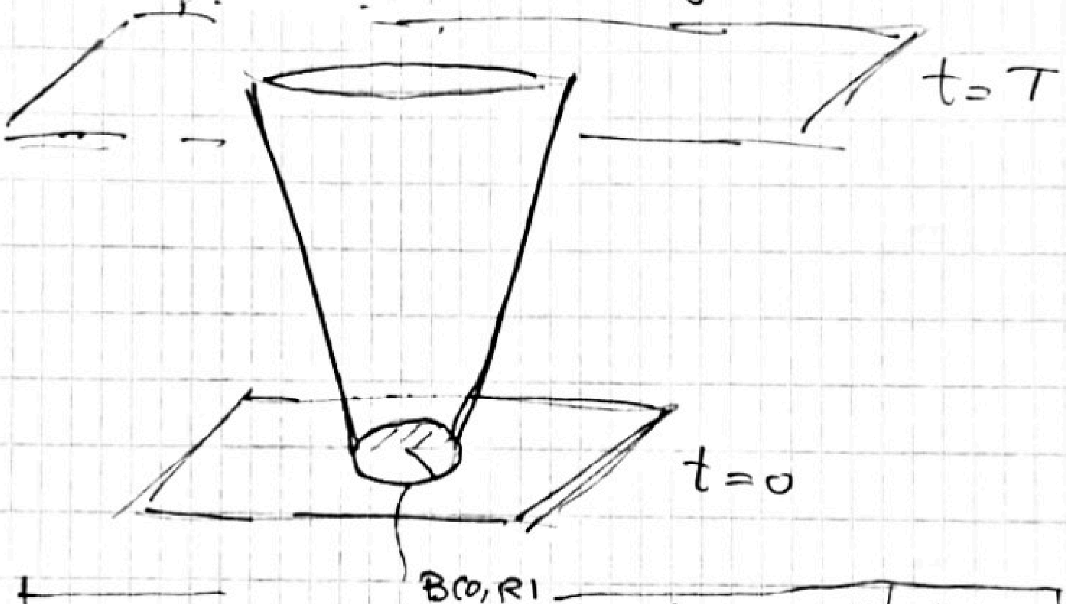
Lecture 21

The Energy Method

(1) $\begin{cases} u_{tt} = \Delta u & , x \in \mathbb{R}^n \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$ $(x,t) \to u(x,t)$, C^2 function.

Assume spt g, spt h in a compact set, $B(0,R)$
We have seen that in dimensions $n=1,2,3$ the domain of influence is as in the figure below.



$\Rightarrow$ $\boxed{x \to u(x,t) \text{ has compact spt } \forall\, t>0.}$ (2)

We will establish in this lecture later that (2) holds in any dimension. So we assume it.

(3) $E(t) = \frac{1}{2} \int_{\mathbb{R}^n} (\underset{\text{kinetic}}{u_t^2} + \underset{\text{potential}}{|\nabla u|^2})\,dx < +\infty$, by (3) is finite.

(4) $\frac{dE(t)}{dt} = \int_{\mathbb{R}^n} (u_t u_{tt} + \nabla u\, \nabla u_t)\,dx = \int_{\mathbb{R}^n} (u_t u_{tt} - \Delta u\, u_t)\,dx$

where we utilized Green's identity, $\left[\int_K (\Delta v \; w)\,dx = -\int_K \nabla v \, \nabla w \, dx + \int_{\partial K} \frac{\partial v}{\partial \nu} w \, dS,\right]$

where the bdry term drops out by (2) [Take K a big ball containing the spt of u].

Continuing (4)

$$= \int_{\mathbb{R}^n} u_t \left[u_{tt} - \Delta u \right] dx = 0.$$

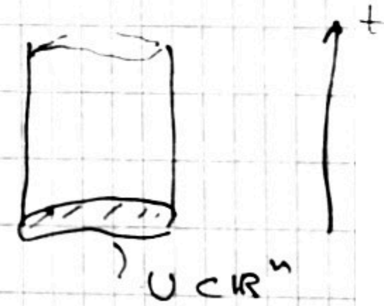
Equation

$\therefore \; t \to E(t)$ constant $\equiv$ Conservation of Energy.

