Lecture 19

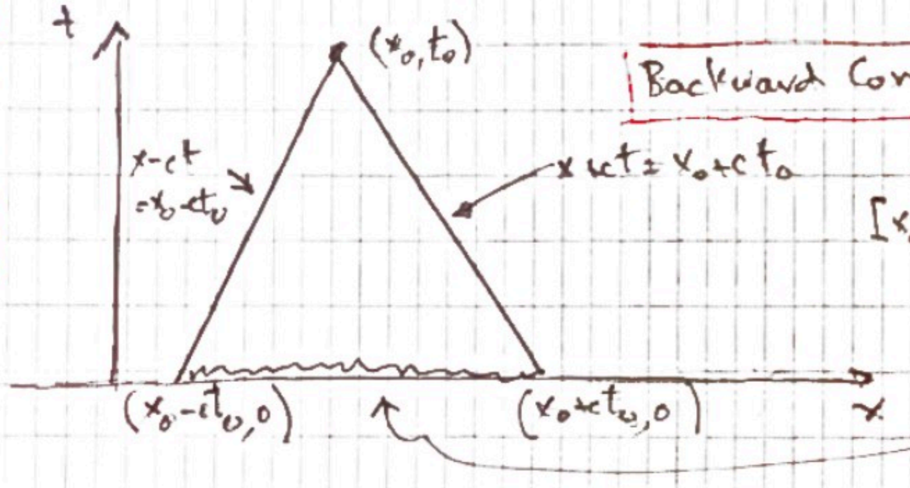
Comparison between $n=1$ and $n=3$

(1) $u_{tt} - c^2 u_{xx} = 0$, $x \in \mathbb{R}$

$$\begin{cases} u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}, \quad g \in C^2,\ h \in C^1$$

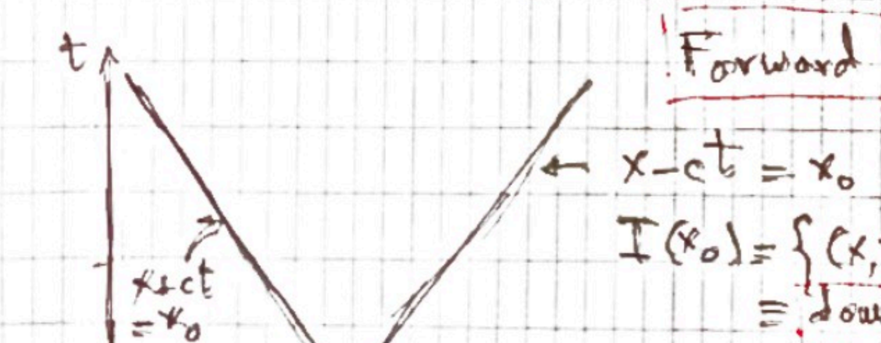
(2) $$u(x_0,t_0) = \frac{1}{2}\left[g(x_0+ct_0) + g(x_0-ct_0)\right] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} h(y)\,dy$$

Backward Cone

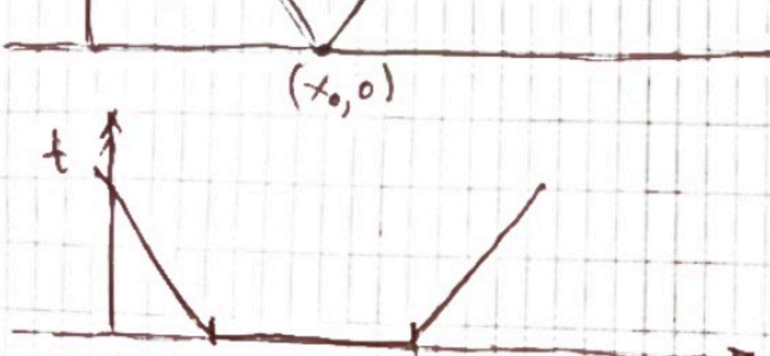


$[x_0-ct_0,\ x_0+ct_0]$ = domain of dependence of (x_0,t_0) = set of pts in $t=0$ that can influence $u(x_0,t_0)$

Forward Cone



$I(x_0) = \{(x,t) : |x - x_0| \le ct,\ t \ge 0\}$ = domain of influence of $(x_0,0)$ = set of pts in x,t whose values $u(x,t)$ may be affected by data at $(x_0,0)$.



