_j of length $\leq \delta^{-\eta} t_0$ in such a way that:

(i) If $\rho, \tilde{\rho} \in [\frac{1}{2}, 2]$ and $|\rho - \tilde{\rho}| \leq 4\delta^{-\frac{\eta}{2}} t_0$ then ρ and $\tilde{\rho}$ belong to a common interval I_j.

(ii) No point belongs to more than two I_j's, and the set $\{C(x, \rho) \in \mathcal{C} \colon \rho$ belongs to two I_j's $\}$ has cardinality $\leq \frac{1}{2}A^{-1}|\mathcal{C}|$.

This can be done because $\delta^{-\frac{\eta}{2}}$ is small compared with $\delta^{-\eta}$. Details are as follows. Let $P = \{\rho \colon \exists C(x, \rho) \in \mathcal{C}\}$. First cover $[\frac{1}{2}, 2]$ with a pairwise disjoint set of intervals γ_j of length $\frac{1}{2}\delta^{-\eta} t_0$. Let $\dot{\gamma}_j$ be the middle half of γ_j. For each j there is an interval $\ddot{\gamma}_j \subset \dot{\gamma}_j$ of length $4\delta^{-\frac{\eta}{2}}t_0$ such that

$$\frac{|\ddot{\gamma} \cap P|}{|\dot{\gamma} \cap P|} \leq \frac{|\ddot{\gamma}_j|}{|\dot{\gamma}_j|} = 16\delta^{\frac{\eta}{2}}.$$

Thus $|\cup_j \ddot{\gamma}_j \cap P| \leq 16\delta^{\frac{\eta}{2}}|\mathcal{C}| < \frac{1}{2}A^{-1}|\mathcal{C}|$. Define the covering $\{I_j\}$ as follows.

$$\begin{aligned} I_0 &= [\text{left endpoint of } \gamma_1,\ \text{right endpoint of } \ddot{\gamma}_1] \\ I_j &= [\text{left endpoint of } \ddot{\gamma}_j,\ \text{right endpoint of } \ddot{\gamma}_{j+1}] \text{ for } 1 \leq j \leq n-1 \\ I_n &= [\text{left endpoint of } \ddot{\gamma}_n,\ \text{right endpoint of } \gamma_n]. \end{aligned}$$

It is easy to see that (i) and (ii) will hold. Now, apply the preceding construction to $\mathcal{C} \cap \{C\colon\ \rho \in I_j\}$ for each j obtaining subfamilies $\mathcal{A}_j$ with $|\mathcal{A}_j| \geq A^{-1}|P \cap I_j|$. Let $\mathcal{A} = \cup_j \mathcal{A}_j$. Then

$$\begin{aligned} |\mathcal{A}| &= \sum_j |\mathcal{A}_j| - |\{\rho \in P\colon\ \rho \text{ belongs to two } I_j\text{'s}\}| \\ &\geq A^{-1}|\mathcal{C}| - \frac{1}{2}A^{-1}|\mathcal{C}| \\ &= \frac{1}{2}A^{-1}|\mathcal{C}|. \end{aligned}$$

Furthermore (25) will hold if $t \leq \delta^{-\frac{\eta}{2}}t_0$. This follows because for any given circle $C \in \mathcal{A}$, all the $\tilde{C}$'s with $|\rho - \tilde{\rho}| \leq 2\delta^{-\frac{\eta}{2}}t_0$ will be contained in a single $\mathcal{A}_j$ by (i). This proves the claim.

Now (25) is obvious if $t \leq \delta$, say, and we have just seen that if it holds for $t \leq t_0$ then it holds for $t \leq \delta^{-\frac{\eta}{2}}t_0$ after passing to a subfamily of proportional size. By $\mathcal{O}(\frac{1}{\eta})$ iterations of this procedure we obtain (25) for all t. Hence by Lemma 3.4 (the case $\alpha = 0$)

$$\left|\left\{x \in C^{\epsilon}\colon\ \mu^{\mathcal{A}_t^C}_{A_0\epsilon}(x) \geq \delta^{-2\eta_0}\frac{\epsilon}{\delta}\lambda^{-2}\right\}\right| \leq \lambda\epsilon$$

for any $C \in \mathcal{A}$. Since η_0 is arbitrary, the Main Lemma now follows by setting $\epsilon = \delta$. □

The rest of the paper is the proof of Lemma 4.1. We always assume that t, λ, ϵ are as in Lemma 4.1, i.e. $t \in [t_0, \delta^{-\eta}t_0], \lambda \in [\lambda_0, \delta^{-\eta}\lambda_0], \epsilon \in [\epsilon_0, \delta^{-\eta}\epsilon_0]$. We consider three cases (i), (ii), (iii), with (iii) being the substantial one.

(i) $\lambda_0\sqrt{\frac{t_0}{\epsilon_0}} \leq \delta^{-200\eta}$. Then $\lambda\sqrt{\frac{t}{\epsilon}} \leq \delta^{-202\eta}$, so (28) already holds for $\mathcal{C}$ by Lemma 3.3(a) provided η is small enough.

(ii) $\lambda_0\sqrt{\frac{t_0}{\epsilon_0}} \geq \delta^{-200\eta}$, but $|\mathcal{C}|\lambda_0^{\frac{2}{3}}(\frac{\epsilon_0}{t_0})^{\frac{2}{3}}\frac{\delta}{\epsilon_0}\delta^{\eta_0+20\eta} < 1$. Then $\lambda\sqrt{\frac{t}{\epsilon}} \geq 1$, hence by Lemma 3.5 there is a subfamily $\mathcal{A}$ with $|\mathcal{A}| \geq \frac{1}{2}|\mathcal{C}|$ and

$$\left|\left\{x \in C^{\epsilon}\colon\ \mu_{A_0\epsilon}^{\mathcal{A}^{C}_{t\epsilon}}(x) \geq \delta^{-\eta}\lambda^{-2}\left(\lambda\sqrt{\frac{t}{\epsilon}}\right)^{\frac{1}{3}}\left(\frac{\delta|\mathcal{C}|}{t}\right)^{\frac{2}{3}}\frac{\epsilon}{\delta}\right\}\right| \leq \lambda\epsilon \tag{29}$$

for all λ, t, ϵ. Under hypothesis (ii),

$$\begin{aligned}
\left(\lambda\sqrt{\frac{t}{\epsilon}}\right)^{\frac{1}{2}}\frac{\delta|\mathcal{C}|}{t} &\leq \left(\lambda\sqrt{\frac{t}{\epsilon}}\right)^{\frac{1}{2}}\lambda_0^{-\frac{2}{3}}\left(\frac{t_0}{\epsilon_0}\right)^{\frac{2}{3}}\frac{\epsilon_0}{\delta}\delta^{-\eta_0-20\eta}\frac{\delta}{t}\\
&\leq \left(\lambda_0\sqrt{\frac{t_0}{\epsilon_0}}\right)^{\frac{1}{2}}\lambda_0^{-\frac{2}{3}}\left(\frac{t_0}{\epsilon_0}\right)^{\frac{2}{3}}\frac{\epsilon_0}{\delta}\frac{\delta}{t_0}\delta^{-\eta_0-20\eta-2\eta}\\
&= \left(\lambda_0\sqrt{\frac{t_0}{\epsilon_0}}\right)^{-\frac{1}{6}}\delta^{-\eta_0-22\eta}\\
&\leq \delta^{-\eta_0}.
\end{aligned}$$

Hence if η is small enough

$$\begin{aligned}
\delta^{-\eta}\lambda^{-2}\left(\lambda\sqrt{\frac{t}{\epsilon}}\right)^{\frac{1}{3}}\left(\frac{\delta|\mathcal{C}|}{t}\right)^{\frac{2}{3}}\frac{\epsilon}{\delta} &\leq \delta^{-\eta}\lambda^{-2}\delta^{-\frac{2}{3}\eta_0}\frac{\epsilon}{\delta}\\
&\leq \delta^{-\eta_0+\eta}\lambda^{-2}\frac{\epsilon}{\delta}
\end{aligned}$$

so that (29) implies (28).

(iii) The remaining case where

$$\lambda_0\sqrt{\frac{t_0}{\epsilon_0}} \geq \delta^{-200\eta} \tag{30}$$

and

$$|\mathcal{C}|\lambda_0^{\frac{2}{3}}\left(\frac{\epsilon_0}{t_0}\right)^{\frac{2}{3}}\frac{\delta}{\epsilon_0}\delta^{\eta_0+20\eta} \geq 1. \tag{31}$$

Then also

$$|\mathcal{C}|\lambda_0^{2}\left(\lambda_0\sqrt{\frac{t_0}{\epsilon_0}}\right)^{-(1+\alpha)}\frac{\delta}{\epsilon_0}\delta^{\eta_0+20\eta} > 1 \tag{32}$$

since the quantity on the left in (32) is obtained from the quantity in (31) by multiplying by $(\lambda_0\sqrt{\frac{t_0}{\epsilon_0}})^{\frac{1}{3}-\alpha}$. We may therefore choose a positive integer r with

$$r^{2-\alpha} \leq |\mathcal{C}|\lambda_0^2 \left(\lambda_0\sqrt{\frac{t_0}{\epsilon_0}}\right)^{-(1+\alpha)} \frac{\delta}{\epsilon_0}\delta^{\eta_0+10\eta} \leq 2r^{2-\alpha}. \tag{33}$$

Note that r is small compared with $|\mathcal{C}|$. We apply Proposition 2.1 with this value of r, taking ϵ in Proposition 2.1 to be $\delta^{-\eta}\epsilon_0$. One or the other of the following must hold:

$$|\mathcal{C}^*| \geq \frac{1}{2}|\mathcal{C}| \tag{34}$$

$$|\cup_i \mathcal{C}_i| \geq \frac{1}{2}|\mathcal{C}|. \tag{35}$$

We first consider the case where (34) holds. Let $\{C_i\}_{i=1}^{r}$ be the dividing circles in Proposition 2.1. We will "throw out" certain circles from $\mathcal{C}$—the subfamily $\mathcal{A}$ will be the circles which are not thrown out. We start by throwing out all circles not in $\mathcal{C}^*$ and also all circles C such that $|\rho - \rho_i| < \frac{1}{100}\frac{\delta|\mathcal{C}|}{r}$ for some i. At least $(\frac{1}{2} - \frac{1}{50})|\mathcal{C}|$ circles remain; we call this family of circles $\mathcal{B}$.