Corollary 1 (Uniqueness) [Corollary of the Energy method]

$$(5) \begin{cases} u_{tt} = \Delta u + f(x,t), \quad x \in U, \quad t \in (0,T) \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases} \text{I.C.'s} \quad u(x,t) = g(x,t), \quad x \in \partial U \times (0,T)$$

Boundary Conditions.



f, g given

Claim: (5) has at most one solution

Pf: Let u, v be slts. Set $w = u - v$. Then

$$(6) \begin{cases} w_{tt} = \Delta w \quad x \in U, \; t \in (0,T) \\ w(x,0) = 0, \quad w_t(x,0) = 0 \\ w = 0 \text{ on } \partial U \times (0,T) \end{cases}$$

$$\frac{d}{dt}\Big[\underbrace{\frac{1}{2}\int_U (w_t^2+|\nabla w|^2)\,dx}_{E(t)}\Big] = \int_U (w_t \cdot w_{tt} + \nabla w\, \nabla w_t)\,dx$$

equation in (6).

$$= \int_U (w_t\, w_{tt} - \Delta w\, w_t)\,dx = 0.$$

where we utilized that $w(x,t)=0$, $x\in\partial U$ implies that $w_t(x,t)=0$, $x\in\partial U$.

$\Rightarrow \quad E(t)=E(0)$,

$E(0)=0$ however $\Rightarrow \quad \frac{1}{2}\int_U (w_t^2+|\nabla w|^2)\,dx = 0$

$\Rightarrow \quad w_t\equiv 0, \quad \nabla w \equiv 0$

$\Rightarrow \quad w(x,t)$ constant in time

$\Rightarrow \quad w(x,t)\equiv 0.$

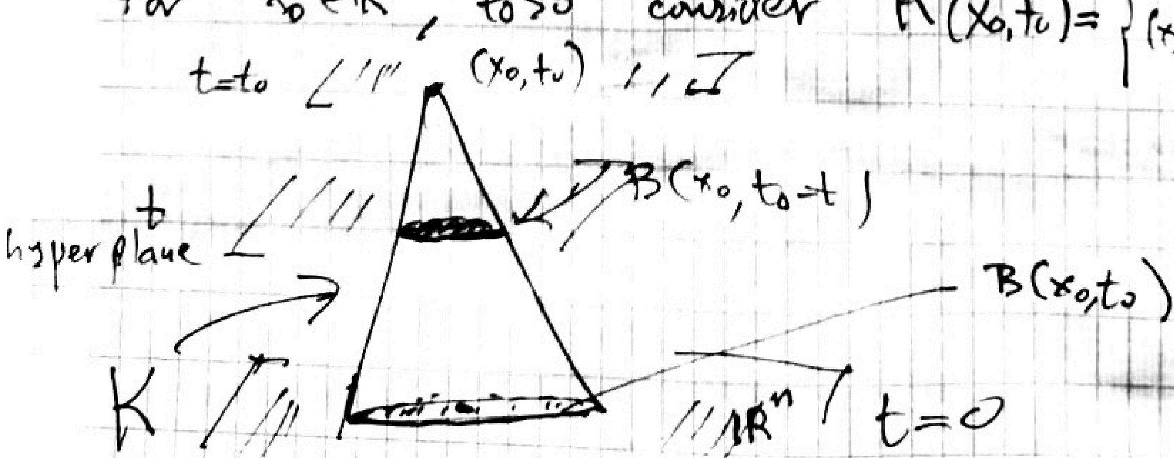
I.C. in (6)

$\blacksquare$

Next via the Energy method we will obtain the domain of dependence for the W.E. in $\mathbb{R}^n$, from which the finite speed of propagation will also follow.