Domain of influence of $E = [a,b]$, $I(E) = \bigcup_{a \le \xi \le b} I(\xi)$

(a) Suppose $\boxed{h \equiv 0}$, $\operatorname{spt} g \subset [a,b]$.

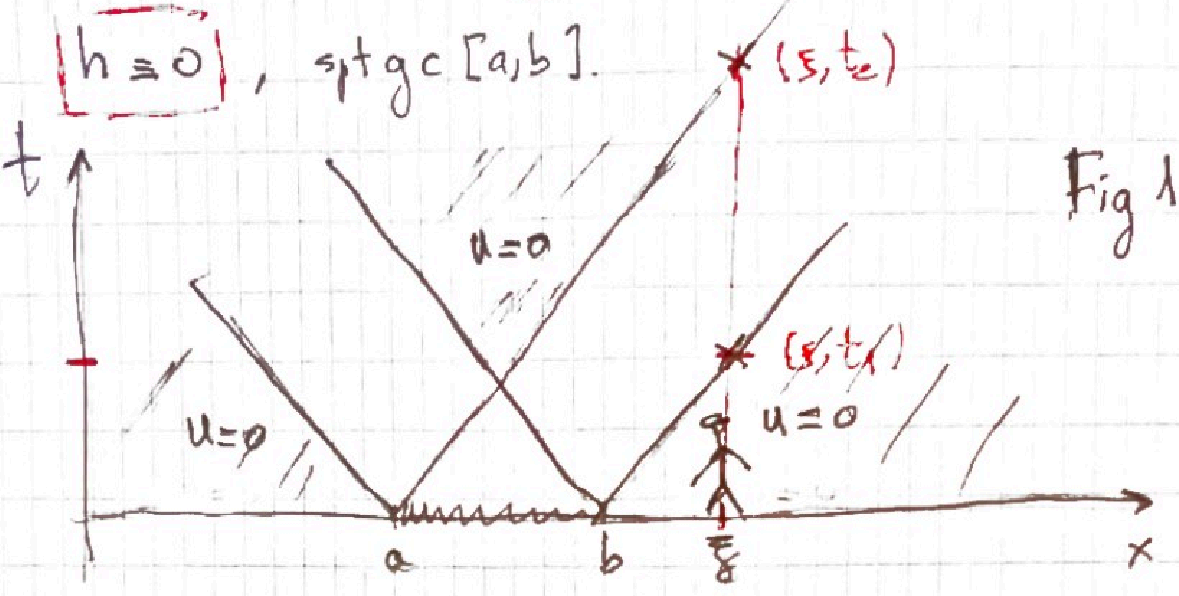


Fig 1

An observer at ξ will first detect a disturbance (signal) at $t_1 = \frac{\xi - b}{c}$, and will last detect a disturbance at

$t_2 = \frac{\xi - a}{c}$.