LEMMA 4.2. *For each dividing circle C_i and each $\tau > \frac{\delta|\mathcal{C}|}{100r}$ there is a family of circles $\mathcal{B}_{i\tau} \subset \mathcal{B}$ with $|\mathcal{B}_{i\tau}| \leq \delta^{\eta}\frac{|\mathcal{C}|}{r}$, such that the set*

$$\bigcup_{\substack{C \in \mathcal{B}^{C_i}_{\tau\, \delta^{-\eta}\epsilon_0} \\ C \notin \mathcal{B}_{i\tau}}} C^{\delta^{-\eta}\epsilon_0} \cap C_i^{\delta^{-\eta}\epsilon_0}$$

is contained in the union of at most

$$A\delta^{3\eta}\frac{\lambda_0}{r}\sqrt{\frac{\tau}{\delta^{-\eta}\epsilon_0}}$$

$\sqrt{\frac{\delta^{-\eta}\epsilon_0}{\tau}}$*-rectangles* $\{R_j^{i\tau}\}_{j=1}^{k}$ *of* $C_i^{\delta^{-\eta}\epsilon_0}$.

Proof. Note we can assume $\tau \leq \delta^{-\eta}t_0$ (otherwise $\mathcal{B}_{\tau}^{C_i} = \emptyset$), and $\mathcal{B} \subset \mathcal{C}$, so the inductive hypothesis (25) or (26) is applicable for any ϵ and λ. To ease the notation we let $\epsilon_1 = \delta^{-\eta}\epsilon_0$ and also define

$$\mu_i(x) = \mu_{A_0\epsilon_1}^{\mathcal{B}^{C_i}_{\tau\,\epsilon_1}}(x).$$

Then (25) or (26) implies

$$\left|\left\{x \in C_i^{\epsilon_1}\colon\ \mu_i(x) \geq \delta^{-\eta_0}\frac{\epsilon_1}{\delta}\lambda^{-2}\max\left(1, \lambda\sqrt{\frac{\tau}{\epsilon_1}}\right)^{\alpha}\right\}\right| \leq \epsilon_1\lambda. \tag{36}$$

This has the form (16) with $\eta_1 = \eta_0$, $t = \tau$, $\beta = \epsilon = \epsilon_1$. We will apply Lemma 3.2 with λ_0 in Lemma 3.2 equal to $\delta^{3\eta}\frac{\lambda_0}{r}$. First we need to know the following:

$$\sqrt{\frac{\epsilon_1}{\tau}} \leq \delta^{\frac{1}{2}\eta_0}\frac{\lambda_0}{r}. \tag{37}$$

This is proved as follows. The choice of r (given by (33)) implies

$$\begin{aligned} r^{2-\alpha} &\approx \left(\lambda_0\sqrt{\frac{\delta|\mathcal{C}|}{\epsilon_0}}\right)^{1-\alpha}\left(\frac{\delta|\mathcal{C}|}{\delta^{-\eta}t_0}\right)^{\frac{1}{2}(1+\alpha)}\delta^{\eta_0+10\eta-\frac{1}{2}(1+\alpha)\eta} \\ &\leq \left(\lambda_0\sqrt{\frac{\delta|\mathcal{C}|}{\epsilon_0}}\right)^{1-\alpha}\delta^{\eta_0+9\eta} \end{aligned}$$

since $\frac{\delta|\mathcal{C}|}{\delta^{-\eta}t_0} \lesssim 1$ by (27). Note in particular that $\lambda_0\sqrt{\frac{\delta|\mathcal{C}|}{\epsilon_0}} > 1$, since $r > 1$. It follows that

$$r \leq \left(\lambda_0\sqrt{\frac{\delta|\mathcal{C}|}{\epsilon_0}}\right)^{\frac{1}{2}}\delta^{\frac{1}{2}\eta_0} \tag{38}$$

since $\frac{1-\alpha}{2-\alpha} \leq \frac{1}{2}$ and $\frac{1}{2-\alpha} \geq \frac{1}{2}$. Equivalently,

$$\frac{\delta|\mathcal{C}|}{\epsilon_0} \geq \left(\frac{\delta^{-\eta_0}r^2}{\lambda_0}\right)^2. \tag{39}$$

Hence

$$\begin{aligned} \sqrt{\frac{\epsilon_1}{\tau}} &\leq \sqrt{\frac{100r\epsilon_1}{\delta|\mathcal{C}|}} \\ &\leq 10\delta^{-\frac{1}{2}\eta}r^{-\frac{3}{2}}\lambda_0\delta^{\eta_0} \\ &\leq \delta^{\frac{1}{2}\eta_0}\frac{\lambda_0}{r} \end{aligned}$$

where the second inequality followed from (39). This proves (37).

We know by (37) that

$$\delta^{3\eta}\frac{\lambda_0}{r}\sqrt{\frac{\tau}{\epsilon_1}} \geq 1. \tag{40}$$

In particular, $\epsilon_1 \le \tau$ so we may apply Lemma 3.2 as previously indicated. We obtain $\sqrt{\frac{\epsilon_1}{\tau}}$-rectangles $\{R_j\}_{j=1}^{k}$ of $C_i^{\epsilon_1}$, with $k \lesssim \delta^{3\eta}\frac{\lambda_0}{r}\sqrt{\frac{\tau}{\epsilon_1}}$ such that (using (40)) at most

$$A\delta^{-\eta_0}\frac{\epsilon_1}{\delta}\left(\delta^{3\eta}\frac{\lambda_0}{r}\right)^{-2}\left(\delta^{3\eta}\frac{\lambda_0}{r}\sqrt{\frac{\tau}{\epsilon_1}}\right)^{\alpha+1} \tag{41}$$

C^{ϵ_1}'s, $C \in \mathcal{B}^{C_i}_{\tau\,\epsilon_1}$, intersect $C_i^{\epsilon_1}$ outside the union of the R_j's. To complete the proof it suffices to show that the quantity (41) is $\le \delta^{\eta}\frac{|\mathcal{C}|}{r}$. For this it suffices to show that

$$r^{2-\alpha} \le A^{-1}\delta^{\eta_0+a\eta}|\mathcal{C}|\lambda_0^2\left(\lambda_0\sqrt{\frac{\tau}{\epsilon_0}}\right)^{-(\alpha+1)}\frac{\delta}{\epsilon_0}$$

for a certain constant A. Here $a = 1 + \frac{7}{2}(1-\alpha)$ and we used the relationship between ϵ_0 and ϵ_1. Since $\tau \le \delta^{-\eta}t_0$, it further suffices if

$$r^{2-\alpha} \le A^{-1}\delta^{\eta_0+b\eta}|\mathcal{C}|\lambda_0^2\left(\lambda_0\sqrt{\frac{t_0}{\epsilon_0}}\right)^{-(\alpha+1)}\frac{\delta}{\epsilon_0} \tag{42}$$

where $b = a + \frac{1}{2}(\alpha+1) = 5 - 3\alpha$. But (42) follows from (33) since $5 - 3\alpha < 10$, so the lemma is proved. □

We throw out the family $\mathcal{B}_{i\tau}$ for each $i \in \{1,\dots,r\}$ and each dyadic value of τ. In doing so, we are throwing out at most $r\log\frac{1}{\delta}\cdot\delta^{\eta}\frac{|\mathcal{C}|}{r}$ circles, so the resulting family $\mathcal{A} = \mathcal{B}\setminus\cup_{i\tau}\mathcal{B}_{i\tau}$ still has cardinality $\ge \frac{1}{4}|\mathcal{C}|$.

We will now prove (28) for $\mathcal{A}$. In fact, we will prove the following clearly stronger statement.

$$|\{x \in C^{\epsilon}\colon\ \exists\, \tilde{C} \in \mathcal{A}^{C}_{t\epsilon} \text{ with } x \in \tilde{C}^{A_0\epsilon}\}| \le \epsilon\lambda_0 \tag{43}$$

if $t \in [t_0, \delta^{-\eta}t_0]$, $\epsilon \in [\epsilon_0, \delta^{-\eta}\epsilon_0]$.

