TAKE HOME EXAM, PDE I, Winter 25-26

(Part II and Part III will be uploaded ASAP)

Part I (Elliptic)

1) (a) Show that for $n=2$ the function $v(x)=\frac{1}{8\pi}r^2\ln r$, $r=|x-\xi|$, is a fundamental solution of Δ^2 (bilaplacian)

(b) Establish for $u\in C^4(\bar\Omega)$, $\xi\in\Omega$, $\Omega\subset\mathbb{R}^n$, open, connected the following identity

$$u(\xi)=\int_\Omega v\,\Delta^2 u\,dx-\int_{\partial\Omega}\left(v\frac{\partial\Delta u}{\partial n}-\Delta u\frac{\partial v}{\partial n}+\Delta v\frac{\partial u}{\partial n}-u\frac{\partial\Delta v}{\partial n}\right)dS$$

(Hint: utilize (25), p34, Evans).

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2) Suppose $L=\Delta+cI$, c constant

(a) Find all spherically symmetric solutions of $Lu=0$

(b) Prove that

$$K(x,\xi)=-\frac{\cos(\sqrt{c}\,r)}{4\pi r},\quad r=|x-\xi|$$

is a fundamental solution to L.

(c) Show that solutions to $Lu=0$ in $\{|x-\xi|<\rho\}$ for $\sin(\sqrt{c}\,\rho)\neq 0$ has the property (MVP)

$$u(\xi)=\frac{\sqrt{c}\,\rho}{\sin(\sqrt{c}\,\rho)}\frac{1}{4\pi\rho^2}\int_{|x-\xi|=\rho}u\,dS_x$$

(For $c=-\gamma<0$ $\frac{\sqrt{c}\,\rho}{\sin(\sqrt{c}\,\rho)}$ is replaced by $\frac{\sqrt{\gamma}\,\rho}{\sinh(\sqrt{\gamma}\,\rho)}$)

Hint: $G(x,\xi)=K(x,\xi)+k\frac{\sin(\sqrt{c}\,r)}{r}$

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3) Prove that for $u \in C^2(D) \cap C(\bar{D})$, $D = \{(x,y) \mid x^2+y^2<1\}$

$$\begin{cases} \Delta u = u^3, & \text{in } D \\ u = 0, & \text{on } \partial D \end{cases} \implies u \equiv 0.$$

4) Suppose $u = u(x,y)$ satisfies

$$a(x,y)\,u_x + b(x,y)\,u_y = -u, \quad (x,y) \in D = \{x^2+y^2<1\},$$

and suppose that

$$a(x,y)\,x + b(x,y)\,y > 0 \quad \text{on } \partial D. \qquad (a(x,y) \text{ smooth fn})$$

Show that $u \equiv 0$ on D.

5) Let $u(x,y) \in C^3(U) \cap C(\bar{U})$, U open, connected, bdd in $\mathbb{R}^2$

Suppose it satisfies

$$\det D^2u(x,y) := \det \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix} \leq 0, \quad (x,y) \in U.$$

Prove that both the maximum and the minimum of $u(x,y)$ are realized on ∂U.

Hint: By contradiction. Assume $\max\limits_{\partial U} u =: m < \max\limits_{\bar{U}} u = M$

$M = u(x_0,y_0)$. Set $g(x,y) := M - \varepsilon\left[(x-x_0)^2+(y-y_0)^2\right]$

α) Show that $g(x,y) > m$ on ∂U, $\varepsilon>0$ small, and so $g(x,y) > u(x,y)$

$\phi(x,y) := u(x,y) - g(x,y)$, $\mu = \max \phi = \phi(x_1,y_1)$

β) Show that $D^2u(x_1,y_1)$ is negative definite — contradiction $\#$