

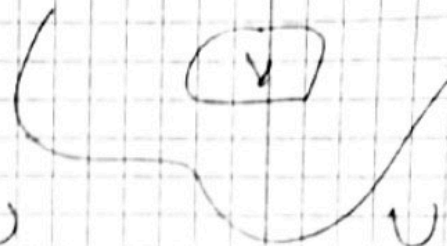
Harnack Inequality

(1)

$$\Delta u \leq 0, \quad x \in U, \quad u \geq 0$$

$$U \subset \mathbb{R}^n$$

(1)



The

Let $V =$ open, bdd, connected, $\bar{V} \subset U$

Then $\exists C, \text{const.}$ depending only on V (not on u !) s.t.

(2)

$$\sup_V u \leq C \inf_V u$$

$\forall$ harmonic u in U , $u \geq 0$

Note 1

$$\forall x, y \in V \Rightarrow$$

$$u(x) \leq \sup_V u \leq C \inf_V u \leq C u(y)$$

(3)

$$\frac{u(x)}{u(y)} \leq C, \quad \text{and} \quad \frac{u(y)}{u(x)} \leq C$$

(values of u uniformly comparable
(all harmonic fns))

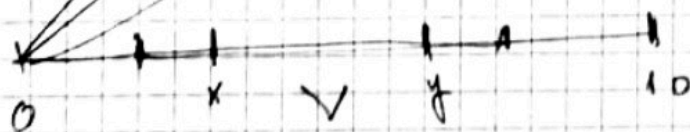
Note 2 ($n=1$)

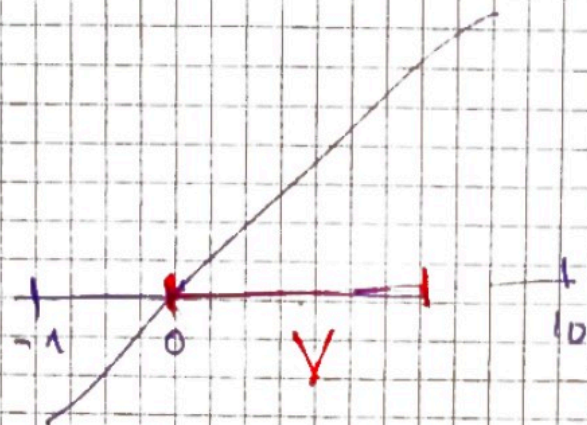
$$U = [0, 10]$$

$$u_1(x) = \lambda x$$

$$\frac{u_1(x)}{u_1(y)} = \frac{x}{y}$$

Harnack verified





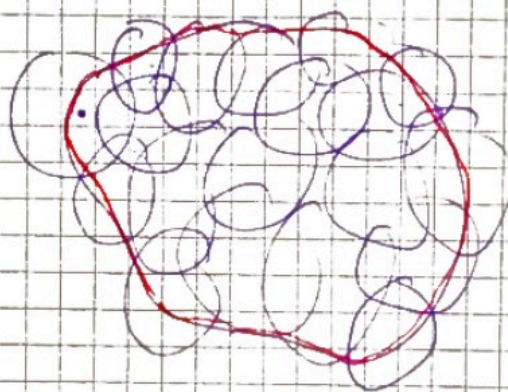
Harnack does not hold.