To show (43) it suffices to show

$$|\{x \in C^{\epsilon_1}\colon\ \exists\, \tilde{C} \in \mathcal{A}^{C}_{t\,\epsilon_1} \text{ with } x \in \tilde{C}^{A_0\epsilon_1}\}| \le \epsilon_0\lambda_0 \tag{44}$$

when $t \in [t_0, \delta^{-\eta}t_0]$. Suppose then that $\tilde{C} \in \mathcal{A}$. Since $\tilde{C} \in \mathcal{C}^*$ we know there is a dividing circle C_i and a (dyadic) value of τ such that $\tilde{C} \in \mathcal{A}^{C_i}_{\tau\,\epsilon_1}$, and this implies $\tilde{C}^{\epsilon_1} \cap C_i^{\epsilon_1} \ne \emptyset$. Since $\tilde{C}$ was not thrown out in passing from $\mathcal{B}$ to $\mathcal{A}$ we know that $\tilde{C}^{\epsilon_1}$ must intersect one of the rectangles $R_j^{i\tau}$ of Lemma 4.2. We now consider the set

$$\{x \in C^{\epsilon_1}\colon\ \exists\tilde{C} \in \mathcal{A}^{C_i}_{\tau\,\epsilon_1} \cap \mathcal{A}^{C}_{t\,\epsilon_1} \text{ with } \tilde{C}^{\epsilon_1} \cap R_j^{i\tau} \ne \emptyset \text{ and } x \in \tilde{C}^{A_0\epsilon_1}\} \overset{def}{=} \Lambda_{i\tau j} \tag{45}$$

with i, τ and the rectangle $R_j^{i\tau}$ held fixed. We will apply Lemma 1.5 with C_1 in Lemma 1.5 equal to C_i, $C_2 = C$, $t_1 = \tau$, $t_2 = t$, $\epsilon = A_0\epsilon_1$, $\lambda = \sqrt{\frac{A_0\epsilon_1}{\tau}}$. We can split the $\tilde{C}$'s into four subfamilies so that hypothesis (i) in Lemma 1.5 holds for any pair belonging to the same subfamily. The lemma then implies that $\Lambda_{i\tau j}$ is contained in the union of a bounded number of γ-rectangles of C^{ϵ_1} with

$$\gamma = \lambda\frac{t_1}{t_2} + \sqrt{\frac{\epsilon}{t_2}} \lesssim \delta^{-\eta}\sqrt{\frac{\epsilon_1}{\tau}}.$$

We used here that $\tau \leq \delta^{-\eta}t$.

In particular, $|\Lambda_{i\tau j}| \lesssim \epsilon_1 \cdot \delta^{-\eta}\sqrt{\frac{\epsilon_1}{\tau}}$ and therefore

$$|\cup_j \Lambda_{i\tau j}| \lesssim \delta^{3\eta}\frac{\lambda_0}{r}\sqrt{\frac{\tau}{\epsilon_1}} \cdot \epsilon_1\delta^{-\eta}\sqrt{\frac{\epsilon_1}{\tau}} \lesssim \delta^{\eta}\epsilon_0\frac{\lambda_0}{r}.$$

The left side of (44) is bounded by $|\cup_{i\tau j} \Lambda_{i\tau j}|$ by the remarks before (45), and

$$|\cup_{i\tau j} \Lambda_{i\tau j}| \leq \sum_{i\tau} |\cup_j \Lambda_{i\tau j}| \lesssim \sum_{i\tau} \delta^{\eta}\epsilon_0\frac{\lambda_0}{r} \lesssim \delta^{\eta}\log\frac{1}{\delta}\epsilon_0\lambda_0.$$

This proves (44), and therefore Lemma 4.1 in the case where (34) holds.

It remains to consider the case where (35) holds. In the next section, we will use the 3-circle technique to prove the following.

LEMMA 4.3. *Assume the inductive hypothesis (25), and that $|\rho - \tilde{\rho}| \leq \delta^{-\eta}t_0$ for all $C, \tilde{C} \in \mathcal{C}$. Also assume that r satisfies*

$$r^2 \leq \delta^{10\eta}\lambda_0\sqrt{\frac{t_0}{\epsilon_0}} \tag{46}$$

and that (35) holds for the decomposition of Proposition 2.1 (with $\epsilon = \delta^{-\eta}\epsilon_0$). Then there is $\mathcal{A} \subset \cup_i \mathcal{C}_i$ with $|\mathcal{A}| \geq \frac{1}{4}|\mathcal{C}|$ such that

$$|\{x \in C^{\epsilon}\colon \mu_{A_0\epsilon}^{\mathcal{A}_{t\,\epsilon}^{C}}(x) > m\}| \leq \lambda\epsilon \tag{47}$$

for all $t \in [t_0, \delta^{-\eta}t_0], \lambda \in [\lambda_0, \delta^{-\eta}\lambda_0], \epsilon \in [\epsilon_0, \delta^{-\eta}\epsilon_0]$. Here

$$m = m(t,\lambda,\epsilon) \overset{def}{=} \max\left(\delta^{-\eta_0+2\eta}\frac{\epsilon}{\delta}\lambda^{-2}, \delta^{-5\eta}\left(\frac{\epsilon}{\delta}\right)^{\frac{4}{5}}\left(\frac{\delta|\mathcal{C}|}{t}\right)^{\frac{3}{5}}\left(\frac{|\mathcal{C}|}{r^4}\right)^{\frac{1}{5}}\lambda^{-\frac{8}{5}}\right).$$

We assume Lemma 4.3 for now and complete the proof of Lemma 4.1. Under our hypothesis (33) on r we have (38) and therefore also (46) provided η

is small. So there is a subfamily $\mathcal{A}$ with $\frac{1}{4}|\mathcal{C}|$ circles so that (47) holds. Fix ϵ, t, λ. If $m = \delta^{-\eta_0+2\eta}\frac{\epsilon}{\delta}\lambda^{-2}$, then we clearly have (28). What remains is to verify that

$$\delta^{-5\eta}\left(\frac{\epsilon}{\delta}\right)^{\frac{4}{5}}\left(\frac{\delta|\mathcal{C}|}{t}\right)^{\frac{3}{5}}\left(\frac{|\mathcal{C}|}{r^4}\right)^{\frac{1}{5}}\lambda^{-\frac{8}{5}} \leq \delta^{-\eta_0+\eta}\frac{\epsilon}{\delta}\lambda^{-2}\left(\lambda\sqrt{\frac{t}{\epsilon}}\right)^{\beta} \tag{48}$$

with $\beta = \frac{2}{5}\frac{\alpha}{2-\alpha}$. Rearranging (33),

$$\begin{aligned}
r^{2-\alpha} &\approx \frac{\delta|\mathcal{C}|}{t_0}\lambda_0^{1-\alpha}\left(\frac{t_0}{\epsilon_0}\right)^{\frac{1}{2}(1-\alpha)}\delta^{\eta_0+10\eta}\\
\frac{|\mathcal{C}|}{r^4} &= \frac{\frac{\delta|\mathcal{C}|}{t_0}\frac{t_0}{\epsilon_0}\frac{\epsilon_0}{\delta}}{r^4}\\
&\approx \left(\frac{\delta|\mathcal{C}|}{t_0}\right)^{1-\frac{4}{2-\alpha}}\frac{\epsilon_0}{\delta}\left(\frac{t_0}{\epsilon_0}\right)^{\frac{\alpha}{2-\alpha}}\lambda_0^{-4(\frac{1-\alpha}{2-\alpha})}\delta^{-\frac{4}{2-\alpha}(\eta_0+10\eta)}.
\end{aligned}$$

Note that $\frac{\delta|\mathcal{C}|}{t_0} \leq \delta^{-\eta}$ by 27, and $\frac{3}{5}+\frac{1}{5}(1-\frac{4}{2-\alpha}) > 0$. It follows that

$$\left(\frac{\delta|\mathcal{C}|}{t}\right)^{\frac{3}{5}}\left(\frac{|\mathcal{C}|}{r^4}\right)^{\frac{1}{5}} \leq \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{5}}\left(\frac{t}{\epsilon}\right)^{\frac{\alpha}{5(2-\alpha)}}\lambda^{-\frac{4}{5}\frac{1-\alpha}{2-\alpha}}\delta^{-\frac{4}{5(2-\alpha)}\eta_0-A\eta}$$

where A is a fixed constant whose value is irrelevant. Hence the left side of (48) is

$$\begin{aligned}
&\lesssim \frac{\epsilon}{\delta}\left(\frac{t}{\epsilon}\right)^{\frac{\alpha}{5(2-\alpha)}}\lambda^{-\frac{8}{5}-\frac{4}{5}\frac{1-\alpha}{2-\alpha}}\delta^{-\frac{4}{5(2-\alpha)}\eta_0-(A+5)\eta}\\
&= \frac{\epsilon}{\delta}\lambda^{-2}\left(\lambda\sqrt{\frac{t}{\epsilon}}\right)^{\beta}\cdot\delta^{-\frac{4}{5(2-\alpha)}\eta_0-(A+5)\eta}.
\end{aligned}$$

The factor $\frac{4}{5(2-\alpha)}$ is $\leq \frac{12}{25} < 1$, so we have (48), provided η is small enough. □

5. Proof of Lemma 4.3. We start by throwing out all circles in $\mathcal{C}^*$ and also the following circles.

I. All circles which belong to a cell $\mathcal{C}_i$ with $|\mathcal{C}_i| \leq \frac{|\mathcal{C}|}{r^3\log\frac{1}{\delta}}$. We will regard the cells with $|\mathcal{C}_i| \leq \frac{|\mathcal{C}|}{r^3\log\frac{1}{\delta}}$ as having been thrown out as well.

II. All circles which enter at least $(\log\frac{1}{\delta})^3 r^2$ of the remaining cells.

We claim that less than $\frac{1}{4}|\mathcal{C}|$ circles have been thrown out in steps I and II. Namely, by 9,

$$\sum_{|\mathcal{C}_i| \leq \frac{|\mathcal{C}|}{r^3\log\frac{1}{\delta}}}|\mathcal{C}_i| \leq r^3 a(r)\frac{|\mathcal{C}|}{r^3\log\frac{1}{\delta}}$$

so that less than $a(\delta^{-1})(\log\frac{1}{\delta})^{-1}$ circles are thrown out at step I. Next we have the following lemma.

