TAKE HOME EXAM, PDE I, Winter 25-26

(Part III will be uploaded ASAP)

Part II (Wave)

1) Prove the Mean Value Property for harmonic functions $u(x)$, $x \in \mathbb{R}^3$ by following the method described below: Harmonic functions are time independent solutions of the wave equation. Utilize the method of spherical means (See (14), p 70 in Evans).

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characteristic fn

2) Consider the (IVP) $\begin{cases} u_{tt} = \Delta u, & (x,t) \in \mathbb{R}^3 \times (0,\infty) \\ u(x,0) \equiv 0, \ x \in \mathbb{R}^3, & u_t(x,0) = \mathbb{1}_{B_\rho(0)}, \ x \in \mathbb{R}^3 \end{cases}$

(a) Plot the graph of the solution $x \to u(x,t)$ for $t = \frac{1}{2}$, $t = 1$, $t = 2$ (as a function of x).

(b) Plot the graph of the solution $t \to u(x,t)$ for $|x| = \frac{1}{2}$, $|x| = 2$ (what a fixed observer detects)

(c) Suppose $|x_0| < \rho$. Suppose you are on the wave along a light ray that emanates at $(x_0, 0)$. That is examine the quantity $u(x_0 + t\vec{v}, t)$, $|\vec{v}| = 1$. Prove that

$$t\, u(x_0 + t\bar{v}, t) \text{ converges as } t \to +\infty.$$

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3) (a) Every solution of the wave equation in 3 dimensions, with data with compact support inside a sphere satisfies $u(x,t) = 0$ at any given point $x \in \mathbb{R}^3$, and sufficiently large t.

(b) Prove that $u(x,t) = O(t^{-1})$ uniformly as $t \to +\infty$ that is $t\, u(x,t)$ is bounded as a function on $\mathbb{R}^3 \times \mathbb{R}^+$.

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4) (a) Every solution of the wave equation in 2 dimensions

with data with compact support inside a circle satisfies
$u(x,t) = O(t^{-1})$ for fixed x, as $t \to +\infty$.

(b) Is $u(x,t) = O(t^{-1})$ true uniformly in x?

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