$$U = [-1, 10]$$

pf

1. Fix $r > 0$. Then by the Borel $\exists N$ integer,

$$\overline{V} \subset \bigcup_{i=1}^N B(x_i, \frac{r}{2}), \quad x_i \in V$$

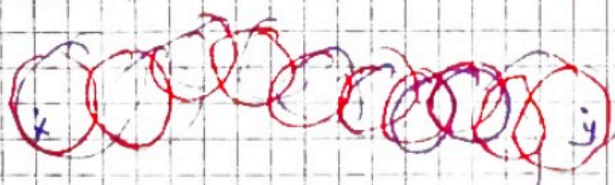


$$N = N(V)$$

Let now $x, y \in V$ and by connectedness

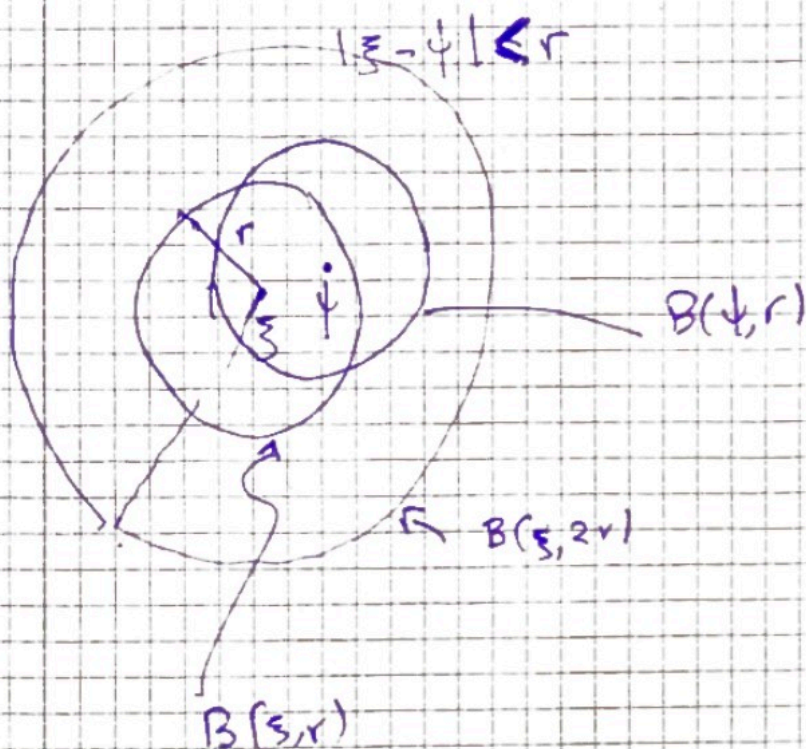
$\exists$ curve γ in V , connecting them.

Let $\{B(x_k, \frac{r}{2})\}$ the subfamily that covers γ and is minimal. Clearly the balls overlap.



(clearly)

(3)

2. Let $\xi, \psi \in V$, with

$$(4) \quad u(\xi) = \frac{1}{|B(\xi, 2r)|} \int_{B(\xi, 2r)} u dz \geq \frac{1}{|B(\xi, 2r)|} \int_{B(\psi, r)} u dz = \frac{1}{\alpha(n)(2r)^n} \int_{B(\psi, r)} u dz = \frac{1}{2^n} u(\psi)$$

$$(B(\psi, r) \subset B(\xi, 2r))$$

 $\Rightarrow$

$$(5) \quad \left\{ \begin{array}{l} 2^n u(\xi) \geq u(\psi) \\ \text{and interchanging } \xi, \psi \\ 2^n u(\psi) \geq u(\xi) \end{array} \right\} \Rightarrow \boxed{2^n u(\psi) \geq u(\xi) \geq \frac{1}{2^n} u(\psi)} \quad (*)$$

3. Coda

$$x \in B(x_i, \frac{1}{2}), \quad y \in B(x_i, \frac{1}{2}), \quad i < j$$

we will apply (*):

$$(6) \quad 2^n u(y) \geq u(x_i), \quad 2^n u(x_i) \geq u(x_{i+1}), \quad \dots, \quad 2^n u(x_j) \geq u(x_i)$$

Hence

$$2^{n(j-i)+2} u(y) \geq u(x)$$

$$n(j-i)+2 \leq n(N-1)+2 \leq n(N+1)$$

$\therefore$

$$2^{n(N+1)} u(y) \geq u(x)$$

Thus since x arbitrary

$$2^{n(N+1)} u(y) \geq \sup_{x \in V} u(x)$$

Next since y is arbitrary

$$2^{n(N+1)} \inf u(y) \geq \sup_{x \in V} u(x),$$

thus This is proved.

