Harmonic Analysis:

Notation: We denote $L^p(\mathbb{R}^n) := L^p(\mathbb{R}^n, \text{Lebesgue measure})$, for all $1 \leq p \leq +\infty$,

and $|E| :=$ Lebesgue measure of E, for all measurable subsets E of $\mathbb{R}^n$.

The aim of Harmonic Analysis is to understand the behaviour of functions $f \in L^p(\mathbb{R}^n)$. For this, we need certain facts for $L^p(\mathbb{R}^n)$, which are very common tools. A very important one is the following:

Thm: The space $C_c(\mathbb{R}^n) := \{ f : \mathbb{R}^n \underset{(\text{or } \mathbb{R})}{\longrightarrow} \mathbb{C} : f \text{ continuous with compact support} \}$ is dense in $L^p(\mathbb{R}^n)$, $\forall\, 1 \leq p \lneq +\infty$.

I.e., $\forall\, 1 \leq p \lneq +\infty$, for any $f \in L^p(\mathbb{R}^n)$: $\forall \varepsilon > 0$, $\exists\, g \in C_c(\mathbb{R}^n)$ s.t. $\|f - g\|_p < \varepsilon$.

We say that a function $f \in L^p(\mathbb{R}^n)$ is supported on $E \subseteq \mathbb{R}^n$ if f is 0 a.e. outside E. And we say that $f \in L^p(\mathbb{R}^n)$ is compactly supported if $\exists$ compact $K \subseteq \mathbb{R}^n$ s.t. f is 0 a.e. outside K.

So, an $f \in C_c(\mathbb{R}^n)$ looks, generally, like this:

<u>"Proof" of Thm:</u> The proof should be obvious once one understands the definition of the Lebesgue integral of f measurable, via approximation of

f by simple functions. Indeed, let $f \in L^p(\mathbb{R}^n)$. We want, $\forall \varepsilon > 0$, to find $g_\varepsilon \in C_c(\mathbb{R}^n)$, s.t. $\|f - g_\varepsilon\|_p < \varepsilon$.

$$\|f-g_\varepsilon\|_p = \left(\int |f-g_\varepsilon|^p\right)^{1/p}$$

<u>Step 1:</u> We find large $r>0$, s.t. $\|f - f\chi_{B(0,r)}\|_p \text{ "}\sim\text{" } 0$
↳ pretty much equal to.

There exists such an R, by the Dominated Convergence Theorem.

<u>Step 2:</u> We find $\tilde{g}$ simple function, supported on $B(0,r)$, s.t.
$\|f\chi_{B(0,r)} - \tilde{g}\|_p \text{ "}\sim\text{" } 0$.

This can be done by the definition of the Lebesgue integral.

Then: $\tilde{g} = \sum_{E \in \mathcal{E}} a_E \cdot \chi_E$, where $\mathcal{E}$ is a finite family of measurable sets and a_E is a constant, $\forall E \in \mathcal{E}$.

Step 3: We find $\tilde{\tilde{g}}$ step function, supported on $B(0,r)$, s.t. $\|\tilde{g} - \tilde{\tilde{g}}\|_p$ "$\sim$" 0.

or on $B(0,2r)$ ...

Note that a step function is of the form $\sum_{R \in \mathcal{R}} c_R \chi_R$, for $\mathcal{R}$ a finite family of rectangles and c_R a constant $\forall R \in \mathcal{R}$.

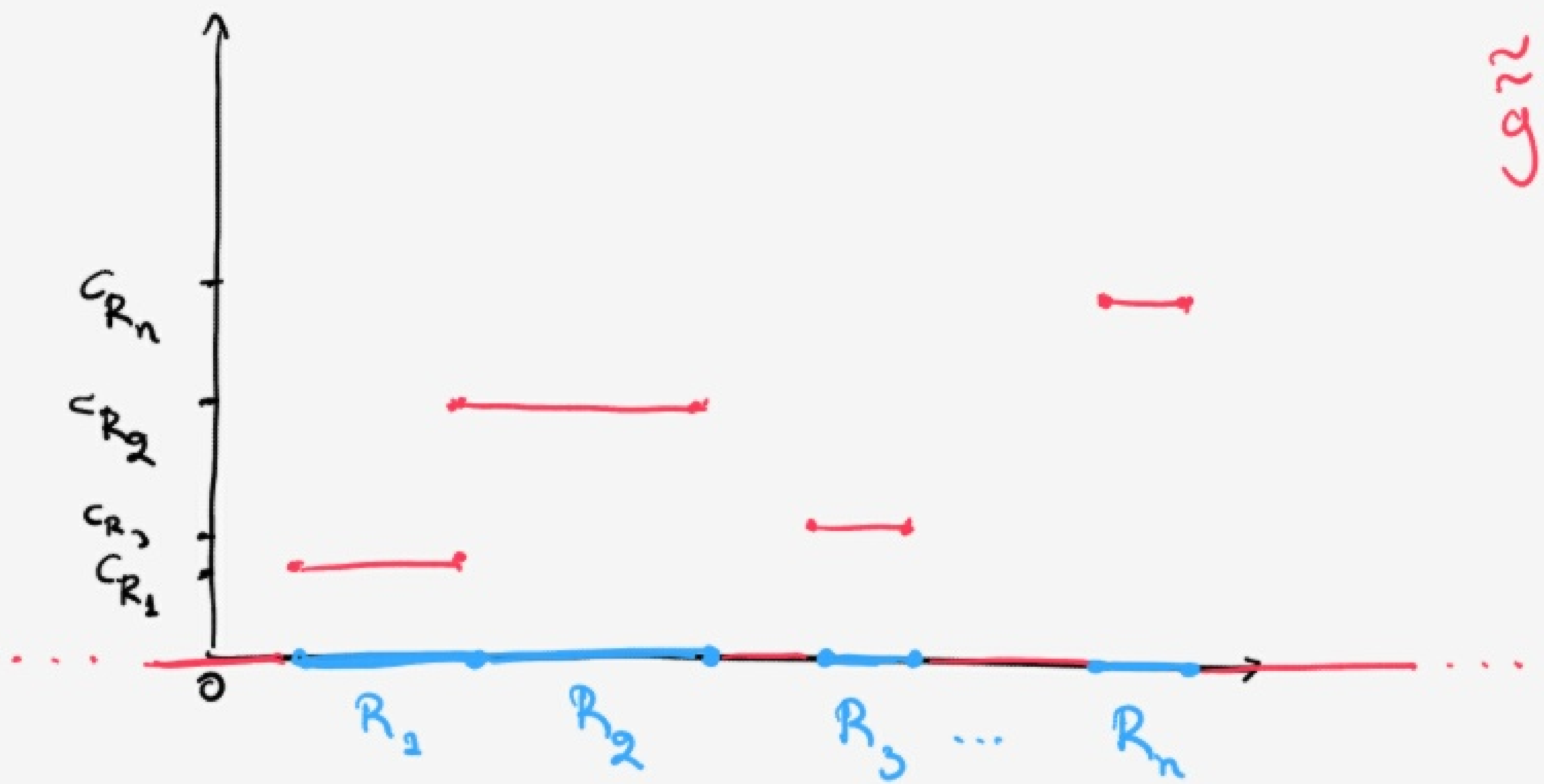
This can be done by approximating each $E \in \mathcal{E}$ by an essentially disjoint union of cubes, $\bigcup_{Q \in \mathcal{Q}_E} Q$, for some finite family $\mathcal{Q}_E$ of cubes, s.t. $\left| \bigcup_{Q \in \mathcal{Q}_E} Q \right|$ "$\sim$" $|E|$. This can be done by approximating $|E|$ by $|\mathcal{U}|$, for some open $\mathcal{U} \supseteq E$ (by the definition of the Lebesgue measure), and then approximating $\mathcal{U}$ by a union of essentially disjoint cubes. Then, for $\tilde{\tilde{g}} := \sum_{\mathcal{E}\in\mathscr{E}} a_{\mathcal{E}}\, \chi_{\left(\bigcup_{Q\in\mathcal{Q}_{\mathcal{E}}} Q\right)}$, $\|\tilde{\tilde{g}} - \tilde{g}\|_p$ "$\sim$" 0.

And note that $\tilde{\tilde{g}}$ is a step function, because it is constant on each $\bigcup_{Q\in\mathcal{Q}_{\mathcal{E}}} Q$, and, since this is a disjoint union (essentially), $\tilde{\tilde{g}}$ is constant on each of the cubes in the union. (Note also that the cubes are finitely many).