(b) Suppose $\boxed{h \not\equiv 0}$, $\operatorname{spt} g, h \subset [a,b]$

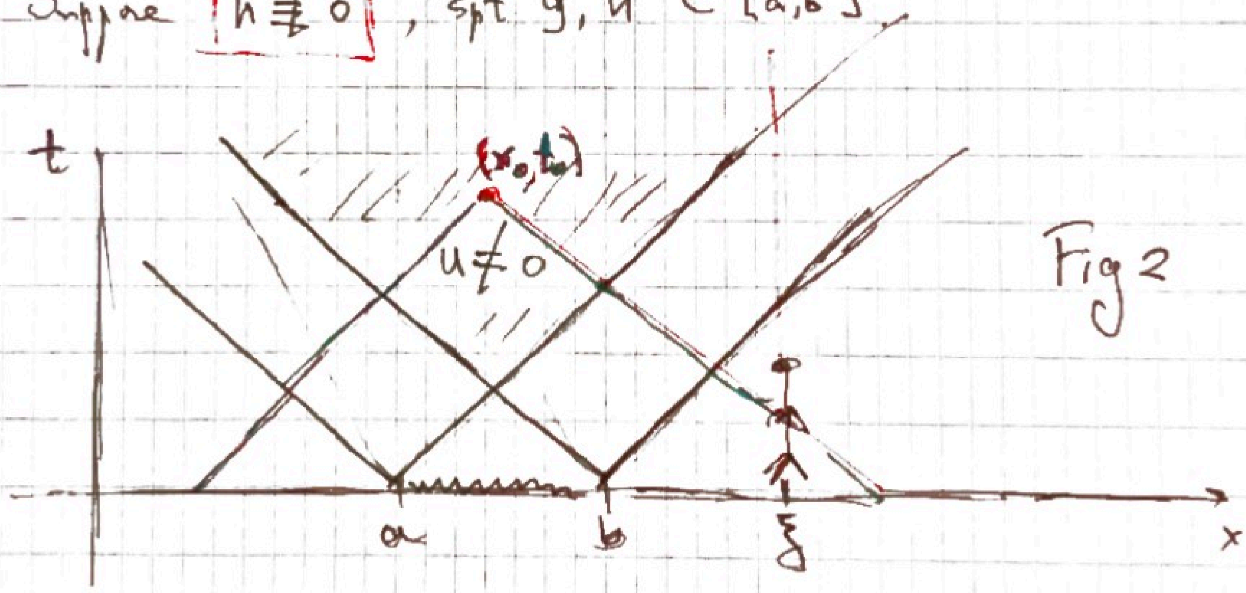
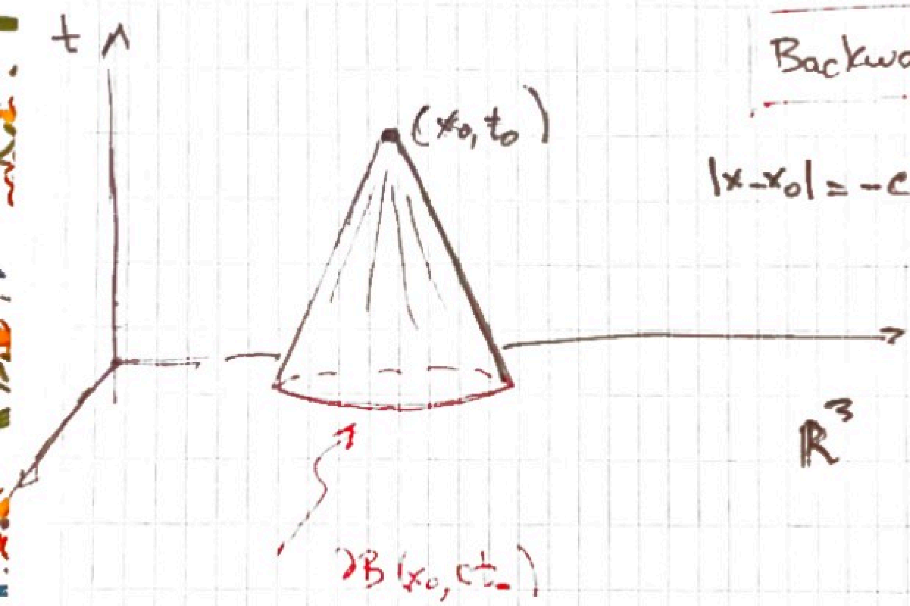


Fig 2

Clearly $u(\xi, t) \not\equiv 0$ for $t > t_2$ due to the term $\int_{\xi - ct}^{\xi + ct} h(y)\,dy$. Hence the observer detects a noise forever after t_2. The signal is not <u>sharp</u>.