Backwards (solid) cone

For $x_0\in\mathbb{R}^n$, $t_0>0$ consider $K(x_0,t_0)=\left\{(x,t)\;\middle|\; \begin{array}{l} 0\le t\le t_0 \\ |x-x_0|\le t_0-t \end{array}\right\}$



Th (Domain of Dependence – Finite speed of Propagation)

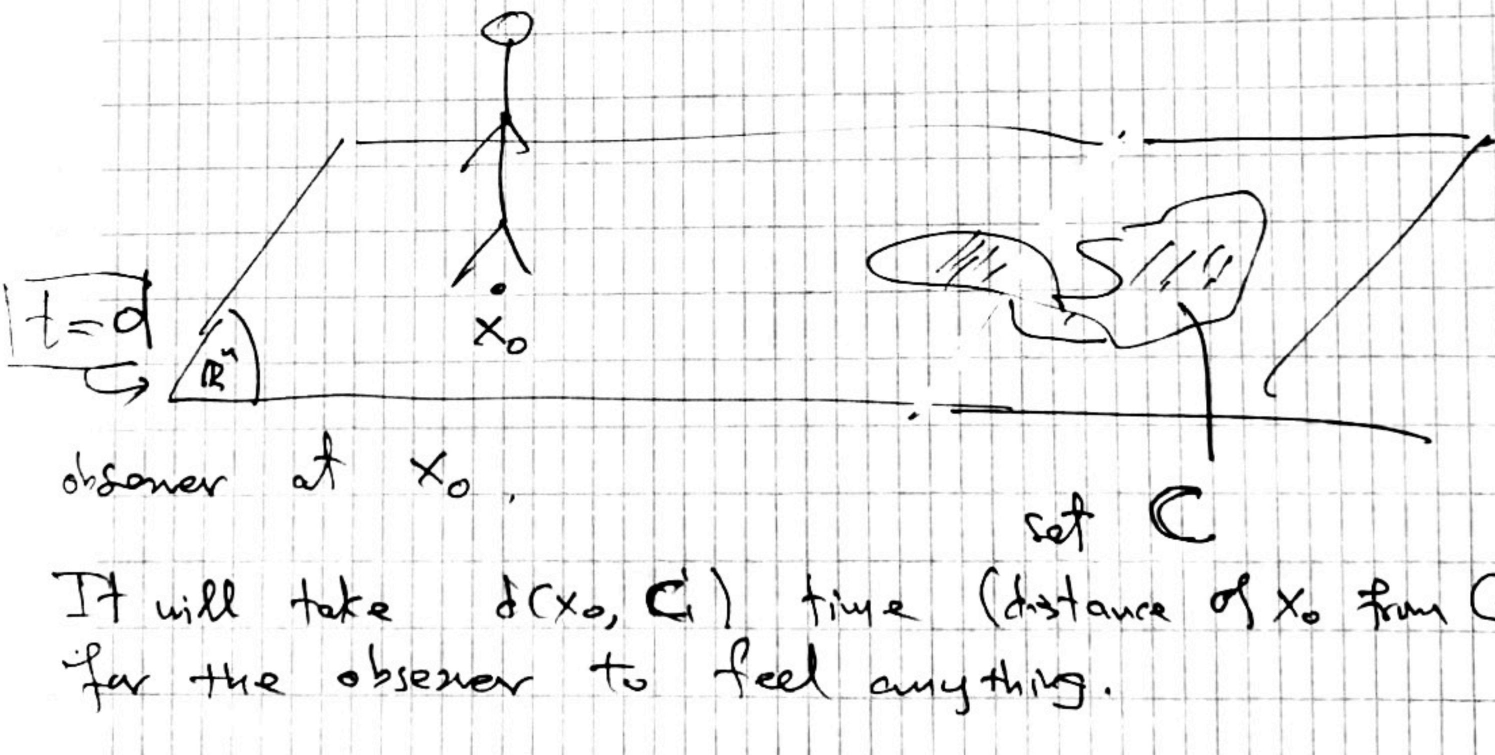
If $u \equiv u_t = 0$ on $B(x_0, t_0) \times \{t=0\}$

$\Rightarrow$

$$u = 0 \quad \text{within the Cone } K(x_0,t_0)$$

Remark (The meaning of this theorem).

Suppose the medium is disturbed in C, and the



observer at x_0,

It will take $d(x_0, C)$ time (distance of x_0 from C) for the observer to feel anything.