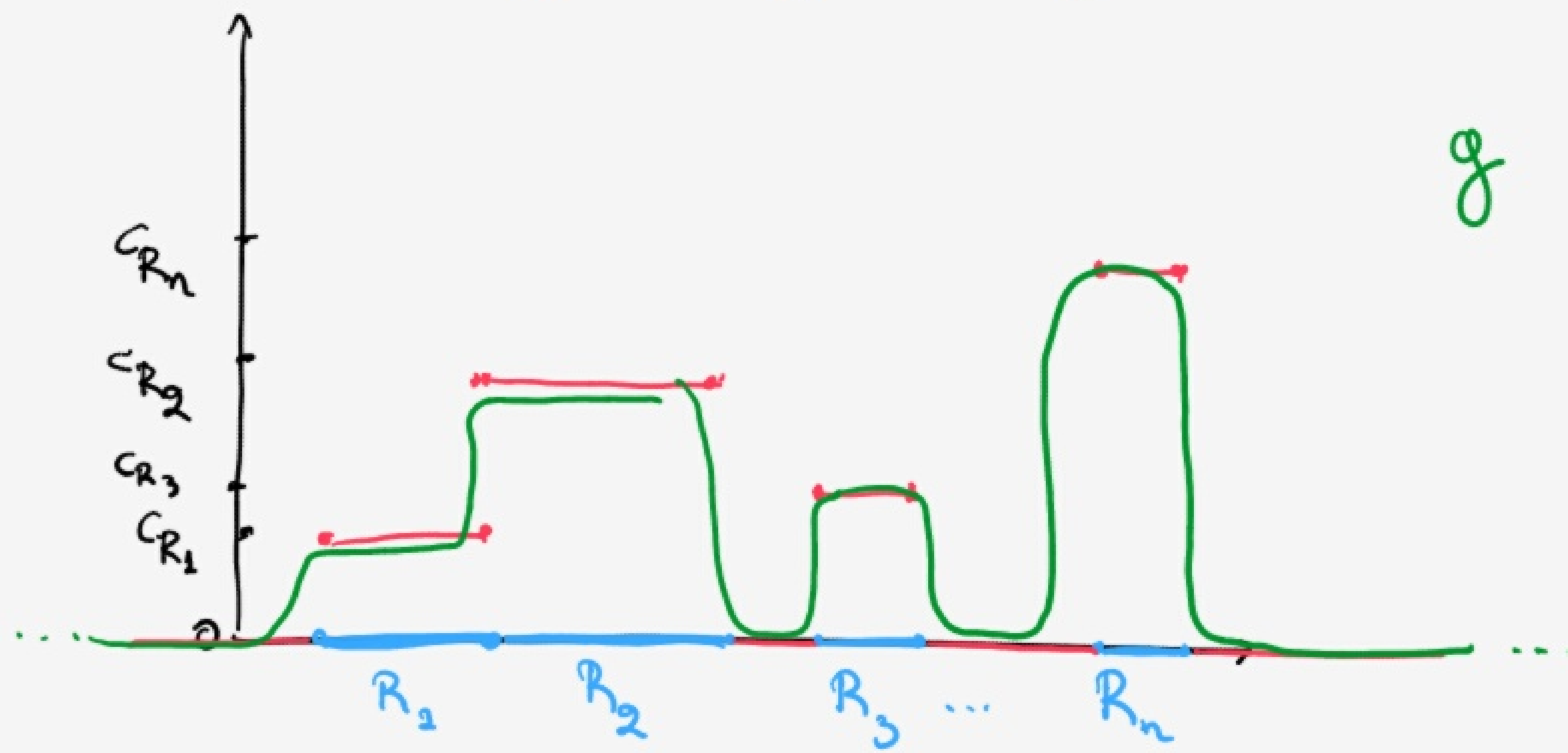
<u>Step 4</u>: Find g continuous, supported on $B(0,2r)$, s.t. $\|\tilde{\tilde{g}} - g\|_p$ "$\sim$" 0.
↳ or $B(0,3r)$...

This is easy: $\tilde{\tilde{g}} = \sum_{R\in\mathcal{R}} \underbrace{c_R}_{\text{constant}} \chi_R$, for a finite family $\mathcal{R}$ of rectangles in $B(0,2r)$.

So, $\tilde{\tilde{g}}$ looks like this:



To construct g, we just change $\tilde{g}$ a little, so that it becomes continuous, with the norm staying pretty much the same:



By the triangle inequality, $\|f-g\|_p$ "$\sim$" 0, for this $g \in C_c(\mathbb{R}^n)$.

Lecture 10 (30/10/2014).

In this lecture, we present the statement of the Lebesgue differentiation theorem, as well as the tool we will use for its proof: the Hardy - Littlewood maximal function.

Def: We say that a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is locally integrable if it is integrable on any ball in $\mathbb{R}^n$, i.e. if $\int_B |f(y)|\,dy < +\infty$, $\forall$ ball B in $\mathbb{R}^n$.

We define $L^1_{loc}(\mathbb{R}^n)$ as the space of locally integrable functions $f : \mathbb{R}^n \to \underset{(\text{or } \mathbb{C})}{\mathbb{R}}$.

ex: • Any constant function $f : \mathbb{R}^n \to \mathbb{R}$ is in $L^1_{loc}(\mathbb{R}^n)$.

Indeed, if c is the constant value of f, then, on any ball B in $\mathbb{R}^n$,

$$\int_B |f(y)|\,dy = \int_B |c|\,dy = |c| \cdot |B| < +\infty.$$

• Any continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is in $L^1_{loc}(\mathbb{R}^n)$.

Indeed, for any ball B in $\mathbb{R}^n$, $\displaystyle\int_B |f(y)|\,dy \leq \int_B \Big(\sup_{y\in B} |f(y)|\Big)\,dy \leq$ (actually, it is $=$, as f cont.)

$$\leq \int_{\bar{B}} \Big(\sup_{y\in\bar{B}} |f(y)|\Big)\,dy = \underbrace{\Big(\sup_{y\in\bar{B}} |f(y)|\Big)}_{<+\infty}\cdot |\bar{B}| < +\infty.$$

$<+\infty$, as $f_{|\bar{B}}$ is a continuous function on a compact set.

• The function $f : \mathbb{R} \longrightarrow \mathbb{R}$, with $f(x) = \frac{1}{x}\cdot \chi_{(0,+\infty)}(x)$ $\forall x \in \mathbb{R}$, does not belong in $L^1_{loc}(\mathbb{R}^n)$, because $|f|$ has infinite integral on any ball containing 0.

Def: Let $f \in L^1_{loc}(\mathbb{R}^n)$. We say that a point $x \in \mathbb{R}^n$ is a *Lebesgue point* for f if

$$f(x) = \lim_{r \downarrow 0} \underbrace{\frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} f(y)\,dy}_{\text{the average of } f \text{ on } B(x,r).},$$

i.e. if $f(x)$ can be approximated by averages of f on balls (in $\mathbb{R}^n$), centered at x, with smaller and smaller radii.

Lebesgue differentiation theorem: Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then, a.e. $x \in \mathbb{R}^n$ is a Lebesgue point for f, i.e: for a.e. $x \in \mathbb{R}^n$,

$$f(x) = \lim_{r \downarrow 0} \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} f(y)\,dy.$$

The Lebesgue differentiation theorem has a very nice and easy application, regarding the local structure of measurable sets:

App: Let $E \subseteq \mathbb{R}^n$ be measurable. Then, for a.e. $x \in E$,

$$\lim_{r \downarrow 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 1.$$

Proof: By the Lebesgue differentiation theorem for the function $\chi_E : \mathbb{R}^n \longrightarrow \mathbb{R}$ (which is in $L^1_{loc}(\mathbb{R}^n)$), we have that, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \downarrow 0} \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} \chi_E(y)\,dy = \chi_E(x), \quad \text{i.e.}$$

$$\lim_{r \downarrow 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = \chi_E(x).$$

Therefore, for a.e. $x \in E$, $\displaystyle\lim_{r \downarrow 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 1$.

The above means that, as $r \downarrow 0$, $|E \cap B(x,r)|$ gets closer to $|B(x,r)|$.

More precisely, $\forall\, 0 < \lambda < 1$, $\exists\, r_0 > 0$ s.t., if $r < r_0$, then

$$\frac{|E \cap B(x,r)|}{|B(x,r)|} > \lambda, \quad \text{i.e. } |E \cap B(x,r)| > \lambda \cdot |B(x,r)|.$$

So, for a.e. $x \in E$, E occupies a large proportion of small balls centered at x (as large proportion as we want, for sufficiently small balls).

Note that this happens for <u>a.e.</u> $x \in E$, but <u>not</u> necessarily for <u>every</u> $x \in E$: for example, when $E = \{x\}$, we cannot hope for E to occupy a large proportion of <u>any</u> ball centered at x! However, the statement of the application still holds for a.e. point of E, as the set of points of E for which the statement does not hold, i.e. $\{x\}$, has measure 0.

Note that, when $f: \mathbb{R} \to \mathbb{R}$ is differentiable, then all points $x \in \mathbb{R}^n$ are Lebesgue points for f', and the derivative at each point is the limit in the definition of a Lebesgue point for f'. Indeed, we know that

$$f'(x) = \lim_{\varepsilon \downarrow 0} \frac{f(x+\varepsilon) - f(x-\varepsilon)}{(x+\varepsilon) - (x-\varepsilon)} = \lim_{\varepsilon \downarrow 0} \frac{\int_{x-\varepsilon}^{x+\varepsilon} f'(y)\,dy}{|(x-\varepsilon, x+\varepsilon)|} = \lim_{\varepsilon \downarrow 0} \frac{1}{|B(x,\varepsilon)|} \cdot \int_{B(x,\varepsilon)} f'(y)\,dy .$$

by definition of derivative.

To understand the $\displaystyle\lim_{r \downarrow 0} \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} f(y)\,dy$ for some $f \in L^1_{loc}(\mathbb{R}^n)$ and some

$x \in \mathbb{R}^n$, we will try to understand the $\underbrace{\sup_{r>0} \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} |f(y)| \, dy}_{\parallel \atop (Mf)(x)}$, $\forall x \in \mathbb{R}^n$.

Def: The (Hardy-Littlewood) maximal function M is an operator, that sends any measurable function $f : \mathbb{R}^n \longrightarrow \underset{(\text{or } \mathbb{C})}{\mathbb{R}}$ to another measurable function $Mf : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$,

with $$Mf(x) = \underbrace{\sup_{r>0} \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} |f(y)| \, dy}_{\substack{\text{the supremum of the averages} \\ \text{of } f \text{ over all balls centered at } x.}} \text{, for all } x \in \mathbb{R}^n.$$

ex: • If $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ is constant, then $Mf(x) = |f|$, for all $x \in \mathbb{R}^n$.

Indeed, if c is the constant value of f, then: for any $x \in \mathbb{R}^n$,

$$\frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} |f(y)|\, dy = \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} |c|\, dy = |c|, \text{ for all } r > 0,$$

so $Mf(x) = |c|$.