(3) $\begin{cases} u_{tt} - c^2 \Delta u = 0, & x \in \mathbb{R}^3 \\ u(x,0) = g(x), & u_t(x,0) = h(x) \end{cases}$

(4) $$u(x_0,t_0) = \fint_{\partial B(x_0, ct_0)} [t_0 h(y) + g(y) + \nabla g(y) \cdot (y - x_0)] \, dS_y$$



Backward cone

$|x - x_0| = -c(t - t_0)$, $t \le t_0$

Ⓘ <u>Difference with $n=1$</u> : Domain of dependence of (x_0, t_0) is the <u>sphere</u> $\partial B(x_0, ct_0)$, a $2 = 3-1$ dim set. ($n=3$)
The Domain of dependence of (x_0, t_0) for $n=1$ is the interval $[x_0 - ct_0, x_0 + ct_0]$, a 1 dim. set. ($n=1$)
<u>Hence</u> for $n=3$ the domain of dependence is 1 dimension less.

Ⓘⓘ <u>Difference with $n=1$</u> : For $n=1$ if $u(x,0)$ is C^k, then
($h \equiv 0$)
$u(x,t)$ is C^k. For $n=3$ if $u(x,0)$ is C^k then $u(x,t)$ by (4) is C^{k-1}. The physical meaning of this the phenomenon

of <u>focussing</u>, possible in dimensions $n \geq 2$.

