TAKE HOME EXAM, PDE1, Winter 25-26

Part III (Heat)

1) a) Consider the B.I.V. problem

$$\begin{cases} u_t - u_{xx} + u = f(x) & , 0<x<l, \ t>0 \\ u(x,0) = g(x) & , 0<x<l \\ u_x(0,t) = u_x(l,t) = 0 & , t \geq 0 \end{cases}$$

Establish the uniqueness of solution via the Energy Method. Assume f and g sufficiently smooth for your argument.

b) Consider the B.I.V. problem

$$\begin{cases} u_t - u_{xx} + u_x = 0 & , 0<x<l, \ t>0 \\ u(x,0) = \phi(x) & , 0<x<l \\ u_x(0,t) = u_x(l,t) = 0 \end{cases}$$

Find appropriate g s.t. $v = gu$ satisfies the equation

$$v_t - v_{xx} + v = 0$$

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2) Suppose $u(x,t) > 0$, and C^2, solution to
$u_t = \mu u_{xx}$, $t>0$.
Show that $\theta = -\frac{2\mu u_x}{u}$ satisfies $\left(\theta = -\frac{2\mu u_x}{u}\right)$

$$\theta_t + \theta\,\theta_x = \mu\,\theta_{xx} \quad , t>0$$

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(→ 2 pgs)

3) Consider the problem

$$\begin{cases} u_t = k\,\Delta u \ , \ k>0 \ \text{constant} \ , \ x \in \Omega \subset \mathbb{R}^n, \ \text{open, bdded}, \ t>0 \\ u(x,t) = g(x) \ , \ x \in \partial\Omega \ , \ t \geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

Suppose v solution to

$$\begin{cases} \Delta v = 0 \ , \ x \in \Omega \\ v = g \ , \ \text{on } \partial\Omega . \end{cases}$$

Show that

$$u(\cdot, t) \xrightarrow{L^2(\Omega)} v \ , \ \text{as } t \to +\infty .$$

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