• For $f: \mathbb{R} \longrightarrow \mathbb{R}$ with $f(x) = \frac{1}{x} \cdot \chi_{(0,+\infty)}(x)$ $\forall x \in \mathbb{R}^n$, we have that $Mf(x) = +\infty$, $\forall x \in \mathbb{R}^n$.

Indeed, the integral of f is $+\infty$ on every ball that contains 0. Now, for each $x \in \mathbb{R}^n$, $\underbrace{B(x, 2|x|)}$ contains 0, so $\int_{B(x,2|x|)} |f(y)|\, dy = +\infty$,

↳ the interval $(x - 2|x|, x + 2|x|)$

so $\displaystyle \frac{1}{|B(x,2|x|)|} \cdot \int_{B(x,2|x|)} |f(y)|\,dy = +\infty$, so $Mf(x) = +\infty$.

Note that, by the examples above, it is obvious that Mf can take very large values. This is not a surprise: by the Lebesgue differentiation theorem, we have that, for a.e. $x \in \mathbb{R}^n$,

$$|f(x)| = \lim_{r \to 0} \left| \frac{1}{|B(x,r)|} \cdot \int_{B(x,y)} f(y)\,dy \right| \leq \sup_{r>0} \left| \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} f(y)\,dy \right| \leq$$

$$\leq \sup_{r>0} \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} |f(y)|\,dy = Mf(x).$$

What perhaps is a surprise, however, is that, when $f \in L^p(\mathbb{R}^n)$, for

$p \geq 1$, then we can control $\|Mf\|_p$ <u>from above</u> by $\|f\|_p$. We will see this in forthcoming lectures.

<u>Sidenote:</u>

When trying to understand how averages of functions on certain objects behave when the objects shrink down to a point (in our case, the objects were balls with a common centre), it is common to try to understand, instead, the maximal operator associated to the objects. This, for example, has been done when the objects are <u>spheres</u> with common centre: then, we consider the spherical maximal function, that considers supremums of averages of a function over all spheres with common centre.

Lecture 11 (4/11/2014).

The aim of this lecture is to see a sketch of the hardest part of the proof of the Lebesgue differentiation theorem: proving that, for any $f \in L^1_{loc}(\mathbb{R}^n)$, the

$$\lim_{r \downarrow 0} \underbrace{\frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} f(y)\,dy}_{= f_r(x)}$$

<u>exists</u>, for <u>a.e. x</u> $\in \mathbb{R}^n$.

The reason this part of the proof is the hardest is that it requires an understanding of the behaviour of the maximal function of f, Mf. In particular, it requires the following:

$\exists$ constant C_n, depending only on the dimension n, s.t.

$$(*) \quad \boxed{\alpha \cdot \left| \{ x \in \mathbb{R}^n : Mf(x) > \alpha \} \right| \leq C_n \cdot \|f\|_1 \,, \quad \forall \alpha \geq 0, \ \forall f \in L^1(\mathbb{R}^n).}$$

$(*)$ will be proved in forthcoming lectures.

It says that, if $f \in L^1(\mathbb{R}^n)$, then the set $\{x \in \mathbb{R}^n : Mf(x) > \alpha\}$ cannot be too large: it has measure $\leq \dfrac{C_n \cdot \|f\|_1}{\alpha}$.

Note that, for any $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ measurable, $\forall \alpha \geq 0$,

$$\alpha \cdot \left| \{x \in \mathbb{R}^n : |f(x)| \geq \alpha \} \right| = \int_{\{x \in \mathbb{R}^n : |f(x)| \geq \alpha\}} \alpha \, dy \leq \int_{\{x \in \mathbb{R}^n : |f(x)| \geq \alpha\}} |f(y)| \, dy \leq \int_{\mathbb{R}^n} |f(y)| \, dy.$$

So, if $f \in L^1(\mathbb{R}^n)$, $\alpha \cdot |\{x \in \mathbb{R}^n : |f(x)| \geq \alpha\}| \leq \|f\|_1$.

So, if, for $f \in L^1(\mathbb{R}^n)$, we had that $Mf \in L^1(\mathbb{R}^n)$ and $\|Mf\|_1 \leq C_n \|f\|_1$, then we would have ⊛. However, for $f \in L^1(\mathbb{R}^n)$ that is not the 0 function, $Mf \notin L^1(\mathbb{R}^n)$, so certainly $\|Mf\|_1 \nleq C_n \cdot \|f\|_1$.

But even though $\|Mf\|_1 \nleq C_n \cdot \|f\|_1$, the weaker inequality ⊛ still holds.

(Note that, in future lectures, we will show that, $\forall\ 1 < p \leq +\infty$, there exists a constant $C_{n,p}$, depending only on n and p, s.t.

$$\|Mf\|_p \leq C_{n,p} \cdot \|f\|_p \quad , \forall f \in L^p(\mathbb{R}^n).$$

So, $p=1$ is the <u>only</u> value of p for which the above does not hold, and ⊛ is the closest we can get to understanding how the integral of Mf behaves for $f \in L^1(\mathbb{R}^n)$.)

Let us now proceed to the sketch of the proof of the Lebesgue differentiation theorem.

Lebesgue differentiation theorem: Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then,

$$\lim_{r \downarrow 0} \underbrace{\frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} f(y)\,dy}_{= f_r(x)} = f(x), \text{ for a.e. } x \in \mathbb{R}^n.$$

Sketch of proof: It suffices to show the theorem for $f \in L^1(\mathbb{R}^n)$. Indeed, $\forall x \in \mathbb{R}^n$, we are interested in averages of f on balls of radius shrinking to 0, therefore, for any fixed $x \in \mathbb{R}^n$, the behaviour of f far from x is irrelevant. We can thus split $\mathbb{R}^n$ in countably many balls, and show the theorem for f restricted on any of these balls (a function in $L^1(\mathbb{R}^n)$ since $f \in L^1_{loc}(\mathbb{R}^n)$). Then, the theorem will be true for f.

We thus assume from now on that $f \in L^1(\mathbb{R}^n)$.

<u>Step 1</u>: We show, using (*), that the

$$\lim_{r \downarrow 0} \underbrace{\frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} f(y)\, dy}_{f_r(x)} \text{ exists, for a.e. } x \in \mathbb{R}^n.$$

For this, we define the oscillation of f at $x \in \mathbb{R}^n$:

$$\Omega f(x) := \left| \limsup_{r \downarrow 0} f_r(x) - \liminf_{r \downarrow 0} f_r(x) \right| .$$

Clearly, $\lim_{r \downarrow 0} f_r(x)$ exists $\Longleftrightarrow \Omega f(x) = 0$.

We thus have to show that $\Omega f(x) = 0$ for a.e. $x \in \mathbb{R}^n$,

i.e. that $|\{x \in \mathbb{R}^n : \Omega f(x) > 0\}| = 0$.

Indeed, fix $\delta > 0$. We will show that $|\{x \in \mathbb{R}^n : \Omega f(x) > \delta\}| = 0$.

For this, let $\varepsilon > 0$. By density of $C_c(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$ (and since $f \in L^1(\mathbb{R}^n)$),

$\exists\, g \in C_c(\mathbb{R}^n)$ s.t. $f = g + h$, for some $h \in L^1(\mathbb{R}^n)$ with $\|h\|_1 < \varepsilon$.

Then, some interesting things hold:

- $\Omega f(x) = \Omega h(x)$, $\forall x \in \mathbb{R}^n$.

  The reason is that the Lebesgue differentiation theorem holds for g, as g is continuous (exercises). More precisely:

$$\limsup_{r\downarrow 0} f_r(x) = \limsup_{r\downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy =$$

$$= \limsup_{r\downarrow 0} \left(\frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} g(y)\,dy + \frac{1}{|B(x,r)|} \int_{B(x,r)} h(y)\,dy \right) =$$

$$\downarrow$$

$$\lim_{r\downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} g(y)\,dy$$

exists

$$= \lim_{r\downarrow 0} \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} g(y)\,dy + \limsup_{r\downarrow 0} \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} h(y)\,dy =$$

$$\downarrow$$

the first limit is $g(x)$

$$= g(x) + \limsup_{r\downarrow 0} h_r(x).$$

Similarly, $\liminf_{r\downarrow 0} f_r(x) = g(x) + \liminf_{r\downarrow 0} h_r(x)$.

Therefore, $\Omega f(x) = \left| \cancel{g(x)} + \limsup_{r \downarrow 0} h_r(x) - \cancel{g(x)} - \liminf_{r \downarrow 0} h_r(x) \right| = \Omega h(x), \ \forall x \in \mathbb{R}^n$.

Thus, $\{x \in \mathbb{R}^n : \Omega f(x) > \delta\} = \{x \in \mathbb{R}^n : \Omega h(x) > \delta\}$.

- $\Omega h(x) \leq 2 M h(x), \ \forall x \in \mathbb{R}^n$.

Indeed, $\Omega h(x) = \left| \limsup_{r \downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} h(y)\, dy - \liminf_{r \downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} h(y)\, dy \right| \leq$

$$\leq \left| \limsup_{r \downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} h(y)\, dy \right| + \left| \liminf_{r \downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} h(y)\, dy \right| \leq$$

$$\leq \limsup_{r \downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |h(y)|\, dy + \liminf_{r \downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |h(y)|\, dy \leq$$

$$\leq 2 \cdot \sup_{r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |h(y)|\, dy = 2 M h(x).$$

(Note that for this we used only the definition of Mh, not that $h \in L^1(\mathbb{R}^n)$ or that $\|h\|_1 < \varepsilon$).