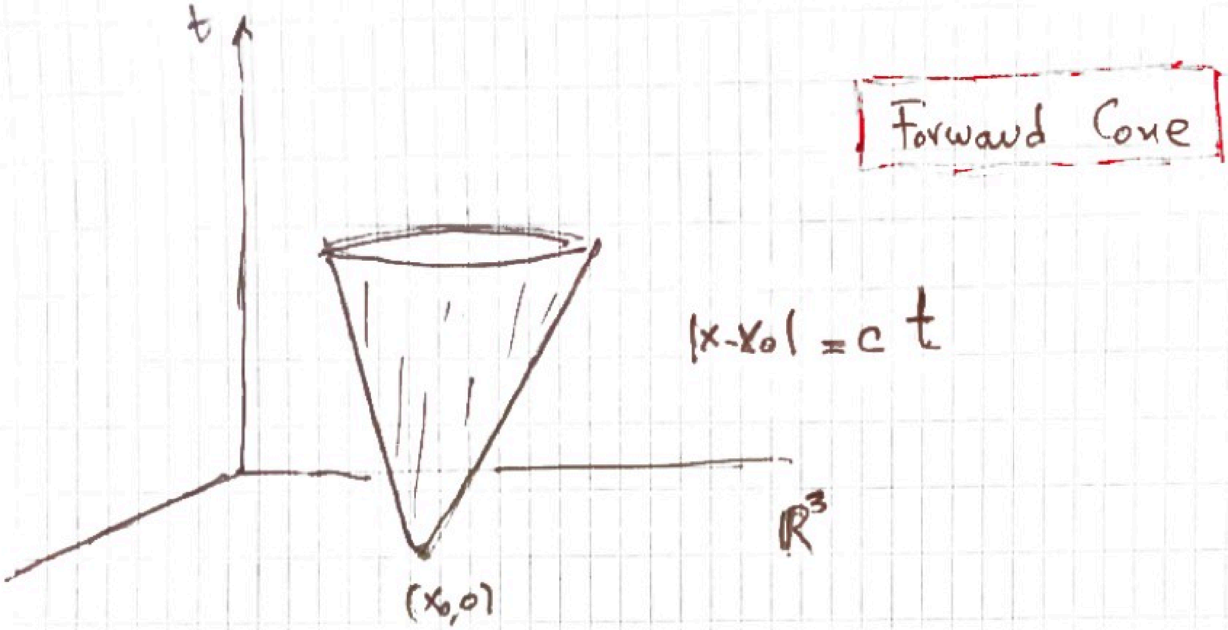


Fig A

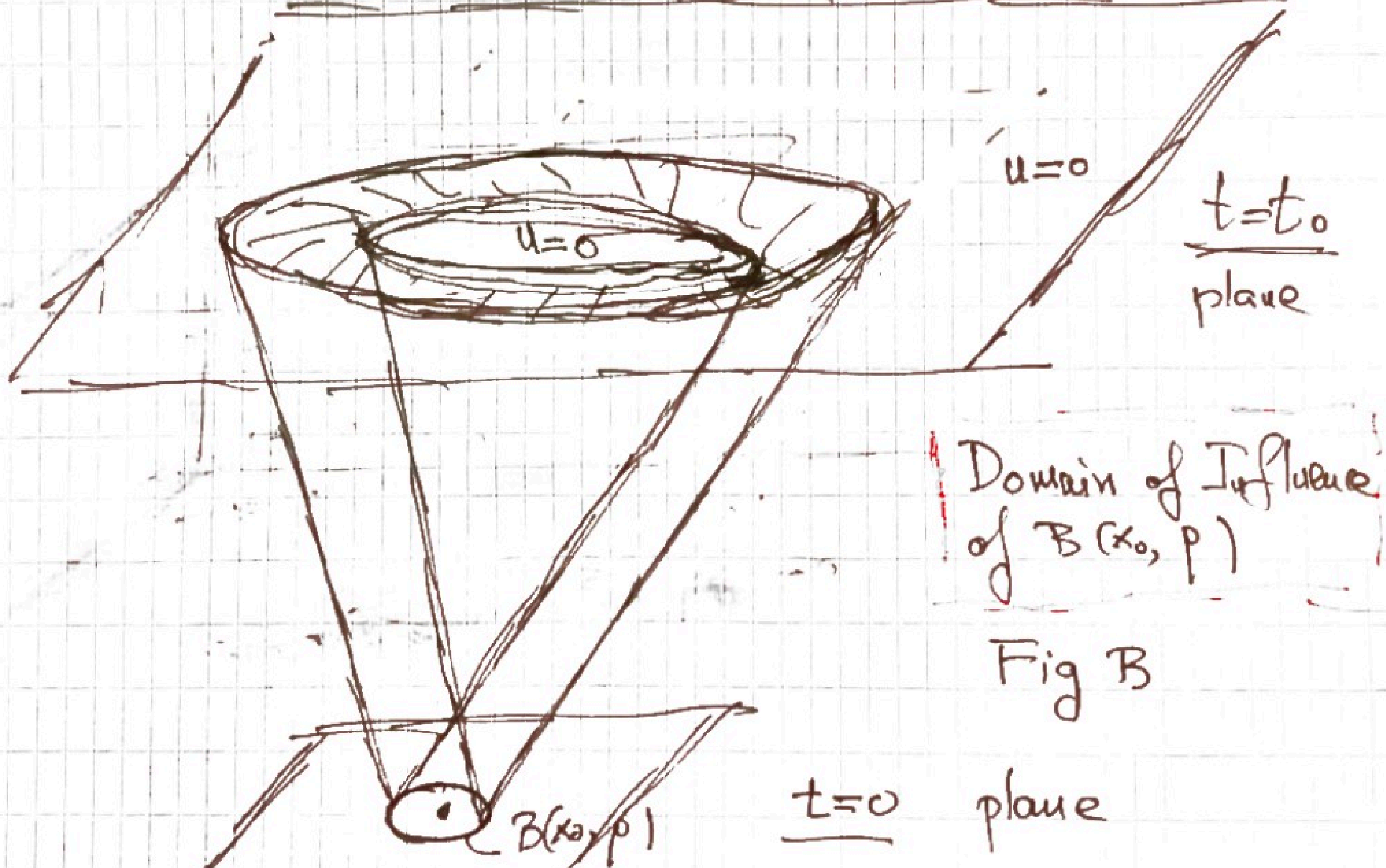


Fig B

spt h, spt $g \subset B(x_0,\rho)$

Signals localized in the annulus Ann $(x_0; ct-\rho, ct+\rho)$

5

III Difference with $n=1$ (for $h \neq 0$)

Fig B on p 4 similar to Fig 1 on p 2

In $\mathbb{R}^3$ the signal is sharp. The observer does not detect any signal for $t > \frac{(|\xi| + \rho)}{c}$.

(To be continued)