Lemma 5.1. *For any ℓ and m there are at most $A\log\frac{1}{\delta}\frac{|\mathcal{C}|^2}{r\ell m}$ circles C which enter at least ℓ cells each containing at least m circles.*

Proof. If $\mathcal{B}$ is the set of circles in question then, using (10),

$$\begin{aligned}|\mathcal{B}|\ell m &\leq \text{number of pairs } (C,\tilde{C}) \text{ such that } C \text{ enters the cell containing } \tilde{C}\\ &= \sum_i n_i|\mathcal{C}_i|\\ &\lesssim \frac{|\mathcal{C}|^2}{r}\log\frac{1}{\delta}.\end{aligned}$$ □

Applying this lemma with $m=\frac{|\mathcal{C}|}{r^3\log\frac{1}{\delta}}$ and $\ell=r^2(\log\frac{1}{\delta})^3$, we see that the set of circles thrown out at step II has cardinality

$$\lesssim \frac{|\mathcal{C}|^2\log\frac{1}{\delta}}{r\cdot r^2\left(\log\frac{1}{\delta}\right)^3\cdot\frac{|\mathcal{C}|}{r^3\log\frac{1}{\delta}}}=\frac{|\mathcal{C}|}{\log\frac{1}{\delta}}.$$

We let the set of remaining circles be $\mathcal{A}$. We now assume that the conclusion of Lemma 4.3 fails for $\mathcal{A}$ and will obtain a contradiction. To begin with we note that $\frac{\delta^{-\eta}\epsilon_0}{t_0}$ is very small by (46). This fact will be used without mention below.

By pigeonholing, we can choose $t\in[t_0,\delta^{-\eta}t_0]$, $\epsilon\in[\epsilon_0,\delta^{-\eta}\epsilon_0]$ and $\lambda\in[\lambda_0,\delta^{-\eta}\lambda_0]$ such that

$$\left|\left\{x\in C^{\epsilon}\colon\ \mu_{A_0\epsilon}^{\mathcal{A}_{t\epsilon}^{C}}(x)\geq m\left(\log\frac{1}{\delta}\right)^{-3}\right\}\right|\geq\lambda\left(\log\frac{1}{\delta}\right)^{-3}\epsilon$$

for at least $(\log\frac{1}{\delta})^{-3}|\mathcal{A}|$ circles $C\in\mathcal{A}$. For each of these circles, there are at least $2(\log\frac{1}{\delta})^{-4}\lambda\sqrt{\frac{t}{\epsilon}}$ disjoint $\sqrt{\frac{\epsilon}{t}}$-rectangles of C^{ϵ} which each intersect at least $m(\log\frac{1}{\delta})^{-3}$ $\tilde{C}^{A_0\epsilon}$'s, $\tilde{C}\in\mathcal{A}_{t\epsilon}^{C}$.

By pigeonholing again, we can find $M\geq m(\log\frac{1}{\delta})^{-3}$ such that, for each of at least $2(\log\frac{1}{\delta})^{-5}|\mathcal{A}|$ circles $C\in\mathcal{A}$, there are at least $2(\log\frac{1}{\delta})^{-5}\lambda\sqrt{\frac{t}{\epsilon}}$ disjoint $\sqrt{\frac{\epsilon}{t}}$-rectangles R of C^{ϵ} such that

$$|\{\tilde{C}\in\mathcal{A}_{t\epsilon}^{C}\colon\ \tilde{C}^{A_0\epsilon}\cap R\neq\emptyset\}|\in[M,2M].$$

Pigeonholing one more time, we can assume that these pairs $(C,\tilde{C})$ have a common value of $\operatorname{sgn}(\tilde{\rho}-\rho)$. There are two symmetric cases for the sign and we will choose +. Thus: There is $M\geq m(\log\frac{1}{\delta})^{-3}$ such that for at least $(\log\frac{1}{\delta})^{-5}|\mathcal{A}|$

circles $C \in \mathcal{A}$, there are at least $(\log \frac{1}{\delta})^{-5}\lambda\sqrt{\frac{t}{\epsilon}}$ disjoint $\sqrt{\frac{\epsilon}{t}}$-rectangles R of C^{ϵ} with the following property:

$$|\{\tilde{C} \in \mathcal{A}^{C}_{t\epsilon}\colon\ \tilde{\rho} > \rho \text{ and } \tilde{C}^{A_0\epsilon} \cap R \neq \emptyset\}| \in \left[\frac{M}{2}, 2M\right]. \tag{49}$$

Pigeonholing again, we have the following lemma. Here, when we refer to a set of n circles with n not an integer, of course we really mean $[n]$ circles.

LEMMA 5.2. *There is a value of k ($\leq r^3 a(r)$) and k cells $\mathcal{C}_i$ for each of which there is a set of $\frac{|\mathcal{A}|}{k(\log\frac{1}{\delta})^6}$ circles $C \in \mathcal{C}_i$ with the following property: each C^{ϵ} has at least $(\log\frac{1}{\delta})^{-5}\lambda\sqrt{\frac{t}{\epsilon}}$ disjoint $\sqrt{\frac{\epsilon}{t}}$-rectangles for which (49) holds.*

We will refer to the cells $\mathcal{C}_i$, circles C and rectangles R in Lemma 5.2 as being *active*. Thus every active circle belongs to an active cell, etc.

The next step is to define, for each active circle C, a certain collection of "bad" active rectangles. Namely, a rectangle R is bad if one of the following two things happens. Here A_1 is a suitable constant.

I. There is $\tau \leq t_0$ such that there are at least $\delta^{-\eta_0-\frac{1}{2}\eta}\lambda^{-2}\frac{\epsilon}{\delta}$ circles $\overline{C} \in \mathcal{A}^{C}_{\tau\, A_1\frac{\epsilon t}{\tau}}$ with

$$\overline{C}^{A_1\epsilon} \cap R \neq \emptyset.$$

II. With $\mathcal{C}_i$ =cell containing C, there are $\tau \leq 2t$ and $\sigma \geq \sqrt{\frac{\epsilon}{\tau}}$ such that at least

$$\delta^{-\frac{1}{2}\eta}\lambda^{-1}\sigma \min\left(|\mathcal{C}_i|, \frac{\tau}{\delta}\right) \tag{50}$$

circles $\overline{C} \in \mathcal{C}_i \cap \mathcal{A}^{C}_{\tau}$ satisfy $\overline{C}^{A_1\epsilon} \cap \overline{R} \neq \emptyset$. Here $\overline{R}$ is the σ-rectangle of $C^{A_1\epsilon}$ concentric with R.

We will now show that for each active circle C at most $\delta^{\frac{\eta}{10}}\lambda\sqrt{\frac{t}{\epsilon}}$ of its active rectangles are bad. We first consider the type I rectangles. Note that the number τ in the definition must be large compared with ϵ, since $\delta^{-\eta_0-\frac{1}{2}\eta}\lambda^{-2}\frac{\epsilon}{\delta} \leq |\mathcal{A}^{C}_{\tau}| \lesssim \frac{\tau}{\delta}$. Since $\tau \leq t_0$, we can invoke the inductive hypothesis (25). Thus for any ϵ_1 and λ_1

$$\left|\left\{x \in C^{\epsilon_1}\colon\ \mu^{\mathcal{A}^{C}_{\tau\,\epsilon_1}}_{A_0\epsilon_1}(x) \geq \delta^{-\eta_0}\frac{\epsilon_1}{\delta}\lambda_1^{-2}\right\}\right| \leq \epsilon_1\lambda_1.$$

Now apply Lemma 3.4 (the case $\alpha = 0$) to conclude that

$$\left|\left\{x \in C^{\epsilon_2}\colon\ \mu^{\mathcal{A}^{C}_{\tau\,\epsilon_1}}_{A_0\epsilon_2}(x) \geq \delta^{-\eta_0-\frac{1}{20}\eta}\frac{\epsilon_2}{\delta}\lambda_1^{-2}\right\}\right| \leq \epsilon_2\lambda_1$$

for any $\epsilon_1, \epsilon_2, \lambda_1$. Hence, by Lemma 3.1, if $\epsilon_2 \leq \min(\tau, \epsilon_1)$ then for any λ_1 there are at most $A\lambda_1 \frac{\sqrt{\tau\epsilon_1}}{\epsilon_2}$ disjoint $\frac{\epsilon_2}{\sqrt{\tau\epsilon_1}}$-rectangles of C^{ϵ_2} which intersect $A\delta^{-\eta_0-\frac{1}{20}\eta}\frac{\epsilon_2}{\delta}\lambda_1^{-2}$ $\overline{C}^{\epsilon_2}$'s, $\overline{C} \in \mathcal{A}^C_{\tau\,\epsilon_1}$. Now set $\epsilon_1 = A_1\frac{\epsilon t}{\tau}$, $\epsilon_2 = A_1\epsilon$ and $\lambda_1 = \delta^{\frac{\eta}{5}}\lambda$. Then $\epsilon_2 \leq \min(\epsilon_1, \tau)$ since $\epsilon << \tau \leq t$. Thus there are at most $A\delta^{\frac{\eta}{5}}\lambda\sqrt{\frac{t}{\epsilon}}$ disjoint $\sqrt{\frac{\epsilon}{t}}$-rectangles of $C^{A_1\epsilon}$ which each intersect $\delta^{-\eta_0-\frac{1}{2}\eta}\frac{\epsilon}{\delta}\lambda^{-2}$ $\overline{C}^{A_1\epsilon}$'s. Summing over the logarithmically many possible dyadic values of τ, it follows that there are at most $\frac{1}{2}\delta^{\frac{\eta}{10}}\lambda\sqrt{\frac{t}{\epsilon}}$ type I rectangles, say.