- Since $\Omega h(x) \leq 2\, Mh(x) \ \forall x \in \mathbb{R}^n$, we have that

  if $\Omega h(x) > \delta$, then $Mh(x) > \frac{\delta}{2}$,

  so $\{x \in \mathbb{R}^n : \Omega h(x) > \delta\} \subseteq \{x \in \mathbb{R}^n : Mh(x) > \frac{\delta}{2}\}$,

  so $|\{x \in \mathbb{R}^n : \Omega h(x) > \delta\}| \leq |\{x \in \mathbb{R}^n : Mh(x) > \frac{\delta}{2}\}| \leq$

  $$\underset{\text{by } (*)}{\leq} \frac{C_n \cdot \|h\|_1}{\frac{\delta}{2}} \leq \left(\frac{2C_n}{\delta}\right) \cdot \varepsilon .$$

  Since ε was arbitrary, we have that, for this fixed δ,

$$|\{x\in\mathbb{R}^n : \Omega h(x) > \delta\}| = 0.$$

So, $$|\underbrace{\{x\in\mathbb{R}^n : \Omega f(x) > \delta\}}_{\substack{\parallel \\ \{x\in\mathbb{R}^n : \Omega h(x) > \delta\}}}| = 0.$$

Since $\delta>0$ was arbitrary, it follows that

$$|\{x\in\mathbb{R}^n : \Omega f(x) > 0\}| = 0,$$

so $\qquad \Omega f(x) = 0$ for a.e. $x\in\mathbb{R}^n$.

So, the proof of Step 1 is complete.

Lecture 12 (6/11/2014)

Sketch of the proof of Lebesgue differentiation theorem (continued):

Step 2: We will prove that, for any $f \in L^1(\mathbb{R}^n)$, there exists a sequence $(r_n)_{n\in\mathbb{N}}$ of radii, with $r_n \downarrow 0$, s.t.

$$\lim_{r_n \downarrow 0} \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} f(y)\,dy = f(x), \quad \text{for a.e. } x \in \mathbb{R}^n.$$

This will be done using properties of convolutions; it is elementary compared to Step 1.

It is clear that Step 1 and Step 2 combined complete the proof.

We now proceed to the proof of a covering lemma, which will prove very useful in our study of the maximal function.

Covering lemma: Let $E \subseteq \mathbb{R}^n$ be measurable. Let $\mathcal{B}$ be a family of balls in $\mathbb{R}^n$, s.t.

$E \subseteq \bigcup_{B \in \mathcal{B}} B$, and $\sup\{\text{radii of all balls in } \mathcal{B}\} < +\infty$. Then:

$\exists\ B_1, B_2, \dots, B_k, \dots$ in $\mathcal{B}$ (perhaps finitely many),

that are pairwise disjoint and

$$\sum_k |B_k| \geq C_n \cdot |E|,$$

for some constant C_n depending only on n $\left(C_n = \frac{1}{5^n}\right.$ would do$\left.\right)$.

Proof: We choose $B_1, B_2, \dots, B_k, \dots$ inductively. The idea is that the ball picked at each step of the inductive process should be disjoint to the balls picked before that step, and as large as possible. More particularly:

Step 1: We pick B_1 to be a ball in $\mathcal{B}$, s.t.

$$(\text{radius of } B_1) \geq \frac{1}{2} \sup \left\{ \text{radii of all balls in } \mathcal{B} \right\}.$$

Step 2: We pick B_2 in $\mathcal{B} \setminus \{B_1\}$, disjoint to B_1, s.t.

$$(\text{radius of } B_2) \geq \frac{1}{2} \sup \left\{ \text{radii of all balls in } \mathcal{B} \setminus \{B_1\} \text{ disjoint to } B_1 \right\}.$$

$\vdots$

Step j: We pick B_j in $\mathcal{B} \setminus \{B_1, B_2, \dots, B_{j-1}\}$, disjoint to $B_1, B_2, \dots, B_{j-1}$, s.t.

$$(\text{radius of } B_j) \geq \frac{1}{2} \sup \left\{ \text{radii of all balls in } \mathcal{B} \setminus \{B_1, \dots, B_{j-1}\} \text{ disjoint to } B_1, \dots, B_{j-1} \right\}.$$

$\vdots$

The process stops right before some step j if we have no options for a ball B_j, i.e. if each ball in $\mathcal{B} \setminus \{B_1, \dots, B_{j-1}\}$ intersects some ball in $\{B_1, \dots, B_{j-1}\}$. In that case, we end up with a finite family of balls. Otherwise, we end up with an infinite sequence of balls.

Case 1: $\sum_k |B_k| = +\infty$. In this case, the proof of the lemma is complete.

Case 2: $\sum_k |B_k| < +\infty$. In this case, we show that

$$E \subseteq \bigcup_k B_k^* ,$$

where, for each k, B_k^* is the ball with the same centre as B_k, and radius $5 \cdot (\text{radius of } B_k)$.

To that end, we will show that $B \subseteq \bigcup_k B_k^*$, $\forall B \in \mathcal{B} \setminus \{B_1, \dots, B_k, \dots\}$.

Indeed, let B be such a ball.

- If $\{B_1, \dots, B_k, \dots\}$ is a finite family of balls, i.e. $\{B_1, \dots, B_k, \dots\} = \{B_1, \dots, B_J\}$, for some $J \in \mathbb{N}$:

the fact that our process stopped right before the $(J+1)$-th step means that there are no balls in $\mathcal{B}\setminus\{B_1,\dots,B_J\}$ disjoint to all of the $B_1,\dots,B_J$.

In particular, B is not disjoint to all of the $B_1,\dots,B_J$,

i.e. $\exists\, j\in\{1,\dots,J\}$ s.t. $B\cap B_j\neq\emptyset$.

- If $\{B_1,\dots,B_k,\dots\}$ is an infinite family of balls, then

$\sum_k |B_k| < +\infty \implies \exists\, k_0\in\mathbb{N} : \frac{1}{2^n}|B| \gneq |B_{k_0}|$, i.e.

$$(\text{radius of } B) \gneq 2\cdot(\text{radius of } B_{k_0}).$$

Now, $(\text{radius of } B_{k_0}) \geq \frac{1}{2}\sup\left\{\text{radii of all balls in } \mathcal{B}\setminus\{B_1,\dots,B_{k_0-1}\} \text{ disjoint to } B_1,\dots,B_{k_0-1}\right\}$,

so $(\text{radius of } B) \gneq \sup\left\{\text{radii of all balls in } \mathcal{B}\setminus\{B_1,\dots,B_{k_0-1}\} \text{ disjoint to } B_1,\dots,B_{k_0-1}\right\}$, so B cannot be disjoint to all of the $B_1, \dots, B_{k_0-1}$,

i.e. $\exists\, j \in \{1, \dots, k_0-1\}$ s.t. $B \cap B_j \neq \emptyset$.

By the bullet points above, we have that $\exists\, j \in \mathbb{N}$ s.t. $B \cap B_j \neq \emptyset$. We consider the smallest j with that property. Then, B is disjoint to $B_1, \dots, B_{j-1}$, thus, since

$$(\text{radius of } B_j) \geq \frac{1}{2} \sup \left\{ \text{radii of all balls in } \mathcal{B} \setminus \{B_1, \dots, B_{j-1}\} \text{ disjoint to } B_1, \dots, B_{j-1} \right\},$$

it follows that

$$(\text{radius of } B_j) \geq \frac{1}{2} (\text{radius of } B) \Longrightarrow (\text{radius of } B) \leq 2 \cdot (\text{radius of } B_j).$$

And since

$$B \cap B_j \neq \emptyset,$$

we have that $B_j^* \supseteq B$.

Therefore, for any $B \in \mathcal{B} \setminus \{B_1, \dots, B_k, \dots\}$, we have that $B \subseteq \bigcup_k B_k^*$.

And clearly $B_i \subseteq \bigcup_k B_k^*$, for all $i \in \{1, \dots, k, \dots\}$.

So, $\bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_k B_k^*$, so $E \subseteq \bigcup_k B_k^* \Longrightarrow$

$$\Longrightarrow |E| \leq \left| \bigcup_k B_k^* \right| \underset{\substack{\text{countable}\\ \text{subadditivity}\\ \text{of measures}}}{\leq} \sum_k \underbrace{|B_k^*|}_{} = 5^n \cdot \sum_k |B_k| ,$$

$$|B_k^*| = C_n \cdot (\text{radius of } B_k^*)^n = C_n \cdot 5^n \cdot (\text{radius of } B_k)^n = 5^n \cdot |B_k|$$

so $\sum_k |B_k| \geq \frac{1}{5^n} \cdot |E|$.

The proof is complete.

Note that we have shown that $\bigcup_k B_k$ contains a large proportion of E (as the balls $B_1, \dots, B_k, \dots$ are disjoint, and thus $\left|\bigcup_k B_k\right| = \sum_k |B_k|$).

However, we did that using properties of the volume of a ball, and eventually we don't know which points of E are inside $\bigcup_k B_k$.

Lecture 13 (11/11/2014)

Theorem 13.1 (on the Hardy – Littlewood maximal function):

(a) $\forall\ 1 \leq p \leq +\infty$, $\forall f \in L^p(\mathbb{R}^n)$, Mf is finite a.e. (i.e. $|\{x \in \mathbb{R}^n : Mf = +\infty\}| = 0$).