Pf

Set $\displaystyle e(t) := \frac{1}{2} \int_{B(x_0,t_0-t)} (u_t^2 + |\nabla u|^2)\, dx$, $u_t = u_t(x,t)$, $\nabla u(x,t)$

$$(7)\quad \frac{de(t)}{dt} = \int_{B(x_0,t_0-t)} (u_t u_{tt} + \nabla u \, \nabla u_t)\, dx - \frac{1}{2} \int_{\partial B(x_0,t_0-t)} (u_t^2 + |\nabla u|^2)\, dS_t$$

1^{st} term in (7) :

(8) $$\int_{B(x_0,t_0-t)} \nabla u \nabla u_t \, dx = -\int_{B(x_0,t_0-t)} \Delta u \, u_t \, dx + \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t \, dS_t$$

$\Rightarrow$ utilizing the equation

$$\frac{de(t)}{dt} = \int_{\partial B(x_0,t_0-t)} \left\{ \frac{\partial u}{\partial \nu} u_t - \left(\frac{1}{2} u_t^2 + |\nabla u|^2 \right) \right\} dS_t \qquad (9).$$

Note now

(10) $$\left| \frac{\partial u}{\partial \nu} u_t \right| \le |\nabla u| \, |u_t| \qquad \left(\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu, \quad |\nu| = 1 \right)$$

$$\le \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2$$

since $u(x,0) = 0 \Rightarrow \nabla u = 0$, $u_t(x,0) = 0$

$\therefore$ $$\frac{de(t)}{dt} \le 0 \Rightarrow e(t) \le e(0) = 0.$$

$$\Rightarrow \int_{B(x_0,t_0-t)} (u_t^2 + |\nabla u|^2) \, dx = 0$$

$$\Rightarrow u_t^2 + |\nabla u|^2 = 0$$

$$\Rightarrow u(x,t) \equiv \text{const}.$$

But $u(x,0) = 0 \Rightarrow u(x,t) \equiv 0.$

$\square$

Problems 18, 19, 20 from Evans

The Nonhomogeneous Problem

(11) $$\begin{cases} u_{tt} - \Delta u = f(x,t) \quad , & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_t(x,0) = 0 \end{cases}$$

Consider the wave equation

(12) $$\begin{cases} u_{tt}(x,t;s) = \Delta u(x,t;s) \\ u(x,t;s) = 0 \\ u_t(x,t;s) = f(x,s) \end{cases}$$

s = parameter
equation is in $x-t$

Define

(13) $$u(x,t) := \int_0^t u(x,t;s)\, ds \qquad (x \in \mathbb{R}^n, t \geq 0)$$

Claim: $u_{tt} = \Delta u + f$.

Motivation

Recall the variation of constants formula for

(14) $$\hat{u}_t = A\hat{u} + \hat{f} \quad , \quad A = \text{operator}$$

(15) $$\hat{u}(t) = e^{At}\hat{u}(0) + \int_0^t e^{A(t-s)} \hat{f}(s)\, ds$$

We write (14) as a 1^{st} order system:

$$(16)\quad \begin{cases} u_t = v \\ v_t = \Delta u + f \end{cases} \qquad \hat{u} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$A\hat{u} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad \hat{f} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$e^{At}\hat{u} = e^{At}\begin{pmatrix} u \\ v \end{pmatrix} \Leftrightarrow \begin{matrix} u_t = v \\ v_t = \Delta u \end{matrix} \Leftrightarrow \; u_{tt} = \Delta u$$

Interested in the nonhomogeneous term in (15).

<u>Exercise</u>: Utilizing this approach derive (13).

<u>Pf of Claim</u>

$$u_{tt}(x,t) = \frac{\partial^2}{\partial t^2}\left(\int_0^t u(x,t;s)\,ds\right) = \frac{\partial}{\partial t}\Big[\underbrace{\cancel{u(x,t;t)}}_{(\text{by } (12))\ 0} + \int_0^t u_t(x,t;s)\,ds\Big]$$

$$= u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds$$

$$= f(x,t) + \int_0^t \Delta u(x,t;s)\,ds$$

$$= f(x,t) + \Delta \int_0^t u(x,t;s)\,ds = \Delta u(x,t) + f(x,t). \qquad \square$$