Now we consider the type II rectangles. Fix τ and σ, $\sigma \geq \sqrt{\frac{\epsilon}{\tau}}$. For each $\overline{C}$ as in II, $\overline{C}^{A_1\epsilon} \cap C^{A_1\epsilon}$ is contained in a bounded number of σ-rectangles of $C^{A_1\epsilon}$ by Lemma 1.1. Since there are $\lesssim \min(|\mathcal{C}_i|, \frac{\tau}{\delta})$ choices for $\overline{C}$ it follows that for any N, at most $A\frac{\min(|\mathcal{C}_i|, \frac{\tau}{\delta})}{N}$ disjoint σ-rectangles can intersect N of these $\overline{C}^{A_1\epsilon}$'s. Let N be the quantity (50), and call the resulting set of σ-rectangles $\{R_j\}$. Then every type II rectangle with the given values of τ and σ must be contained in the triple of one of the R_j's, and the triple of any given R_j contains at most $A\sigma\sqrt{\frac{t}{\epsilon}}$ type II rectangles. We conclude that there are at most

$$A\sigma\sqrt{\frac{t}{\epsilon}} \cdot \frac{\min\left(|\mathcal{C}_i|, \frac{\tau}{\delta}\right)}{\delta^{-\frac{1}{2}\eta}\lambda^{-1}\sigma\min\left(|\mathcal{C}_i|, \frac{\tau}{\delta}\right)} = A\delta^{\frac{\eta}{2}}\lambda\sqrt{\frac{t}{\epsilon}}$$

type II rectangles with the given τ and σ so there are less than $\frac{1}{2}\delta^{\frac{\eta}{10}}\lambda\sqrt{\frac{t}{\epsilon}}$ type II rectangles in all. This completes the estimate for the number of bad rectangles.

Define a rectangle to be good if it is not bad. The preceding estimate implies that Lemma 5.2 is still valid if one replaces "rectangle" by "good rectangle" in the statement. We now consider the set of all pairs $(C, \tilde{C})$ such that C is an active circle and $\tilde{C} \in \mathcal{A}^C_{t\epsilon}$ is a circle with $\tilde{\rho} > \rho$ such that $\tilde{C}^{A_0\epsilon}$ intersects some good rectangle of C^{ϵ}. We call these active pairs. By Lemma 5.2 there are at least

$$k \cdot \frac{|\mathcal{A}|}{k\left(\log\frac{1}{\delta}\right)^6} \cdot \left(\log\frac{1}{\delta}\right)^{-5}\lambda\sqrt{\frac{t}{\epsilon}} \cdot \frac{M}{2} \approx \left(\log\frac{1}{\delta}\right)^{-11}|\mathcal{A}|\lambda\sqrt{\frac{t}{\epsilon}}M$$

active pairs. This is because the four factors on the left side are lower bounds for respectively the number of active cells, the number of active circles per cell, the number of good rectangles per circle and the number of active pairs such that $\tilde{C}^{A_0\epsilon}$ intersects the given rectangle. We have used that for fixed C each $\tilde{C}^{A_0\epsilon}$ intersects a bounded number of $\sqrt{\frac{\epsilon}{t}}$-rectangles of C^{ϵ} by Lemma 1.1.

Consequently, for some $n \leq |\mathcal{A}|$ there must be n circles $\tilde{C} \in \mathcal{A}$ each of which forms the second member of at least $(\log\frac{1}{\delta})^{-12}\frac{|\mathcal{A}|}{n}\lambda\sqrt{\frac{t}{\epsilon}}M$ active pairs. Now we do some more pigeonholing.

There is a number ℓ_0 such that at least $\frac{n}{\log \frac{1}{\delta}}$ of these n circles each enter between ℓ_0 and $2\ell_0$ active cells. By Lemma 5.1, we have

$$\frac{n}{\log \frac{1}{\delta}} \lesssim \frac{|\mathcal{A}|^2 \log \frac{1}{\delta}}{r\ell_0 \frac{|\mathcal{A}|}{k\left(\log \frac{1}{\delta}\right)^6}} \tag{51}$$

$$\ell_0 \lesssim \left(\log \frac{1}{\delta}\right)^8 \frac{k}{r} \frac{|\mathcal{A}|}{n}$$

We may further choose a number $\ell \leq \ell_0$ such that for each of at least $\frac{n}{(\log \frac{1}{\delta})^2}$ of the $\frac{n}{\log \frac{1}{\delta}}$ circles $\tilde{C}$ in the preceding paragraph, there are at least ℓ active cells $\mathcal{C}_i$ such that $\tilde{C}$ is the second member of at least $(\log \frac{1}{\delta})^{-13} \frac{|\mathcal{A}|}{\ell n} \lambda \sqrt{\frac{t}{\epsilon}} M$ active pairs $(C, \tilde{C})$ with $C \in \mathcal{C}_i$. We will call these circles $\tilde{C}$ *opposing circles*.

Fix an opposing circle $\tilde{C}$. We are going to get a lower bound on the cardinality of the following set $\mathcal{T}_{\tilde{C}}$ for some choice of $\sigma > A_2\sqrt{\frac{\epsilon}{t}}$ and $\tau \leq 2t$. Here A_2 is a large enough constant.

$\mathcal{T}_{\tilde{C}} \overset{def}{=} \{(C_1, C_2, C_3)$: C_i belong to the same cell, $(C_i, \tilde{C})$ is an active pair for each i, $\frac{\tau}{2} \leq |\rho_1 - \rho_2| \leq 2\tau$, $\frac{\tau}{2} \leq |\rho_1 - \rho_3| \leq 2\tau$, and $\operatorname{dist}(C_i^\epsilon \cap \tilde{C}^{A_0\epsilon}, C_j^\epsilon \cap \tilde{C}^{A_0\epsilon}) \in [\sigma, A_2\sigma]$ for each (i,j) with $i \neq j\}$.

Claim. For any given $\sqrt{\frac{\epsilon}{t}}$-rectangle $\tilde{R}$ of $\tilde{C}^{A_0\epsilon}$ there are at most $\delta^{-\eta_0 - 2\eta}\lambda^{-2}\frac{\epsilon}{\delta}$ active pairs $(C, \tilde{C})$ such that $\tilde{R}$ intersects some good rectangle of C^ϵ.

Proof. Suppose there are N such pairs $\{(C_i, \tilde{C})\}_{i=1}^N$. For each i, we fix a good $\sqrt{\frac{\epsilon}{t}}$-rectangle R_i of C_i^ϵ which intersects $\tilde{R}$. It follows by the definition of type I bad rectangles that for any $\tau \leq t_0$ there are at most $\delta^{-\eta_0 - \frac{1}{2}\eta}\lambda^{-2}\frac{\epsilon}{\delta}$ circles $C \in \mathcal{A}^{C_i}_{\tau\, A_1 \frac{\epsilon t}{\tau}}$ with $C^{A_1\epsilon} \cap R_i \neq \emptyset$.

On the other hand, for a suitable constant A_3 which should be chosen before A_1, $C_i^{A_3\epsilon}$ will contain $\tilde{R}$ for each i, by Lemma 1.2. Accordingly, for each i and j

$$\tilde{C}^{A_3\epsilon} \cap C_i^{A_3\epsilon} \cap C_j^{A_3\epsilon} \neq \emptyset. \tag{52}$$

At least $N\frac{t_0}{8t} \geq \frac{1}{8}N\delta^\eta$ circles C_i must have radii ρ_i which lie in a fixed interval of length t_0. Hence we can fix $\tau \leq t_0$ and $i \in \{1, \ldots, N\}$ such that for $\gtrsim \frac{N\delta^\eta}{\log \frac{1}{\delta}}$ choices of j, we have $|\rho_i - \rho_j| \in [\tau, 2\tau]$. If A_1 is large enough then using (52) and Lemma 1.3, we get $\Delta(C_i, C_j) \leq A_1\frac{\epsilon t}{\tau}$ for all such j, i.e. $C_j \in \mathcal{A}^{C_i}_{\tau\, A_1 \frac{\epsilon t}{\tau}}$. Also $C_j^{A_1\epsilon}$ intersects R_i, since $C_j^{A_1\epsilon}$ contains $\tilde{R}$. Combining with the upper bound from the previous paragraph, we get

$$\frac{N\delta^{\eta}}{\log\frac{1}{\delta}} \lesssim \delta^{-\eta_0-\frac{\eta}{2}}\lambda^{-2}\frac{\epsilon}{\delta}$$

and the claim follows. □

Before proceeding with the main argument we need a lemma.

LEMMA 5.3. *Fix* $\eta > 0$ *and* A *with* A *sufficiently large depending on* η. *Suppose that* $\{\gamma_i\}_{i=1}^{N}$ *is a set of pairwise disjoint arcs of the unit circle with* $|\gamma_i| = \omega$ *for all i, and assume* $N > A^2\omega^{-\eta}$. *Then for some* $\sigma > A\omega$ *there are at least* $(N\omega^{\eta})^3$ *triples* $\gamma_{i_1}, \gamma_{i_2}, \gamma_{i_3}$ *with* $\operatorname{dist}(\gamma_{i_j}, \gamma_{i_k}) \in [\sigma, A\sigma]$ *for each* $j, k \in \{1, 2, 3\}$ *with* $j \neq k$.