(b) There exists a constant C_n, depending only on n, s.t.

$$a \cdot |\{x \in \mathbb{R}^n : Mf(x) > a\}| \leq C_n \cdot \|f\|_1, \ \forall a > 0, \ \forall f \in L^1(\mathbb{R}^n).$$

(c) $\forall\ 1 \lneq p \leq +\infty$, there exists a constant $C_{n,p}$, depending only on n and p, s.t.

$$\|Mf\|_p \leq C_{n,p} \cdot \|f\|_p, \ \forall f \in L^p(\mathbb{R}^n).$$

Proof: For all three parts of the theorem, we need the following claim (which is, essentially, (b)):

<u>Claim 1</u>: Let $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function. Then,

$$\alpha \cdot |\underbrace{\{x\in\mathbb{R}^n : Mf(x) > \alpha\}}_{= E_\alpha}| \leq 5^n \cdot \int_{\mathbb{R}^n} |f|, \ \forall \alpha > 0.$$

Indeed: By the inner regularity of the Lebesgue measure,

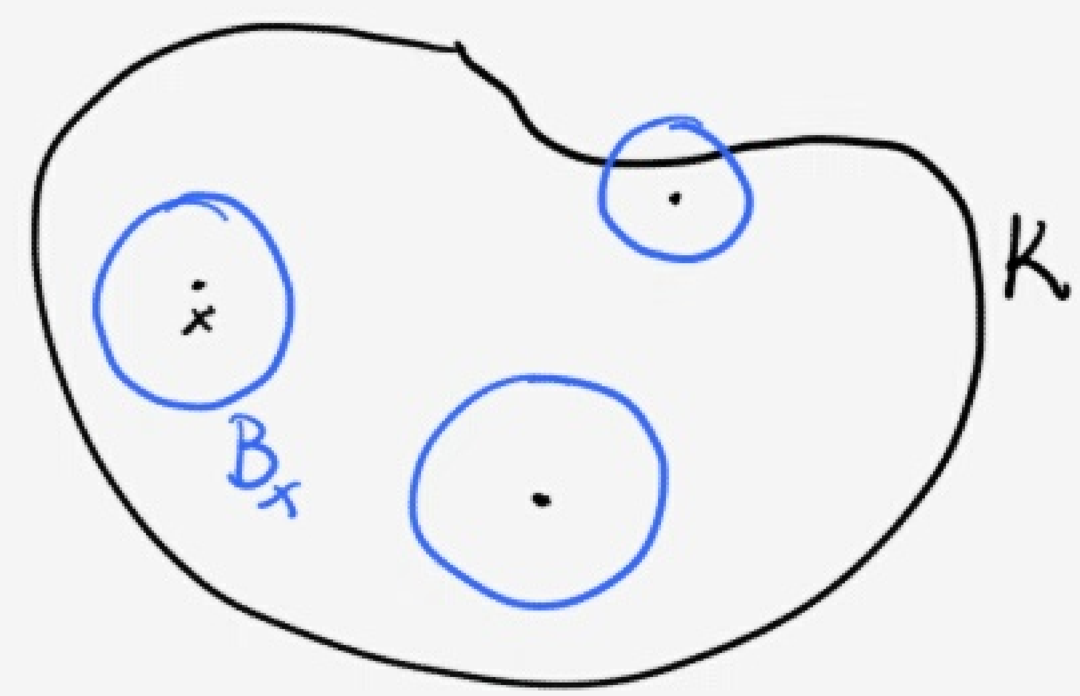
$$|E_\alpha| = \sup\{|K| : K \text{ compact subset of } E_\alpha\}.$$

So, it suffices to show that $\alpha \cdot |K| \leq 5^n \cdot \int_{\mathbb{R}^n} |f|$, $\forall \alpha > 0$, $\forall K$ compact $\subseteq E_\alpha$.

Fix $\alpha > 0$, $K \subseteq E_\alpha$ compact.

$\forall x \in K$, $Mf(x) > \alpha$, so there exists ball B_x, centered at x, s.t.

$$\frac{1}{|B_x|} \cdot \int_{B_x} |f| > \alpha \iff \alpha \cdot |B_x| < \int_{B_x} |f|. \quad (*)$$



In particular, $K \subseteq \bigcup_{x \in K} B_x$, and since K is a compact set,

$\exists\, x_1, \dots, x_m \in K$, s.t. $K \subseteq \bigcup_{k=1}^{m} B_{x_k}$.

Now, $\sup\{\text{radii of } B_{x_1}, \dots, B_{x_m}\} < +\infty$, so, by the Covering Lemma of the previous lecture, there exist pairwise disjoint balls

$B_1, \dots, B_J$ in $\{B_{x_1}, \dots, B_{x_m}\}$, s.t.

$$\sum_{k=1}^{J} |B_k| \geq \frac{1}{5^n} \cdot |K|.$$

So, $a \cdot |K| \leq a \cdot 5^n \cdot \sum_{k=1}^{J} |B_k| = 5^n \cdot \sum_{k=1}^{J} a \cdot |B_k| \overset{\text{by } (*)}{\leq} 5^n \cdot \sum_{k=1}^{J} \int_{B_k} |f| \overset{B_1,\dots,B_J \text{ pairwise disjoint}}{=\!=}$

$= 5^n \int_{\bigcup_{k=1}^{J} B_k} |f| \leq 5^n \cdot \int_{\mathbb{R}^n} |f|$. The proof of Claim 1 is complete.

Claim 1 implies the following:

- (b) holds. Indeed, if $f \in L^1(\mathbb{R}^n)$, Claim 1 implies that

$$\alpha \cdot \left|\{x \in \mathbb{R}^n : Mf(x) > \alpha\}\right| \leq C_n \cdot \|f\|_1, \quad \forall \alpha > 0,$$

for $C_n = 5^n$, a constant that depends only on n.

- (a) holds for $p=1$. Indeed, for any $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ measurable,

$$\{x \in \mathbb{R}^n : Mf(x) = +\infty\} \subseteq \{x \in \mathbb{R}^n : Mf(x) > k\}, \quad \forall k \in \mathbb{N}.$$

So, $\quad \left|\{x \in \mathbb{R}^n : Mf(x) = +\infty\}\right| \leq \left|\{x \in \mathbb{R}^n : Mf(x) > k\}\right|, \ \forall k \in \mathbb{N}.$

So, by (b): $\quad \forall k \in \mathbb{N}, \ \left|\{x \in \mathbb{R}^n : Mf(x) = +\infty\}\right| \leq \dfrac{C_n \cdot \|f\|_1}{k} \xrightarrow[k \to +\infty]{} 0,$

thus $\left|\{x \in \mathbb{R}^n : Mf(x) = +\infty\}\right| = 0.$

- <u>Claim 2</u>: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ measurable.

Then, $\alpha \cdot \left|\{x\in\mathbb{R}^n : Mf(x) > \alpha\}\right| \leq 2\cdot 5^n \cdot \displaystyle\int_{\{x\in\mathbb{R}^n : |f(x)| > \frac{\alpha}{2}\}} |f(y)|\,dy$.

Indeed, note that $\displaystyle\int_{\{x\in\mathbb{R}^n : |f(x)| > \frac{\alpha}{2}\}} |f(y)|\,dy = \int_{\mathbb{R}^n} |f_1|$, where

$$f_1 = f \cdot \chi_{\{x\in\mathbb{R}^n : |f(x)| > \frac{\alpha}{2}\}}, \text{ i.e.}$$

$$f_1(x) = \begin{cases} f(x), & \text{if } |f(x)| > \frac{\alpha}{2} \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that

$|f(x)| \leq |f_1(x)| + \frac{\alpha}{2}$, $\forall x\in\mathbb{R}^n$ (if x is s.t. $|f(x)| \leq \frac{\alpha}{2}$, then $|f(x)| \leq |f_1(x)| + \frac{\alpha}{2}$, while if x is s.t. $|f(x)| > \frac{\alpha}{2}$, then $f(x) = f_1(x)$, so $|f(x)| \leq |f_1(x)| + \frac{\alpha}{2}$).

Therefore, $Mf(x) \leq Mf_1(x) + \frac{a}{2}$, $\forall x \in \mathbb{R}^n$.

$$\left(\forall x \in \mathbb{R}^n,\ \forall r > 0,\ \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} |f(y)|\,dy \leq \right.$$

$$\leq \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} \left[|f_1(y)| + \frac{a}{2} \right] dy = \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} |f_1(y)|\,dy + \frac{a}{2},$$

$$\left. \text{so} \quad Mf(x) \leq Mf_1(x) + \frac{a}{2} \right).$$

So, if $Mf(x) > a$, then $Mf_1(x) > \frac{a}{2} \Longrightarrow$

$$\Longrightarrow \{x \in \mathbb{R}^n : Mf(x) > a\} \subseteq \{x \in \mathbb{R}^n : Mf_1(x) > \frac{a}{2}\}$$

$$\Longrightarrow |\{x \in \mathbb{R}^n : Mf(x) > a\}| \leq |\{x \in \mathbb{R}^n : Mf_1(x) > \frac{a}{2}\}| \underset{\text{by Claim 1}}{\leq} \frac{5^n \cdot \int_{\mathbb{R}^n} |f_1|}{\frac{a}{2}} =$$

$$= \frac{1}{a} \cdot (2 \cdot 5^n) \cdot \int_{\mathbb{R}^n} |f_1| = \frac{1}{a} \cdot (2 \cdot 5^n) \cdot \int_{\{x \in \mathbb{R}^n : |f(x)| > \frac{a}{2}\}} |f(y)| \, dy ,$$

so $$a \cdot \left| \{x \in \mathbb{R}^n : Mf(x) > a\} \right| \leq 2 \cdot 5^n \cdot \int_{\{x \in \mathbb{R}^n : |f(x)| > \frac{a}{2}\}} |f(y)| \, dy .$$

Claim 2 will be used to prove (c), which will, in turn, imply (a) for $1 \lneq p \leq +\infty$.

