Lecture 20

RK on the Euler-Poisson-Darboux method of Spherical Means

Recall $\quad U(x;r,t) = \fint_{\partial B(x,r)} u(y,t)\,dS \quad$ (radial, centered at fixed x)

satisfies also the wave equation:

$$(1) \qquad U_{tt} = U_{rr} + \frac{n-1}{r} U_r$$

This is expected since the Laplacian Δ is invariant under rotations, and rotating in all possible directions (centered at x) and taking the average leads to the radial Laplacian.

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The Hadamard Method of descent: $n=2$

Idea: Increasing the dimension

Consider

$$(2) \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \qquad \begin{cases} u(x_1,x_2,0) = g(x_1,x_2) \\ u_t(x_1,x_2,0) = h(x_1,x_2) \end{cases}$$

$u = u(x_1, x_2, t)$

Set

$$(3) \qquad \tilde{u}(x_1,x_2,x_3,t) := u(x_1,x_2,t)$$

Trivially

$$(4) \qquad \frac{\partial^2 \tilde{u}}{\partial t^2} = \frac{\partial^2 \tilde{u}}{\partial x_1^2} + \frac{\partial^2 \tilde{u}}{\partial x_2^2} + \frac{\partial^2 \tilde{u}}{\partial x_3^2}$$

Set

$$\tilde{x} = (x_1, x_2, x_3) \quad , x = (x_1, x_2)$$

Define

$$\tilde{u}(\tilde{x}, 0) = \tilde{g}(\tilde{x}) = g(x)$$
$$\tilde{u}_t(\tilde{x}, 0) = \tilde{h}(\tilde{x}) = h(x)$$

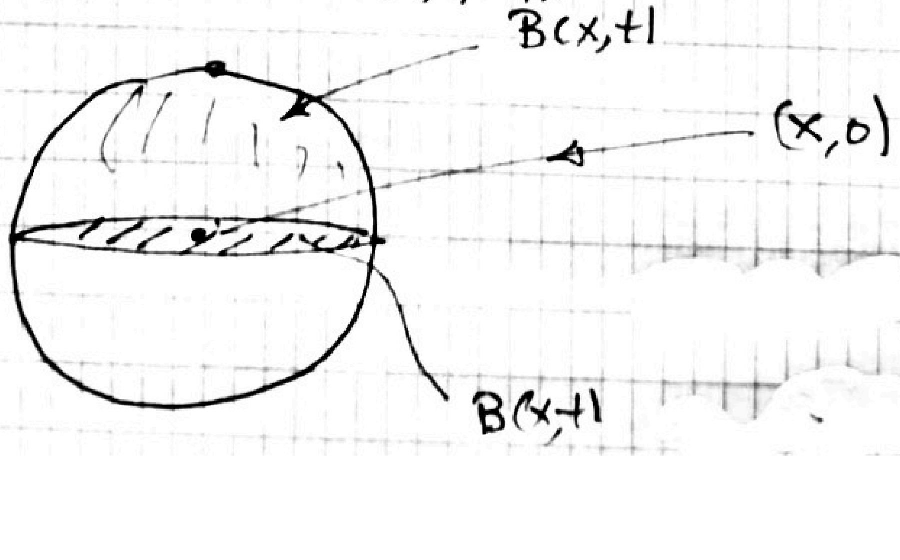
Kirchhoff $\Longrightarrow$

(5) $$u(x,t) = \tilde{u}(\tilde{x}, t)$$

(by 1st line, p 9) $$= \frac{\partial}{\partial t}\left(t \fint_{\partial \tilde{B}(\tilde{x},t)} \tilde{g}\, d\tilde{S} \right) + t \fint_{\partial \tilde{B}(\tilde{x},t)} \hat{h}\, dS \qquad (5')$$

First term:

(6) $$\fint_{\partial \tilde{B}(\tilde{x},t)} \tilde{g}\, d\hat{S} = \frac{1}{4\pi t^2} \int_{\partial \tilde{B}(\hat{x},t)} \tilde{g}\, d\hat{S}$$



Continuing from (6):

upper + lower hemisphere.

$$(7)\quad \frac{1}{4\pi t^2}\int_{\partial\tilde{B}(\tilde{x},t)} \tilde{g}\, dS = \frac{2}{4\pi t^2}\int_{B(x,t)} g(y)\left(1+|\nabla\gamma(y)|^2\right)^{1/2} dy$$

Calculating

$$\left(1+|\nabla\gamma|^2\right)^{1/2} = \frac{t}{\left(t^2-|y-x|^2\right)^{1/2}}$$

we conclude

$$(8)\quad \fint_{\partial\tilde{B}(\tilde{x},t)} \tilde{g}\, d\tilde{S} = \frac{1}{2\pi t}\int_{B(x,t)} \frac{g(y)}{\left(t^2-|y-x|^2\right)^{1/2}}\, dy$$

$$= \frac{t}{2}\fint_{B(x,t)} \frac{g(y)}{\left(t^2-|y-x|^2\right)^{1/2}}\, dy$$

Finally from (5')

$$(9)\quad u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2\fint_{B(x,t)} \frac{g(y)}{\left(t^2-|y-x|^2\right)^{1/2}}\, dy\right)$$

$$+\frac{t^2}{2}\fint_{B(x,t)} \frac{h(y)}{\left(t^2-|y-x|^2\right)^{1/2}}\, dy$$

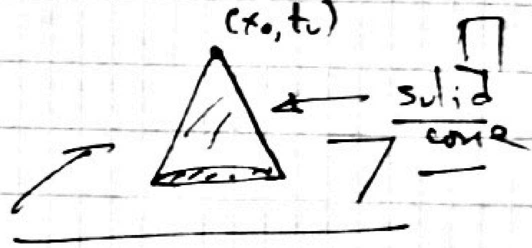
For performing the differentiation in the 1^{st} term we first change variables

$$B(x,t) \longrightarrow B(0,1),$$

differentiate, then convert back to $B(x,t)$.

Conclusion (Poisson's formula)

$$(10) \quad u(x_0,t_0) = \frac{1}{2} ⨍_{B(x_0,t_0)} \frac{t g(y) + t^2 h(y) + t \nabla g(y) \cdot (y-x)}{(t - |y-x_0|^2)^{1/2}} \, dy$$



Remarkable Fact

Only in $\mathbb{R}^3$ signals are sharp.

Throwing a stone in a lake, generates concentric waves, and an observer on the surface feels the disturbance forever, although diminishing in time.

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