Proof. We normalize so that the unit circle has length 1. Choose an arc I of the unit circle which satisfies the following:

$$I \text{ contains at least } \tfrac{1}{3}N|I|^{\eta} \text{ arcs } \gamma_j \tag{53}$$

and is as short as possible subject to this condition. Then $\frac{1}{3}N|I|^{\eta}\omega \leq |I|$, so $|I| \geq (\frac{1}{3}N\omega)^{\frac{1}{1-\eta}} \geq A^2\omega$. Also, some semicircle must satisfy (53), so $|I| \leq \frac{1}{2}$. Consequently we can cover I with $\leq A$ disjoint arcs J_i whose lengths are between $A^{-1}|I|$ and $A^{-1}|I| + \omega$, in such a way that each γ_j in (53) is contained in one of the J_i's. Each J_i contains at most

$$N\left(\frac{|I|}{A} + \omega\right)^{\eta} \leq N(A^{-1} + A^{-2})^{\eta}|I|^{\eta} \leq \frac{1}{10}N|I|^{\eta}$$

γ_j's, by minimality of I, and assuming we have chosen A large enough. On the other hand the union of all the J_i's containing less than $\frac{1}{2}A^{-1}N|I|^{\eta}$ γ_j's will clearly contain $\leq \frac{1}{2}N|I|^{\eta}$ γ_j's. We conclude there must be five different J_i's which each contain at least $\frac{1}{2}A^{-1}N|I|^{\eta}$ γ_j's. If we order these in (say) the clockwise direction then the first, third and fifth are mutually at distance between $\frac{|I|}{A}$ and $|I|$. Take the triples with one entry γ_j contained in each of these three J_i's. This gives at least $(\frac{1}{2}A^{-1}N|I|^{\eta})^3 \geq N^3(\frac{1}{2}A^{-1}|I|)^{3\eta} \geq N^3\omega^{3\eta}$ triples so the proof of the lemma is complete. □

We now continue with our opposing circle $\tilde{C}$. Fix one of the $\geq \ell$ active cells in its definition, and let $\nu \geq (\log\frac{1}{\delta})^{-13}\frac{|\mathcal{A}|}{\ell n}\lambda\sqrt{\frac{t}{\epsilon}}M$ be the number of active pairs $(C, \tilde{C})$ with $C \in \mathcal{C}_i$. By step II in the definition of $\mathcal{A}$, we know that $\ell \leq (\log\frac{1}{\delta})^3 r^2$, and by the assumption on r in Lemma 4.3 this means that $\ell \leq \delta^{9\eta}\lambda\sqrt{\frac{t}{\epsilon}}$.

Accordingly

$$\begin{aligned}\nu &\geq \delta^{-8.1\eta} M \frac{|\mathcal{A}|}{n} \\ &\geq \delta^{-(\eta_0+6\eta)} \frac{\epsilon}{\delta} \lambda^{-2}.\end{aligned} \tag{54}$$

For the second inequality we used that $n \leq |\mathcal{A}|$ and that

$$M \geq \left(\log \frac{1}{\delta}\right)^{-3} m \geq \left(\log \frac{1}{\delta}\right)^{-3} \delta^{-\eta_0+2\eta} \frac{\epsilon}{\delta} \lambda^{-2}.$$

Comparing (55) with the claim above, we conclude that for any given $\sqrt{\frac{\epsilon}{t}}$-rectangle $\tilde{R}$ of $\tilde{C}^{A_0\epsilon}$ there are at most $\delta^{4\eta}\nu$ active pairs $(C, \tilde{C})$ with $C \in \mathcal{C}_i$ such that $\tilde{R}$ intersects a good rectangle of C^ϵ.

By pigeonholing, there is a number N and a collection of N disjoint $\sqrt{\frac{\epsilon}{t}}$-rectangles $\tilde{R}_j$ of $\tilde{C}^{A_0\epsilon}$ such that for each j, there are at least $\frac{\nu}{N \log \frac{1}{\delta}}$ circles $C \in \mathcal{C}_i$ such that $(C, \tilde{C})$ forms an active pair and $\tilde{R}_j$ intersects a good rectangle of C^ϵ. The estimate in the preceding paragraph implies that $N \geq \delta^{-3\eta}$, so we may apply Lemma 5.3 with the given η and an appropriate large constant A. We conclude that there is $\sigma > A\sqrt{\frac{\epsilon}{t}}$, and at least $(N(\sqrt{\frac{\epsilon}{t}})^\eta)^3$ triples of rectangles $\tilde{R}_j$ at mutual distance $\in [\sigma, A\sigma]$. For each such triple of $\tilde{R}_j$'s we obtain $(\frac{\nu}{N \log \frac{1}{\delta}})^3$ triples of circles $C_{l_1}, C_{l_2}, C_{l_3} \in \mathcal{C}_i$ such that $(C_{l_p}, \tilde{C})$ is an active pair for $p = 1, 2, 3$. Thus altogether we have at least

$$\left(N \left(\sqrt{\frac{\epsilon}{t}}\right)^\eta\right)^3 \cdot \left(\frac{\nu}{N \log \frac{1}{\delta}}\right)^3 \geq \delta^{2\eta} \left(\frac{|\mathcal{A}|}{\ell n} \lambda \sqrt{\frac{t}{\epsilon}} M\right)^3$$

triples $C_{l_1}, C_{l_2}, C_{l_3}$. We may order each triple so that $|\rho_{l_1} - \rho_{l_2}| \geq |\rho_{l_1} - \rho_{l_3}| \geq |\rho_{l_2} - \rho_{l_3}|$, hence $|\rho_{l_1} - \rho_{l_3}| \geq \frac{1}{2}|\rho_{l_1} - \rho_{l_2}|$. It is now clear that there is a value of τ so that for at least a $(\log \frac{1}{\delta})^{-1}$ proportion of these triples we have $|\rho_{l_1} - \rho_{l_2}| \in [\frac{1}{2}\tau, 2\tau]$ and $|\rho_{l_1} - \rho_{l_3}| \in [\frac{1}{2}\tau, 2\tau]$.

This procedure was carried out with respect to the fixed cell $\mathcal{C}_i$. There are at least ℓ choices for $\mathcal{C}_i$, and by pigeonholing we can find at least $(\log \frac{1}{\delta})^{-2}\ell$ of them which correspond to a common value of σ and τ. Defining $\mathcal{T}_{\tilde{C}}$ with respect to this σ and τ we see that

$$|\mathcal{T}_{\tilde{C}}| \geq \frac{\ell}{(\log \frac{1}{\delta})^2} \cdot \delta^{2\eta} \left(\log \frac{1}{\delta}\right)^{-1} \left(\frac{|\mathcal{A}|}{\ell n} \lambda \sqrt{\frac{t}{\epsilon}} M\right)^3. \tag{55}$$

This is our promised lower bound for $|\mathcal{T}_{\tilde{C}}|$. Next, there are at least $\frac{n}{(\log \frac{1}{\delta})^2}$ choices of the opposing circle $\tilde{C}$. By pigeonholing, we can find $\frac{n}{(\log\frac{1}{\delta})^4}$ which correspond to a common value of σ and τ in (55). With this σ and τ, we define

$$\mathcal{Q} = \{(C_1, C_2, C_3, \tilde{C}):\ (C_1, C_2, C_3) \in \mathcal{T}_{\tilde{C}}\}.$$

It is then immediate that $|\mathcal{Q}|$ is bounded below by $\frac{n}{(\log\frac{1}{\delta})^4}$ times the quantity in 55. This gives

$$|\mathcal{Q}| \geq \delta^{4\eta}|\mathcal{A}|\left(\frac{|\mathcal{A}|}{\ell n}\right)^2\left(M\lambda\sqrt{\frac{t}{\epsilon}}\right)^3. \tag{56}$$

We will now get an upper bound for $|\mathcal{Q}|$ which will contradict (56).

The first step is to count the number of *allowable triples*, i.e. triples (C_1, C_2, C_3) for which some $\tilde{C}$ with $(C_1, C_2, C_3, \tilde{C}) \in \mathcal{Q}$ can exist. Namely, C_1 is an arbitrary active circle, i.e. may be chosen in at most $|\mathcal{A}|$ ways. Once C_1 has been chosen, C_2 must belong to the same cell as C_1 and must also satisfy $|\rho_2 - \rho_1| \leq 2\tau$, hence may be chosen in $\lesssim \min(\frac{|\mathcal{A}|}{k}, \frac{\tau}{\delta})$ ways. Now we look how many ways there are to choose C_3.

By definition of $\mathcal{Q}$, the distance between any two of the $\tilde{C}^{A_0\epsilon} \cap C_i^{\epsilon}$'s, $i \in \{1, 2, 3\}$ is $\leq A_2\sigma$. Hence by Lemma 1.4 $C_1^{A_0\epsilon}$ must intersect both $C_2^{A_0\epsilon}$ and $C_3^{A_0\epsilon}$ within a common $A_4\sigma$-rectangle $\bar{R}$. Furthermore, this rectangle must contain a good $\sqrt{\frac{\epsilon}{t}}$- rectangle of C_1^{ϵ}, since it contains $\tilde{C}^{A_0\epsilon} \cap C_1^{\epsilon}$. We conclude the following property for C_3: $C_3^{A_0\epsilon}$ intersects $C_1^{A_0\epsilon}$ within an $A_4\sigma$-rectangle R of $C_1^{A_0\epsilon}$ which also contains (i) a point of $C_1^{A_0\epsilon} \cap C_2^{A_0\epsilon}$ and (ii) a good rectangle of C_1^{ϵ}.

Because of (i) and Lemma 1.1, the rectangle R must be contained in one of a bounded number (actually just two) of $A_5 \max(\sigma, \sqrt{\frac{\epsilon}{\tau}})$-rectangles. Because of (ii) we may then apply the definition of type II bad rectangles with σ replaced by $A_5 \max(\sigma, \sqrt{\frac{\epsilon}{\tau}})$ to conclude that there are

$$\lesssim \delta^{-\frac{1}{2}\eta}\lambda^{-1}\max\left(\sigma, \sqrt{\frac{\epsilon}{\tau}}\right)\min\left(\frac{|\mathcal{A}|}{k}, \frac{\tau}{\delta}\right)$$

choices for C_3.