Lecture 14 (14/11/2014)

In this lecture we will complete the proof of Theorem 13.1, using Claim 2: that

$$|\{x\in\mathbb{R}^n : Mf(x)>\alpha\}| \leq \frac{C_n}{\alpha}\cdot\int_{\{x\in\mathbb{R}^n : |f(x)|>\frac{\alpha}{2}\}} |f| \quad , \forall \alpha>0, \ \forall f:\mathbb{R}^n\to\mathbb{R} \text{ measurable.}$$

Proof of Theorem 13.1 (continued):

We first show (c), $\forall\ 1\lneq p\leq+\infty$.

For $p=+\infty$: $\|Mf\|_\infty = \|f\|_\infty$, $\forall f\in L^\infty(\mathbb{R}^n)$ (exercises, simple).

For $1\lneq p\lneq+\infty$: Let $f\in L^p(\mathbb{R}^n)$.

We define $\lambda(\alpha):=|\{x\in\mathbb{R}^n : Mf(x)>\alpha\}|$, $\forall \alpha>0$.

Then, $$\int_{\mathbb{R}^n}(Mf)^p = -\int_0^{+\infty}\alpha^p\, d\lambda(\alpha).$$

This is obvious by the process of approximating the Lebesgue integral via simple functions. More precisely,

$$\int_{\mathbb{R}^n} (Mf)^p = \sup_{\substack{a_1<a_2<\ldots<a_m \\ \text{in } (0,+\infty), \\ m\in\mathbb{N}}} \Big\{ a_2^p \cdot |\{x : a_1 < Mf(x) \leq a_2\}| + a_3^p \cdot |\{x : a_2 < Mf(x) \leq a_3\}| + \ldots + a_m^p \cdot |\{x : a_{m-1} < Mf(x) \leq a_m\}| \Big\} =$$

$$= \sup_{\substack{a_1<a_2<\ldots<a_m \\ \text{in } (0,+\infty), \\ m\in\mathbb{N}}} \Big\{ a_2^p \cdot \big(\lambda(a_1) - \lambda(a_2)\big) + a_3^p \cdot \big(\lambda(a_2) - \lambda(a_3)\big) + \ldots + a_m^p \cdot \big(\lambda(a_{m-1}) - \lambda(a_m)\big) \Big\} =$$

$$= - \sup_{\substack{a_1<a_2<\ldots<a_m \\ \text{in } (0,+\infty), \\ m\in\mathbb{N}}} \cdot \Big\{ a_2^p \cdot \big(\lambda(a_2) - \lambda(a_1)\big) + a_3^p \cdot \big(\lambda(a_3) - \lambda(a_2)\big) + \ldots + a_m^p \cdot \big(\lambda(a_m) - \lambda(a_{m-1})\big) \Big\} = - \int_0^{+\infty} a^p \, d\lambda(a).$$

So: $$\int_{\mathbb{R}^n} (Mf)^p = -\int_0^{+\infty} a^p \, d\lambda(a) \underset{d\lambda(a) = \lambda'(a)\,da}{=} -\int_0^{+\infty} a^p \lambda'(a)\, da \underset{\substack{\text{integration}\\ \text{by parts}}}{=} -\left[a^p \lambda(a)\right]_0^{+\infty} + \int_0^{+\infty} (a^p)' \lambda(a)\, da.$$

Now, $-\left[a^p \lambda(a)\right]_0^{+\infty} = 0$. Indeed:

$$-\left[a^p \lambda(a)\right]_0^{+\infty} = \lim_{\varepsilon \to 0} \varepsilon^p \lambda(\varepsilon) - \lim_{M \to +\infty} M^p \lambda(M). \text{ And:}$$

$$M^p \lambda(M) = M^p \cdot \left|\{x \in \mathbb{R}^n : Mf(x) > M\}\right| \underset{\text{Claim 2}}{\leq} C_n \cdot M^{p-1} \cdot \int_{\{x \in \mathbb{R}^n : |f(x)| > \frac{M}{2}\}} |f| =$$

$$= C_n \cdot 2^{p-1} \cdot \int_{\{x \in \mathbb{R}^n : |f(x)| > \frac{M}{2}\}} \left(\frac{M}{2}\right)^{p-1} \cdot |f| \leq C_n \cdot 2^{p-1} \cdot \int_{\{x \in \mathbb{R}^n : |f(x)| > \frac{M}{2}\}} |f|^p \xrightarrow[M \to +\infty]{} 0, \text{ by}$$

the Dominated Convergence Theorem,
as $f \in L^p(\mathbb{R}^n)$,

while $\varepsilon^p \lambda(\varepsilon) = \varepsilon^p \cdot \left| \{ x \in \mathbb{R}^n : Mf(x) > \varepsilon \} \right| \underset{\text{Claim 2}}{\leq} C_n \cdot \varepsilon^{p-1} \cdot \int_{\{x \in \mathbb{R}^n : |f(x)| > \frac{\varepsilon}{2}\}} |f| =$

$$= C_n \cdot \varepsilon^{p-1} \cdot \int_{\{x \in \mathbb{R}^n : \frac{\varepsilon}{2} < |f(x)| \leq \sqrt{\varepsilon}\}} |f| + C_n \cdot \varepsilon^{p-1} \cdot \int_{\{x \in \mathbb{R}^n : |f(x)| > \sqrt{\varepsilon}\}} |f|$$

$$= C_n 2^{p-1} \cdot \int_{\{x \in \mathbb{R}^n : \frac{\varepsilon}{2} < |f(x)| \leq \sqrt{\varepsilon}\}} \left(\frac{\varepsilon}{2}\right)^{p-1} |f| \leq$$

$$\leq C_n \cdot 2^{p-1} \cdot \int_{\{x \in \mathbb{R}^n : |f(x)| \leq \sqrt{\varepsilon}\}} |f|^p \xrightarrow[\varepsilon \to 0]{} 0$$

by the Dominated Convergence theorem.

$$= C_n \cdot \varepsilon^{p-1} \cdot \frac{1}{\sqrt{\varepsilon}^{\,p-1}} \cdot \int_{\{x \in \mathbb{R}^n : |f(x)| > \sqrt{\varepsilon}\}} \sqrt{\varepsilon}^{\,p-1} \cdot |f| \leq$$

$$\leq C_n \cdot \varepsilon^{\frac{p-1}{2}} \cdot \int_{\{x \in \mathbb{R}^n : |f(x)| > \sqrt{\varepsilon}\}} |f|^p \leq$$

$$\leq C_n \cdot \varepsilon^{\frac{p-1}{2}} \cdot \int_{\mathbb{R}^n} |f|^p \xrightarrow[\varepsilon \to 0]{} 0.$$

Thus, $$\int_{\mathbb{R}^n} (Mf)^p = \int_0^{+\infty} (a^p)' \lambda(a)\, da = p \cdot \int_0^{+\infty} a^{p-1} \cdot |\{x \in \mathbb{R}^n : Mf(x) > a\}|\, da \underset{\text{Claim 2}}{\leq}$$

$$\leq C_n \cdot p \cdot \int_{a=0}^{+\infty} a^{p-1} \cdot \frac{1}{a} \cdot \int_{\{x \in \mathbb{R}^n : |f(x)| > \frac{a}{2}\}} |f(x)|\, dx\, da$$

$$= C_n \cdot p \cdot \int_{a=0}^{+\infty} \int_{\{x \in \mathbb{R}^n : |f(x)| > \frac{a}{2}\}} a^{p-2} \cdot |f(x)|\, dx\, da. \quad (*)$$

By Tonelli's theorem, we can change the order of integration.
Now, for every fixed $a \in (0, +\infty)$, we integrate on the set of $x \in \mathbb{R}^n$ with $|f(x)| > \frac{a}{2}$.
So, when we change variables, for every fixed $x \in \mathbb{R}^n$ we have to integrate on the set of $a \in (0, +\infty)$ with $|f(x)| > \frac{a}{2}$, i.e. $a < 2|f(x)|$. Thus:

$$\int_{\mathbb{R}^n} (Mf)^p \leq \circledast = C_n \cdot p \cdot \int_{x\in\mathbb{R}^n} \int_{a=0}^{2|f(x)|} a^{p-2} \cdot |f(x)| \, da \, dx =$$

$$= C_n \cdot p \cdot \int_{x\in\mathbb{R}^n} |f(x)| \cdot \left[\int_{a=0}^{2|f(x)|} \frac{\left(a^{p-1}\right)'}{p-1} \, da \right] dx =$$

$$= \frac{C_n \cdot p}{p-1} \cdot \int_{x\in\mathbb{R}^n} |f(x)| \cdot \left(2|f(x)|\right)^{p-1} dx =$$

$$= \frac{2^{p-1} C_n \cdot p}{p-1} \cdot \int_{x\in\mathbb{R}^n} |f(x)|^p \, dx = \frac{2^{p-1} C_n \cdot p}{p-1} \cdot \|f\|_p^p < +\infty .$$

So, $Mf \in L^p(\mathbb{R}^n)$, and $\|Mf\|_p \leq \underbrace{\left(\frac{2^{p-1} C_n \cdot p}{p-1} \right)^{1/p}}_{\text{a constant depending only on } n \text{ and } p.} \cdot \|f\|_p .$

The proof of (c) is complete.

It is clear that, for $1 \leq p \leq +\infty$ and $f \in L^p(\mathbb{R}^n)$, the fact that $Mf \in L^p(\mathbb{R}^n)$ implies that Mf is finite a.e., i.e. $|\{x \in \mathbb{R}^n : Mf(x) = +\infty\}| = 0$.

Indeed, for $p = +\infty$ this is true by the definition of $L^\infty(\mathbb{R}^n)$, while for $1 \leq p < +\infty$: $\displaystyle\int_{\mathbb{R}^n} (Mf)^p \geq \int_{\{x \in \mathbb{R}^n : Mf(x) > +\infty\}} Mf = +\infty$ if $|\{x \in \mathbb{R}^n : Mf(x) = +\infty\}| > 0$, a contradiction.