□

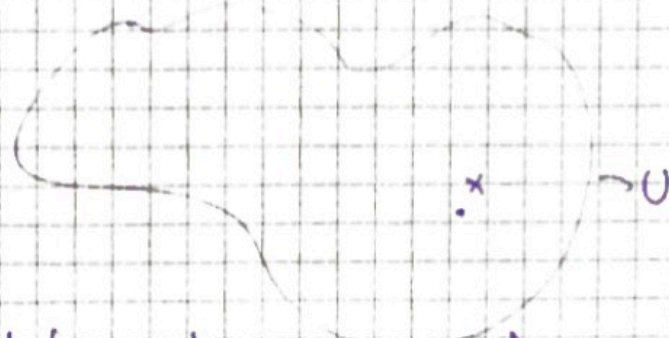
Green's Function

(5)

Potential theory - Electrostatics

Given an open, bdd connected $U \subset \mathbb{R}^n$

place a unit charge at $x \in U$



The potential that this generates is $\Phi(y-x)$

Consider now the following problem:

$$(1) \quad \begin{cases} \Delta_y \phi^x(y) = 0, & y \in U \\ \phi^x(y) = \Phi(y-x) & \text{on } \partial U \end{cases}$$

One is tempted to say + let $\phi^x(y) = \Phi(y-x)$,
but this is not quite true:

$$\Delta_y \Phi(y-x) = 0, \quad y \neq x$$

but $y \rightarrow \Phi(y-x)$ is singular at $y=x$!

Notice that $y \rightarrow \Phi(y-x)$ is nice on ∂U . Hence
we expect that (1) should have a nice solution.

$$\text{Def: } \boxed{G(x,y) := \Phi(y-x) - \phi^x(y)} \quad \text{-- Green's function}$$

Physical Interpretation

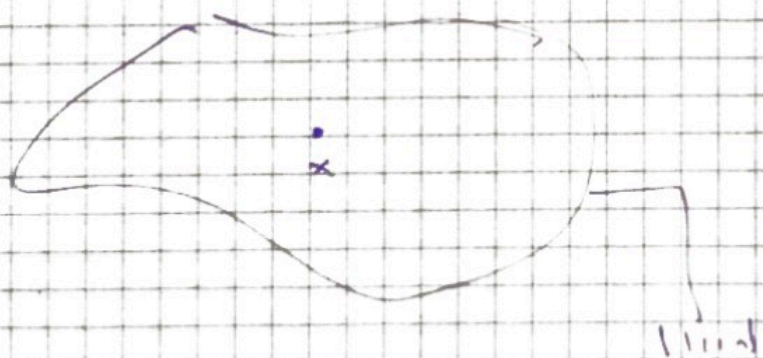
Thus formally

$$\begin{aligned}\Delta_y G(x, y) &= \Delta_y \Phi(y-x) - \Delta_y \phi^x(y) = \Delta_y \Phi(y-x) \\ &= -\delta_x(y), \quad y \in U\end{aligned}$$

For $y \in \partial U$

$$G(x, y) = \Phi(y-x) - \phi^x(y) = 0 \text{ on } \partial U$$

Thus $G(x, y)$ is the potential ^{produced} ~~induced~~ by placing a positive unit charge at x inside ~~the~~ a perfectly conducting metal surface ∂U , grounded at zero potential



The charge at x induces a charge distribution on ∂U , and these two together generate the potential $G(x, y)$ in U , that is zero on ∂U . The surface charge density $\sigma(\xi)$ on ∂U at ξ is

$$\sigma(\xi) = -\epsilon_0 \frac{\partial G(x, \xi)}{\partial n_\xi}$$

Thus G encodes the induced charge distributions^(*) needed to keep the conductor at 0 potential.

□