Hence we may bound the number of allowable triples by

$$A\delta^{-\frac{1}{2}\eta}\lambda^{-1}\max\left(\sigma, \sqrt{\frac{\epsilon}{\tau}}\right)|\mathcal{A}|\min\left(\frac{|\mathcal{A}|}{k}, \frac{\tau}{\delta}\right)^2. \tag{57}$$

Next, for each allowable triple (C_1, C_2, C_3), there are at most $A\sigma^{-2}\frac{\epsilon}{\delta}$ choices for $\tilde{C}$ with $(C_1, C_2, C_3, \tilde{C}) \in \mathcal{Q}$. This follows from Lemma 1.6. In addition, there are at most AM choices for $\tilde{C}$. This follows from Lemma 1.7, since we have also assumed the upper bound $2M$ in (49). Combining these bounds with (57), we get

$$(58)\quad |\mathcal{Q}| \lesssim \delta^{-\frac{1}{2}\eta}\lambda^{-1}\max\left(\sigma, \sqrt{\frac{\epsilon}{\tau}}\right)|\mathcal{A}|\min\left(\frac{|\mathcal{A}|}{k}, \frac{\tau}{\delta}\right)^2 \min\left(\sigma^{-2}\frac{\epsilon}{\delta}, M\right).$$

The rest of the proof is just a calculation to show that (56) and (58) are incompatible. Namely, if we combine (56) and (58) and make some obvious manipulations we obtain

$$(59)\quad \left(M\lambda\sqrt{\frac{t}{\epsilon}}\right)^3\left(\frac{|\mathcal{A}|}{\ell n}\right)^2 \le \delta^{-5\eta}\lambda^{-1}\min\left(\frac{|\mathcal{A}|}{k}, \frac{\tau}{\delta}\right)^2\max\left(\sigma, \sqrt{\frac{\epsilon}{\tau}}\right) \times \min\left(\sigma^{-2}\frac{\epsilon}{\delta}, M\right).$$

We now consider cases. If $\sigma \le \sqrt{\frac{\epsilon}{\tau}}$ then we estimate as follows:

$$\min\left(\frac{|\mathcal{A}|}{k}, \frac{\tau}{\delta}\right)^2\sqrt{\frac{\epsilon}{\tau}} \le \left(\frac{|\mathcal{A}|}{k}\right)^{\frac{3}{2}}\sqrt{\frac{\epsilon}{\delta}}$$

since the left-hand side is largest when $\tau = \frac{\delta|\mathcal{A}|}{k}$. So the right side of (59) is

$$\le \delta^{-5\eta}\lambda^{-1}\left(\frac{|\mathcal{A}|}{k}\right)^{\frac{3}{2}}\sqrt{\frac{\epsilon}{\delta}}M$$

and (59) implies

$$\begin{aligned} M^2 &\le \delta^{-5\eta}k^{\frac{1}{2}}|\mathcal{A}|^{\frac{3}{2}}\left(\frac{\ell n}{k|\mathcal{A}|}\right)^2\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{\epsilon}{t}\right)^{\frac{3}{2}}\lambda^{-4} \\ &\le \delta^{-6\eta}\frac{k^{\frac{1}{2}}|\mathcal{A}|^{\frac{3}{2}}}{r^2}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{\epsilon}{t}\right)^{\frac{3}{2}}\lambda^{-4} \\ &= \delta^{-6\eta}\frac{k^{\frac{1}{2}}}{r^2}\left(\frac{\epsilon}{\delta}\right)^2\left(\frac{\delta|\mathcal{A}|}{t}\right)^{\frac{3}{2}}\lambda^{-4}. \end{aligned}$$

We used (51) to get the second inequality. Since $k \le r^3 a(r)$ and $\frac{\delta|\mathcal{A}|}{t} \le \delta^{-\eta}$, we conclude that

$$\begin{aligned} M^2 &\le \delta^{-8\eta}r^{-\frac{1}{2}}\left(\tfrac{\epsilon}{\delta}\right)^2\lambda^{-4} \\ M &\le \delta^{-4\eta}r^{-\frac{1}{4}}\tfrac{\epsilon}{\delta}\lambda^{-2} \end{aligned}$$

which contradicts the choice of M if η is small.

On the other hand if $\sigma \geq \sqrt{\frac{\epsilon}{\tau}}$ then since

$$\sigma \min\left(\sigma^{-2}\frac{\epsilon}{\delta}, M\right) \leq \sqrt{\frac{M\epsilon}{\delta}}$$

we may estimate the right side of (59) by

$$\delta^{-5\eta}\lambda^{-1}\left(\frac{|\mathcal{A}|}{k}\right)^2\sqrt{\frac{\epsilon}{\delta}}M^{\frac{1}{2}}.$$

Then (59) implies [again using (51)]

$$\begin{aligned} M^{\frac{5}{2}} &\leq \delta^{-5\eta}\left(\frac{\ell n}{k}\right)^2\left(\frac{\epsilon}{t}\right)^{\frac{3}{2}}\sqrt{\frac{\epsilon}{\delta}}\lambda^{-4} \\ &\leq \delta^{-6\eta}\left(\frac{|\mathcal{A}}{r}\right)^2\left(\frac{\epsilon}{t}\right)^{\frac{3}{2}}\sqrt{\frac{\epsilon}{\delta}}\lambda^{-4} \\ &= \delta^{-6\eta}\left(\frac{|\mathcal{A}|}{r^4}\right)^{\frac{1}{2}}\left(\frac{|\mathcal{A}|\delta}{t}\right)^{\frac{3}{2}}\left(\frac{\epsilon}{\delta}\right)^2\lambda^{-4} \end{aligned}$$

$$M \leq \delta^{-3\eta}\left(\frac{|\mathcal{A}|}{r^4}\right)^{\frac{1}{5}}\left(\frac{|\mathcal{A}|\delta}{t}\right)^{\frac{3}{5}}\left(\frac{\epsilon}{\delta}\right)^{\frac{4}{5}}\lambda^{-\frac{8}{5}}$$

which again contradicts the choice of M. Lemma 4.3 has therefore been proved.

Appendix 1. Here we will prove the following result.

COROLLARY 5.4. *If* $\alpha \leq 1$ *and if E is a Borel set in the plane which contains circles centered at all points of a Borel set with Hausdorff dimension at least* α, *then E has Hausdorff dimension at least* $1 + \alpha$.

Proof. Using measure theory we can find a compact subset $\tilde{E}$ of E, a compact set F with Hausdorff dimension at least α, and a number ρ such that, for each $x \in F$, there is a circle centered at x with radius between ρ and 2ρ which intersects $\tilde{E}$ in a set whose angle measure at x is at least π. We may assume that $\rho = 1$.

Let Π be the projection of F on the x_1 axis. By rotating coordinates and applying Marstrand's projection theorem, we may assume that Π has Hausdorff dimension $\geq \alpha$. Fix $\beta < \alpha$ and a measure μ supported on Π with

$$\mu(I) \lesssim |I|^\beta \tag{60}$$

for all intervals I. Let $\{D(x_j, r_j)\}$ be a covering of $\tilde{E}$ by discs with small radius , define

$$A_\delta = \bigcup_{\{j:\ r_j \in [\delta, 2\delta]\}} D_j$$

and let χ_δ be the characteristic function of A_δ. Let $\mathcal{M}_\delta$ be as in (1). Then the hypothesis implies that $\sum_k \mathcal{M}_{2^{-k}}(\chi_{2^{-k}}) \gtrsim 1$ at all points of Π. Consequently, $\sum_k \int \mathcal{M}_{2^{-k}}(\chi_{2^{-k}}) d\mu \gtrsim 1$. Now we want to replace μ by Lebesgue measure. Namely, we claim that (for any function χ)

$$\int \mathcal{M}_\delta \chi d\mu \lesssim \delta^{-\frac{1}{3}(1-\beta)} \|\mathcal{M}_{2\delta}\chi\|_{L^3(\mathbb{R}, dx)}. \tag{61}$$

This is proved as follows: if I is an interval of length $\frac{\delta}{2}$ then for any $x, y \in I$,

$$\mathcal{M}_\delta f(x) \lesssim \mathcal{M}_{2\delta} f(y). \tag{62}$$

Now subdivide $\mathbb{R}$ in intervals $\{I_j\}$ of length $\frac{\delta}{2}$. Then

$$\begin{aligned}
\int \mathcal{M}_\delta \chi d\mu &\lesssim \sum_j \min_{x \in I_j} \mathcal{M}_{2\delta}\chi(x) \cdot \mu(I_j) \\
&\lesssim \delta^{-\frac{1}{3}} \left(\sum_j |I_j| \min_{x \in I_j} (\mathcal{M}_{2\delta}\chi(x))^3 \right)^{\frac{1}{3}} \left(\sum_j \mu(I_j)^{\frac{3}{2}} \right)^{\frac{2}{3}} \\
&\leq \delta^{-\frac{1}{3}} \|\mathcal{M}_{2\delta}\chi\|_3 \left(\sum_j \mu(I_j)^{\frac{3}{2}} \right)^{\frac{2}{3}}
\end{aligned}$$

where we used (62) and Holder's inequality. The last factor here may be estimated as follows:

$$\sum_j \mu(I_j)^{\frac{3}{2}} \lesssim \delta^{\frac{\beta}{2}} \sum_j \mu(I_j) \lesssim \delta^{\frac{\beta}{2}}$$

by (60) and the fact that μ is a finite measure. This proves (61).