The proof of Theorem 13.1 is complete.

The last observation from the proof of Theorem 13.1 is really Fact 2 from our list of important facts about $L^p(\mathbb{R}^n)$:
Let $1 \leq p \leq +\infty$. If $f \in L^p(\mathbb{R}^n)$, then f is finite a.e.

So, to show that a measurable function $f: \mathbb{R}^n \to \mathbb{R}$ is finite a.e., it suffices to show that $\int_{\mathbb{R}^n} |f|^p < +\infty$ for <u>some</u> $p \in [1, +\infty)$, or that

$$f \in L^\infty(\mathbb{R}^n).$$

The converse is clearly not true.

Fact 6 from list of important facts about $L^p(\mathbb{R}^n)$:

Let K be a subset of $\mathbb{R}^n$, with $\boxed{|K| < +\infty.}$

Then, $\forall\ 1 \leq p < q \leq +\infty$,

$$\|f\|_p \leq C_{n,p,q,K} \cdot \|f\|_q \,, \quad \forall f \in L^q(\mathbb{R}^n) \text{ supported on } K,$$

where $C_{n,p,q,K}$ is a constant depending only on n, p, q, K.

In particular, if $|K| = 1$, then $\|f\|_p \leq \|f\|_q$, $\forall f \in L^q(\mathbb{R}^n)$ supported on K.

Proof: With Hölder's inequality. Indeed:

$$\int_{\mathbb{R}^n} |f|^p = \int_K |f|^p \cdot 1 \overset{(*)}{\leq} \left(\int_K \left(|f|^p\right)^{\frac{q}{p}} \right)^{\frac{p}{q}} \cdot \left(\int_K 1^{\left(\frac{q}{p}\right)'} \right)^{\frac{1}{\left(\frac{q}{p}\right)'}} =$$

Hölder's inequality for $\frac{q}{p}$ (>1), $\left(\frac{q}{p}\right)'$

the conjugate of $\frac{q}{p}$.

$$= \left(\int_K |f|^q \right)^{\frac{p}{q}} \cdot \left(\int_K 1 \right)^{\frac{1}{\left(\frac{q}{p}\right)'}} = \|f\|_q^p \cdot |K|^{\frac{1}{\left(\frac{q}{p}\right)'}}, \text{ so } \|f\|_p \leq \underbrace{|K|^{\frac{1}{p\cdot\left(\frac{q}{p}\right)'}}}_{= C_{n,p,q,K}} \cdot \|f\|_q .$$

Note that the use of Hölder's inequality, e.g. in (*), is a very common trick in harmonic analysis to increase the power of an integrand, i.e. to move from a p-norm to a q-norm, for $q > p$.

Lecture 15 (18/11/2014)

In this lecture, we present fact 5 in our list of facts about $L^p(\mathbb{R}^n)$, which will be used to prove Step 2 in the proof of the Lebesgue Differentiation Theorem. Here is a reminder of the sketch of the proof:

LDT: Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \downarrow 0} \underbrace{\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy}_{= f_r(x)} = f(x).$$

Sketch of proof: It suffices to show this for $f \in L^1(\mathbb{R}^n)$. Then:

Step 1: The $\lim_{r \downarrow 0} f_r(x)$ exists for a.e. $x \in \mathbb{R}^n$.

We completed the proof of this when we proved the "weak L^1" estimate for Mf : $\alpha \cdot \left|\{x\in\mathbb{R}^n : Mf(x) > \alpha\}\right| \leq c_n \cdot \|f\|_1$, $\forall \alpha > 0$.

<u>Step 2</u>: $\lim\limits_{r\downarrow 0} f_r(x) = f(x)$ for a.e. x.

This will be much easier than Step 1. We will use Fact 5, to show that there exists a sequence $(r_n)_{n\in\mathbb{N}}$ of radii, with $r_n \downarrow 0$, s.t. $\lim\limits_{n\to+\infty} f_{r_n}(x) = f(x)$ for a.e. x. The proof will immediately be completed.

<u>Fact 5</u>: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\mathbb{R}^n)$, for some $p\in[1,+\infty]$.

Suppose that $f_n \xrightarrow[\|\cdot\|_p]{} f$, for some $f\in L^p(\mathbb{R}^n)$.

Then, there exists a subsequence $(f_{k_n})_{n\in\mathbb{N}}$, with

$$f_{k_n}(x) \xrightarrow[n\to+\infty]{} f(x) \text{ for a.e. } x\in\mathbb{R}^n.$$

Fact 5 is, in a loose sense, a "converse" of theorems like the Dominated Convergence Theorem and the Monotone Convergence Theorem.

More precisely, the DCT and the MCT ensure that, under certain hypotheses, $f_n(x) \xrightarrow[n\to+\infty]{} f(x)$ for a.e. x (pointwise convergence)

implies

$$\int |f_n - f|^p \xrightarrow[n\to+\infty]{} 0 \quad \text{(convergence in } L^p\text{)}.$$

i.e. $f_n \xrightarrow[\|\cdot\|_p]{} f$

Fact 5 ensures that convergence in L^p implies pointwise convergence (for a subsequence, however).

<u>Proof of Fact 5</u>: We first prove it for $f=0$, $p=1$. The rest of the proof will

be immediate. So:

<u>Claim 1</u>: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(\mathbb{R}^n)$, with $f_n \xrightarrow[\|\cdot\|_1]{} 0$.

Then, there exists a subsequence $(f_{k_n})_{n\in\mathbb{N}}$, with

$$f_{k_n}(x) \xrightarrow[n\to+\infty]{} 0 \quad \text{for a.e. } x.$$

<u>Proof</u>: Since $f_n \xrightarrow[\|\cdot\|_1]{} 0$, i.e. $\|f_n\|_1 \xrightarrow[n\to+\infty]{} 0$, there exists a subsequence $(f_{k_n})_{n\in\mathbb{N}}$, $\|f_{k_n}\|_1 < \frac{1}{2^n}$, $\forall n\in\mathbb{N}$.

- $\left(f_{k_n}(x)\right)_{n\in\mathbb{N}}$ converges, for a.e. x.

For this, we use the following fact:

Let $(X, \|\cdot\|_X)$ be Banach. Then, every absolutely convergent series converges,

i.e.: if $x_1, x_2, \ldots \in X$ s.t. $\sum_{n=1}^{+\infty} \|x_n\|_X < +\infty$, then $\sum_{n=1}^{+\infty} x_n$ converges in X

(i.e., $\exists\ x \in X$ s.t. the partial sums of $\sum_{n=1}^{+\infty} x_n$ converge to x w.r.t. $\|\cdot\|_X$).

Now: $f_{k_n} = f_{k_1} + (f_{k_2} - f_{k_1}) + (f_{k_3} - f_{k_2}) + \ldots + (f_{k_n} - f_{k_{n-1}}) =$

$= f_{k_1} + \sum_{m=2}^{n} (f_{k_m} - f_{k_{m-1}}) = S_n$, the n-th partial sum of the series $f_{k_1} + \sum_{m=2}^{+\infty} (f_{k_m} - f_{k_{m-1}})$.

We want to show that $(f_{k_n}(x))_{n \in \mathbb{N}}$ converges for a.e. x,

i.e. that $(S_n(x))_{n \in \mathbb{N}}$ converges for a.e. x, i.e. that the series $f_{k_1}(x) + \sum_{m=2}^{+\infty} \left(f_{k_m}(x) - f_{k_{m-1}}(x) \right)$ converges for a.e. x.

It suffices to show that $|f_{k_1}(x)| + \sum_{m=2}^{+\infty} |f_{k_m}(x) - f_{k_{m-1}}(x)|$ converges for a.e. x

($(\mathbb{R}, |\cdot|)$ is Banach, so every absolutely convergent (w.r.t. $|\cdot|$) series converges),

i.e. that $|f_{k_1}(x)| + \sum_{m=2}^{+\infty} |f_{k_m}(x) - f_{k_{m-1}}(x)| < +\infty$, for a.e. x.

It suffices to show that $|f_{k_1}| + \sum_{m=2}^{+\infty} |f_{k_m} - f_{k_{m-1}}| \in L^1(\mathbb{R}^n)$

(by Fact 2).

It suffices to show that $\| |f_{k_1}| \|_1 + \sum_{m=2}^{+\infty} \| |f_{k_m} - f_{k_{m-1}}| \|_1 < +\infty$

($(L^1(\mathbb{R}^n, \|\cdot\|_1)$ is Banach, so, if a series is absolutely convergent w.r.t. $\|\cdot\|_1$, then it converges in $L^1(\mathbb{R}^n)$.)

And indeed, $\| |f_{k_1}| \|_1 + \sum_{m=2}^{+\infty} \| |f_{k_m} - f_{k_{m-1}}| \|_1 =$

$$= \| f_{k_1} \|_1 + \sum_{m=2}^{+\infty} \underbrace{\| f_{k_m} - f_{k_{m-1}} \|_1}_{\leq \| f_{k_m} \|_1 + \| f_{k_{m-1}} \|_1} <$$

by the way $(f_{k_n})_{n\in\mathbb{N}}$ was chosen

$$< \frac{1}{2^1} + \sum_{m=2}^{+\infty} \left(\frac{1}{2^m} + \frac{1}{2^{m-1}} \right) <$$

$$< \frac{1}{2} + \sum_{m=2}^{+\infty} 2 \cdot \frac{1}{2^{m-1}} = \frac{1}{2} + \sum_{m=2}^{+\infty} \frac{1}{2^{m-2}} < +\infty .$$

So, $\left(f_{k_n}(x)\right)_{n\in\mathbb{N}}$ converges for a.e. $x\in\mathbb{R}^n$.