We therefore have

$$\begin{aligned}
1 &\lesssim \sum_k \int \mathcal{M}_{2^{-k}}(\chi_{2^{-k}}) d\mu \\
&\lesssim \sum_k \|\mathcal{M}_{2^{-k+1}}(\chi_{2^{-k}})\|_3 2^{k\frac{(1-\beta)}{3}} \\
&\lesssim \sum_k 2^{k(\epsilon + \frac{1-\beta}{3})} |A_{2^{-k}}|^{\frac{1}{3}}
\end{aligned}$$

for any $\epsilon > 0$, where the last line followed from (1) and the previous line from (61). Consequently $|A_{2^{-k}}| \gtrsim 2^{-k(4\epsilon+1-\beta)}$ for some k. This holds for every covering, so $\dim E \geq 1 + \beta - 4\epsilon$, and since $\beta < \alpha$ and $\epsilon > 0$ are arbitrary we're done. □

We remark that if $\alpha \leq 1$ then a set in $\mathbb{R}^n$ which contains a sphere centered at each point of a set of dimension $\geq \alpha$ must have dimension $\geq n - 1 + \alpha$. When $n \geq 3$ this is much easier since one has an $L^2 \to L^2$ estimate $\|\mathcal{M}_\delta f\|_2 \lesssim (\log \frac{1}{\delta})^{\frac{1}{2}} \|f\|_2$, which can be proved similarly to Theorem 1′ of [4].

Appendix 2. Here we give the proof of (2). We will use x-space Fourier transforms,

$$\hat{f}(\xi) \overset{def}{=} \int_{\mathbb{R}^2} f(x) e^{-2\pi i x\cdot\xi} dx.$$

Let f and u be as in (2). We may assume that $\hat{f}$ is real, else we consider odd and even parts. Also, if $\operatorname{supp}\hat{f}$ is contained in a fixed compact set then (recall we are using inhomogeneous Sobolev spaces) the result is obvious, so we may fix a sufficiently large constant A and assume that $\hat{f}$ vanishes when $|\xi| \leq A$.

Let $u^t(x) = u(t,x)$ and let $m_f^t(x) = \int_{-\pi}^{\pi} f(x + te^{i\theta}) \frac{d\theta}{2\pi}$. Then (denoting real part of z by $\mathrm{re}z$)

$$\widehat{u^t}(\xi) = \cos(t|\xi|)\hat{f}(\xi) \tag{63}$$

$$\widehat{m_f^t}(\xi) = \mathrm{re}\,(a(t|\xi|)(t|\xi|)^{-\frac{1}{2}} e^{2\pi i t|\xi|})\hat{f}(\xi) \tag{64}$$

where $a(r)$ obeys the following: $c \overset{def}{=} \lim_{r\to\infty} a(r)$ exists and is nonzero, and (letting $a^{(j)}$ = jth derivative of a) $|a^{(j)}(r)| \lesssim r^{-(j+1)}$ for any fixed $j \in \mathbb{Z}^+$. (64) follows from the stationary phase asymptotics for the Fourier transform of surface measure on the unit sphere $S^1 \subset \mathbb{R}^2$.

Now consider the functions $g = (-\Delta)^{\frac{1}{4}} f$, $h = (-\Delta)^{-\frac{1}{4}} f$, where Δ is the $\mathbb{R}^2$ laplacian. Thus $\hat{g}(\xi) = (2\pi|\xi|)^{\frac{1}{2}}\hat{f}(\xi)$, $\hat{h}(\xi) = (2\pi|\xi|)^{-\frac{1}{2}}\hat{f}(\xi)$, so using the asymptotics (64) we obtain

$$\widehat{m_g^t}(\xi) = \left(\frac{2\pi}{t}\right)^{\frac{1}{2}} \mathrm{re}\,(a(t|\xi|)e^{2\pi i t|\xi|})\hat{f}(\xi)$$

$$\frac{d}{dt}\widehat{m_h^t}(\xi) = \left(\frac{2\pi}{t}\right)^{\frac{1}{2}} \mathrm{re}\,((ia(t|\xi|) + b(t,\xi))e^{2\pi i t|\xi|})\hat{f}(\xi)$$

where b obeys the estimates $|D_\xi^\alpha b(t,\xi)| \lesssim |\xi|^{-(1+|\alpha|)}$ for any fixed multiindex α, uniformly in $t \in [1,2]$. Note the factor of i in the second formula. It follows therefore that

$$e^{2\pi it|\xi|}\hat{f}(\xi) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}} \left(\widehat{m_g^t}(\xi) - i\frac{d}{dt}\widehat{m_h^t}(\xi)\right) \cdot (c^{-1} + J_t(\xi))\hat{f}(\xi)$$

where the function J_t obeys estimates $|D_\xi^\alpha J_t(\xi)| \lesssim |\xi|^{-(1+|\alpha|)}$ uniformly in $t \in [1,2]$. Here we used that $\hat{f}(\xi)$ vanishes when $|\xi|$ is small. It follows that J_t defines an L^∞ bounded multiplier so (taking real parts and using (63)) we may conclude that

$$\begin{aligned} \|u^t\|_\infty &\lesssim \|m_g^t\|_\infty + \left\|\frac{d}{dt}m_h^t\right\|_\infty \\ &\lesssim \|m_g^t\|_\infty + \|m_{|\nabla h|}^t\|_\infty \end{aligned} \tag{65}$$

where the last line follows by differentiating under the integral sign in the definition of the spherical means.

Next, Theorem 1 may be interpreted in terms of Sobolev spaces: on $t \in [1,2]$, $\|m_{|f|}^t\|_{L_t^3L_x^\infty} \leq C_\epsilon\|f\|_{3,\epsilon}$ for any f and any fixed $\epsilon > 0$. The proof is standard, namely: (i) $f = \sum_{j=0}^\infty f_j$ with $\operatorname{supp}\widehat{f_0} \subset D(0,1)$, $\operatorname{supp}\widehat{f_j} \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ and $\sum_j \|f_j\|_{3,\frac{\epsilon}{2}} \lesssim \|f\|_{3,\epsilon}$, using a dyadic partition of unity in Fourier space. (ii) If R is large then (using dilations) there are Schwartz functions ρ_R and χ_R with bounded L^1 norms, $\operatorname{supp}\chi_R \subset D(0, CR^{-1})$, $\|\chi_R\|_\infty \lesssim R^2$, and $\hat{\rho}_R(\xi)\hat{\chi}_R(\xi) = 1$ when $\frac{R}{2} \leq |\xi| \leq 2R$. Consequently if $\operatorname{supp}\hat{f} \subset \{\frac{R}{2} \leq |\xi| \leq 2R\}$ and if we set $\tilde{f} = \rho_R * f$ then $\|m_{|f|}^t\|_{L_x^\infty} \lesssim M_\delta\tilde{f}(t)$, where $\delta \approx \frac{1}{R}$, and $\|\tilde{f}\|_3 \lesssim \|f\|_3 \lesssim R^{-\frac{\epsilon}{2}}\|f\|_{3,\frac{\epsilon}{2}}$. (iii) Let f_j be as in (i) and define $\tilde{f}_j$ as in (ii), $R = 2^j$. Apply Theorem 1 to conclude that $\|M_\delta\tilde{f}_j\|_3 \lesssim R^{\frac{\epsilon}{4}}\|\tilde{f}_j\|_3$. Thus by (ii), $\|m_{|f_j|}^t\|_{L_t^3L_x^\infty} \lesssim R^{-\frac{\epsilon}{4}}\|f_j\|_{3,\frac{\epsilon}{2}}$. Now sum over j to obtain the result.

We conclude using (66) that $\|u^t\|_{L_t^3L_x^\infty} \lesssim \|g\|_{3,\epsilon} + \|\nabla h\|_{3,\epsilon}$. Clearly $\|g\|_{3,\epsilon} \lesssim \|f\|_{3,\frac{1}{2}+\epsilon}$, and also $\|\nabla h\|_{3,\epsilon} \lesssim \|f\|_{3,\frac{1}{2}+\epsilon}$ by L^3 boundedness of the Riesz transforms. The proof is complete.

Department of Mathematics, 253–37 Caltech, Pasadena, CA 91125
Electronic mail: WOLFF@CCO.CALTECH.EDU

REFERENCES

[1] J. Bourgain, Besicovitch type maximal operators and applications to Fourier analysis, *Geom. Funct. Anal.* **1** (1990), 147–187.

[2] K. L. Clarkson, New applications of random sampling in computational geometry, *Discrete Comput. Geom.* **2** (1987), 195–222.

[3] K. L. Clarkson, H. Edelsbrunner, L. J. Guibas, M. Sharir, and E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, *Discrete Comput. Geom.* **5** (1990), 99–160.
[4] L. Kolasa and T. Wolff, On some variants of the Kakeya problem, *Pacific J. Math.* (to appear).
[5] J. M. Marstrand, Packing circles in the plane, *Proc. London Math. Soc.* **55** (1987), 37–58.
[6] H. Pecher, Nonlinear small data scattering for the wave and Klein-Gordon equations, *Math. Z.* **185** (1984), 261–270.
[7] W. Schlag, A generalization of Bourgain's circular maximal theorem, preprint, 1995.
[8] W. Schlag and C. Sogge, Local smoothing estimates related to the circular maximal theorem, preprint, 1996.
[9] M. Sharir and P. Agarwal, *Davenport-Schinzel Sequences and their Geometric Applications*, Cambridge University Press, 1995.