- $f_{k_n}(x) \xrightarrow[n\to+\infty]{} 0$ for a.e. $x\in\mathbb{R}^n$.

  Indeed, we have shown that $\left(f_{k_n}(x)\right)_{n\in\mathbb{N}}$ converges for a.e. x, so

  $\exists\ h:\mathbb{R}^n\to\mathbb{R}$ measurable, with $f_{k_n}(x) \xrightarrow[n\to+\infty]{} h(x)$ for a.e. x

  (h is the pointwise limit of the sequence $(f_{k_n})_{n\in\mathbb{N}}$).

  We now show that $h\equiv 0$ almost everywhere:

  $|f_{k_n}(x)| \xrightarrow[n\to+\infty]{} |h(x)|$ for a.e. x,

  and $|f_{k_n}(x)| \leq |f_{k_1}(x)| + \sum_{m=2}^{+\infty} |f_{k_m}(x) - f_{k_{m-1}}(x)| \quad \forall x\in\mathbb{R}^n$, where $|f_{k_1}| + \sum_{m=2}^{+\infty} |f_{k_m} - f_{k_{m-1}}| \in L^1(\mathbb{R}^n)$ (we just showed this).

So, by the DCT, $\underbrace{\int_{\mathbb{R}^n} |f_{k_n}|}_{\|f_{k_n}\|_1} \longrightarrow \int_{\mathbb{R}^n} |h|$.

But $\|f_{k_n}\|_1 \xrightarrow[n\to+\infty]{} 0$. So, $\int_{\mathbb{R}^n} |h| = 0$.

So, $h \equiv 0$ a.e.

Thus, $f_{k_n}(x) \xrightarrow[n\to+\infty]{} 0$ for a.e. x.

Lecture 16 (20/11/2014).

In this lecture we will complete the proof of Fact 5, and then use Fact 5 to complete the second (and final) step of the Lebesgue Differentiation Theorem.

Finally, we will give an informal introduction to the Hausdorff dimension.

Fact 5 (Convergence in L^p implies pointwise convergence for a subsequence): Let $1 \le p \le +\infty$.

If $f_n \xrightarrow[\|\cdot\|_p]{} f$, then there exists $(f_{k_n})_{n\in\mathbb{N}}$ subsequence of $(f_n)_{n\in\mathbb{N}}$, with

$$\Updownarrow$$

$$\|f_n - f\|_p \xrightarrow[n\to+\infty]{} 0$$

$$f_{k_n}(x) \xrightarrow[n\to+\infty]{} f(x) \text{ for a.e. } x\in\mathbb{R}^n.$$

Proof: For $p=+\infty$, the above is obvious: uniform convergence implies pointwise convergence a.e. So, we may assume that $1 \le p < +\infty$.

We have proved

Claim 1: If $f_n \xrightarrow[\|\cdot\|_1]{} 0$ $\Leftrightarrow \int |f_n| \longrightarrow 0$, then $\exists\ (f_{k_n})_{n\in\mathbb{N}}$ subsequence of $(f_n)_{n\in\mathbb{N}}$, with

$$f_{k_n}(x) \xrightarrow[n\to+\infty]{} 0 \text{ for a.e. } x.$$

(i.e., Fact 5 holds for $p=1$, $f\equiv 0$).

Claim 2: If $f_n \xrightarrow[\|\cdot\|_p]{} 0$ $\Updownarrow$ $\int |f_n|^p \longrightarrow 0$, then $\exists\ (f_{k_n})_{n\in\mathbb{N}}$ subsequence of $(f_n)_{n\in\mathbb{N}}$, with

$$f_{k_n}(x) \xrightarrow[n\to+\infty]{} 0 \text{ for a.e. } x.$$

(i.e., Fact 5 holds for all $1\leq p<+\infty$, $f\equiv 0$).

Proof: $f_n \xrightarrow[\|\cdot\|_p]{} 0 \Leftrightarrow \int |f_n|^p \xrightarrow[n\to+\infty]{} 0 \underset{\text{Claim 1}}{\Longrightarrow} \exists\ \left(|f_{k_n}|^p\right)_{n\in\mathbb{N}}$ subsequence of $(|f_n|^p)_{n\in\mathbb{N}}$, with $|f_{k_n}(x)|^p \xrightarrow[n\to+\infty]{} 0$ for a.e. x,

i.e $f_{k_n}(x) \xrightarrow[n\to+\infty]{} 0$ for a.e. x.

<u>Claim 3</u>: fact 5 holds.

<u>Proof</u>: $f_n \xrightarrow[\|\cdot\|_p]{} f \iff \underbrace{\|f_n - f\|_p}_{= g_n} \xrightarrow[n\to+\infty]{} 0 \overset{\text{Claim 2}}{\Longrightarrow} \exists\ (g_{k_n})_{n\in\mathbb{N}}$ subsequence

of $(g_n)_{n\in\mathbb{N}}$, s.t. $g_{k_n}(x) \xrightarrow[n\to+\infty]{} 0$ for a.e. x,

i.e. $f_{k_n}(x) - f(x) \xrightarrow[n\to+\infty]{} 0$ for a.e. x,

i.e. $f_{k_n}(x) \xrightarrow[n\to+\infty]{} f(x)$ for a.e. x.

LDT: Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then,

$$\underbrace{\lim_{r \downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy}_{\substack{\| \\ f_r(x)}} = f(x), \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Proof: We may assume that $f \in L^1(\mathbb{R}^n)$.

Step 1:

The $\lim_{r \downarrow 0} f_r(x)$ exists for a.e. $x \in \mathbb{R}^n$ (proved).

Step 2:

$\lim_{r \downarrow 0} f_r(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$.

Proof: We will show that $f_r \xrightarrow[\|\cdot\|_1]{r \downarrow 0} f$, i.e. $\|f_r - f\|_1 \xrightarrow[r \downarrow 0]{} 0$ (each $f_r : \mathbb{R}^n \to \mathbb{R}$ is measurable).

Then, by Fact 5, there exists a sequence $(r_n)_{n \in \mathbb{N}}$ of radii, with $r_n \downarrow 0$, s.t. $f_{r_n}(x) \xrightarrow[n \to +\infty]{} f(x)$ for a.e. $x \in \mathbb{R}^n$.

In other words,

$$\lim_{r_n \downarrow 0} f_{r_n}(x) = f(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Since $\lim_{r \downarrow 0} f_r(x)$ exists for a.e. $x \in \mathbb{R}^n$ by Step 1, it has to be that

$$\lim_{r \downarrow 0} f_r(x) = f(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

So, it suffices to prove that $\|f_r - f\|_1 \xrightarrow[r \downarrow 0]{} 0$. Indeed:

$$\|f_r - f\|_1 = \int_{x\in\mathbb{R}^n} |f_r(x) - f(x)|\, dx = \int_{x\in\mathbb{R}^n} \left| \frac{1}{|B(x,r)|} \cdot \int_{B(x,r)} f(y)\, dy - f(x) \right| dx =$$

$$f(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(x)\, dy$$

$$= \int_{x\in\mathbb{R}^n} \left| \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy - \frac{1}{|B(x,r)|} \int_{B(x,r)} f(x)\, dy \right| dx =$$

$$|B(x,r)| = C_n r^n$$

$$= \frac{1}{C_n r^n} \cdot \int_{x\in\mathbb{R}^n} \left| \int_{B(x,r)} (f(y) - f(x))\, dy \right| dx \leq$$

$$\leq \frac{1}{c_n r^n} \cdot \int_{x\in\mathbb{R}^n} \underbrace{\int_{B(x,r)} |f(y)-f(x)|\, dy}_{\substack{\text{in here, } x \\ \text{is considered fixed,} \\ \text{so we set } y=x+r\cdot z \\ \text{for } z\in B(0,1)\text{, and} \\ \text{then } dy = r^n dz}}\, dx = \frac{1}{c_n \cancel{r^n}} \cdot \int_{x\in\mathbb{R}^n} \int_{z\in B(0,1)} |f(x+rz)-f(x)|\, \cancel{r^n}\, dz\, dx =$$

$$\underset{\substack{\text{change of} \\ \text{order of inte-} \\ \text{gration}}}{=} \frac{1}{c_n} \cdot \int_{z\in B(0,1)} \left(\int_{x\in\mathbb{R}^n} |f(x+rz)-f(x)|\, dx \right) dz = \frac{1}{c_n} \cdot \int_{z\in B(0,1)} \|f(\cdot+rz)-f(\cdot)\|_1\, dz.$$

Now, $\|f(\cdot+rz)-f(\cdot)\|_1 \xrightarrow{r\downarrow 0} 0$ (exercises),

and $\|f(\cdot+rz)-f(\cdot)\|_1 \leq \|f(\cdot+rz)\|_1 + \|f(\cdot)\|_1 = 2\|f\|_1 \;\; \forall r>0,$

where $\int_{z\in B(0,1)} 2\|f\|_1\, dz = |B(0,1)| \cdot 2\|f\|_1 < +\infty.$

So, by the Dominated Convergence Theorem,

$$\int_{z\in B(0,1)} \| f(\cdot+rz) - f(\cdot)\|_1 \, dz \xrightarrow[r\downarrow 0]{} 0 ,$$

so $\|f_r - f\|_1 \xrightarrow[r\downarrow 0]{} 0$.

The proof is complete.

Hausdorff dimension

A set E may be a subset of $\mathbb{R}^n$, but saying that it is truly "n-dimensional" may make no actual sense. For example, we feel that a line segment should be a 1-dimensional object, no matter whether we see it as a subset of $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, etc. The sphere $S^2 \subseteq \mathbb{R}^3$ locally looks like $\mathbb{R}^2$, so we feel that it should be